Hamiltonian Formulation

- Newtonian $\rightarrow$ Lagrangian $\rightarrow$ Hamiltonian
    - Describe same physics and produce same results
    - Difference is in the viewpoints
        - Flexibility of coordinate transformation

- Hamiltonian formalism linked to the development of
    - Hamilton-Jacobi theory
    - Classical perturbation theory
    - Quantum mechanics
    - Statistical mechanics

Hamiltonian

Lagrangian describing a system where angular momentum is conserved, does not depend on time explicitly, i.e.

$$\frac{dL}{dt} = 0$$

we can express the dynamics in terms of the $2n + 1$ variables q_i, p_i, and t.

The Lagrangian: $L = L(q_i, \dot{q}_i, t)$

Therefore, differentiating w.r.t. time:

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i \qquad (1)$$

Conservation Theorems and Physical Significance of Hamiltonian

- Some problems involve *cyclic coordinates* — Hamiltonian procedure is adapted to treatment of such problems

Cyclic coordinates

A coordinate q_j which does not appear in the *Lagrangian*.

Then Lagrange's equations $\rightarrow$ its conjugate momentum p_j is constant.

Then $$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j} = 0$$

$\Rightarrow$ A cyclic coordinate will also be absent from H

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \qquad \text{(vi)}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \text{(vii)}$$

Equations (iv) and (vi) give:

$$\boxed{p_\theta = mR^2\dot{\theta} = \text{Constant}}$$

$\Rightarrow$ angular momentum about the *z*-axis is constant of motion

From equations (v) and (vi): $m\ddot{z} = -kz$

or $$\boxed{\ddot{z} + \omega_0^2 z = 0}$$

where $\omega_0^2 = \dfrac{k}{m}$

Therefore, motion in *z*-direction is *simple harmonic motion.*

Example-2:

Use the Hamiltonian method to find the equations of motion for a spherical pendulum of mass m and length b .

Solution:

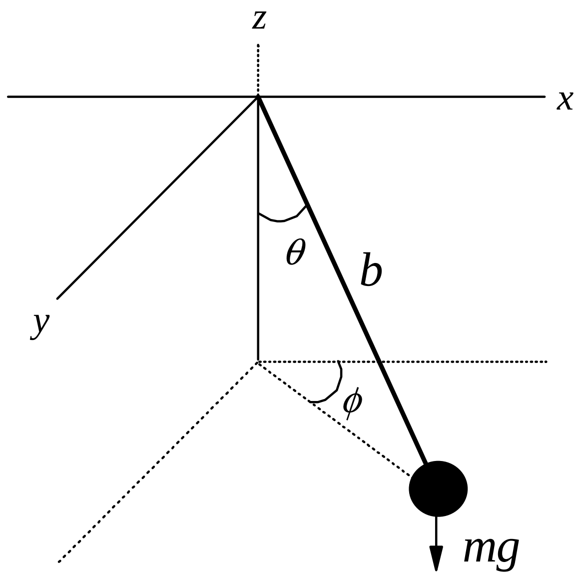
The generalized coordinates are θ and ϕ .

Using spherical polar coordinates:

$$x = b\sin\theta\cos\phi$$

$$y = b\sin\theta\sin\phi$$

$$z = b\cos\theta$$



K.E. is given by: $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

and P.E. : $V = -mgz$

Perform simple calculations to transform the K.E. and P.E. equations using spherical coordinates.

Example-3:

Consider a particle of mass m moving freely in a conservative force field, whose potential function is V. Find the Hamiltonian function and show that the canonical equations of motion reduce to Newton's equations. Use rectangular coordinates.

Solution:

For a particle moving freely in a conservative field:

K.E.: $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

P.E.: $V = V(x, y, z)$

The Lagrangian is: $L = T - V$

$$= \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)$$

Generalised momenta are then:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$