2 The Definition

As I've already stated, I am assuming that you have seen complex numbers to this point and that you're aware that $i = \sqrt{-1}$ and so $i^2 = -1$. This is an idea that most people first see in an algebra class (or wherever they first saw complex numbers) and $i = \sqrt{-1}$ is defined so that we can deal with square roots of negative numbers as follows,

$$\sqrt{-100} = \sqrt{(100)(-1)} = \sqrt{100}\,\sqrt{-1} = \sqrt{100}\,i = 10\,i$$

What I'd like to do is give a more *mathematical* definition of a complex numbers and show that $i^2 = -1$ (and hence $i = \sqrt{-1}$ can be thought of as a consequence of this definition. We'll also take a look at how we define arithmetic for complex numbers.

What we're going to do here is going to seem a little backwards from what you've probably already seen but is in fact a more accurate and mathematical definition of complex numbers. Also note that this section is not really required to understand the remaining portions of this document. It is here solely to show you a different way to define complex numbers.

So, let's give the definition of a complex number.

Given two real numbers a and b we will define the complex number z as,

$$z = a + bi$$

Note that at this point we've not actually defined just what i is at this point. The number a is called the **real part** of z and the number b is called the **imaginary part** of z and are often denoted as,

$$\text{Re}\,z = a \qquad\qquad \text{Im}\,z = b$$

There are a couple of special cases that we need to look at before proceeding. First, let's take a look at a complex number that has a zero real part,

$$z = 0 + bi = bi$$

In these cases, we call the complex number a **pure imaginary** number.

Next, let's take a look at a complex number that has a zero imaginary part,

$$z = a + 0i = a$$

In this case we can see that the complex number is in fact a real number. Because of this we can think of the real numbers as being a subset of the complex numbers.

We next need to define how we do addition and multiplication with complex numbers. Given two complex numbers $z_1 = a + bi$ and $z_2 = c + di$ we define addition and multiplication as follows,

$$z_1 + z_2 = (a + c) + (b + d)\,i$$
$$z_1 z_2 = (ac - bd) + (ad + cb)\,i$$

Therefore, the multiplicative inverse of the complex number z is,

$$z^{-1} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}\, i \tag{6}$$

As you can see, in this case, the “exponent” of -1 is not in fact an exponent! Again, you really need to *forget* some notation that you’ve become familiar with in other math courses.

So, now that we have the definition of the multiplicative inverse we can finally define division of two complex numbers. Suppose that we have two complex numbers z_1 and z_2 then the division of these two is defined to be,

$$\frac{z_1}{z_2} = z_1\, z_2^{-1} \tag{7}$$

In other words, division is defined to be the multiplication of the numerator and the multiplicative inverse of the denominator. Note as well that this actually does match with the process that we used above. Let’s take another look at one of the examples that we looked at earlier only this time let’s do it using multiplicative inverses. So, let’s start out with the following division.

$$\frac{6 + 3i}{10 + 8i} = (6 + 3i)\,(10 + 8i)^{-1}$$

We now need the multiplicative inverse of the denominator and using Equation 6 this is,

$$(10 + 8i)^{-1} = \frac{10}{10^2 + 8^2} - \frac{8}{10^2 + 8^2}\, i = \frac{10 - 8i}{164}$$

Now, we can do the multiplication,

$$\frac{6 + 3i}{10 + 8i} = (6 + 3i)\,(10 + 8i)^{-1} = (6 + 3i)\,\frac{10 - 8i}{164} = \frac{60 - 48i + 30i - 24i^2}{164} = \frac{21}{41} - \frac{9}{82} i$$

Notice that the second to last step is identical to one of the steps we had in the original working of this problem and, of course, the answer is the same.

As a final topic let’s note that if we don’t want to remember the formula for the multiplicative inverse we can get it by using the process we used in the original multiplication. In other words, to get the multiplicative inverse we can do the following

$$(10 + 8i)^{-1} = \frac{1}{(10 + 8i)}\,\frac{10 - 8i}{(10 - 8i)} = \frac{10 - 8i}{10^2 + 8^2}$$

As you can see this is essentially the process we used in doing the division initially.

(b) $z_1 - z_2 = 13 - 2i \qquad \Rightarrow \qquad \overline{z_1 - z_2} = \overline{13 - 2i} = 13 + 2i$

(c) $\overline{z}_1 - \overline{z}_2 = \overline{5+i} - \left(\overline{-8+3i}\right) = 5 - i - (-8 - 3i) = 13 + 2i$

We can see that results from **(b)** and **(c)** are the same as the fact implied they would be.

There is another nice fact that uses conjugates that we should probably take a look at.However, instead of just giving the fact away let's derive it.We'll start with a complex number $z = a + bi$ and then perform each of the following operations.

$$\begin{aligned} z + \overline{z} &= a + bi + (a - bi) \\ &= 2a \end{aligned} \qquad \begin{aligned} z - \overline{z} &= a + bi - (a - bi) \\ &= 2bi \end{aligned}$$

Now, recalling that Re $z = a$ and Im $z = b$ we see that we have,

$$\text{Re}z = \frac{z + \overline{z}}{2} \qquad \text{Im}z = \frac{z - \overline{z}}{2i} \tag{13}$$

Modulus

The other operation we want to take a look at in this section is the **modulus** of a complex number.Given a complex number $z = a + bi$ the modulus is denoted by $|z|$ and is defined by

$$|z| = \sqrt{a^2 + b^2} \tag{14}$$

Notice that the modulus of a complex number is always a real number and in fact it will never be negative since square roots always return a positive number or zero depending on what is under the radical.

Notice that if z is a real number (*i.e.* $z = a + 0i$) then,

$$|z| = \sqrt{a^2} = |a|$$

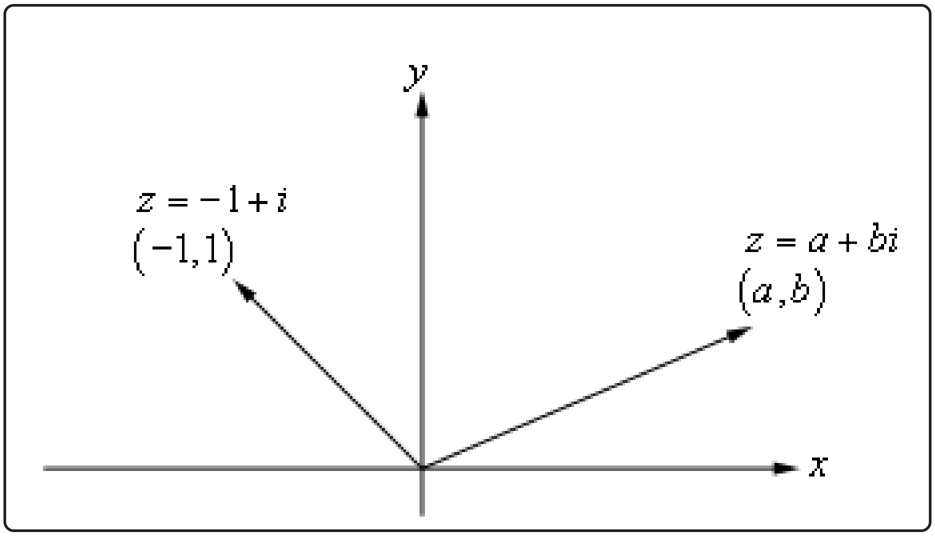
where the $|\cdot|$ on the z is the modulus of the complex number and the $|\cdot|$ on the a is the absolute value of a real number (recall that in general for any real number a we have $\sqrt{a^2} = |a|$).So, from this we can see that for real numbers the modulus and absolute value are essentially the same thing.

We can get a nice fact about the relationship between the modulus of a complex number and its real and imaginary parts.To see this let's square both sides of Equation 14 and use the fact that Re $z = a$ and Im $z = b$.Doing this we arrive at

$$|z|^2 = a^2 + b^2 = (\text{Re}\, z)^2 + (\text{Im}\, z)^2$$

Since all three of these terms are positive we can drop the Im z part on the left which gives the following inequality,

$$|z|^2 = (\text{Re}\, z)^2 + (\text{Im}\, z)^2 \geq (\text{Re}\, z)^2$$



In this interpretation we call the x-axis the **real axis** and the y-axis the **imaginary axis**. We often call the xy-plane in this interpretation the **complex plane**.

Note as well that we can now get a geometric interpretation of the modulus. From the image above, we can see that $|z|=\sqrt{a^2+b^2}$ is nothing more than the length of the vector that we're using to represent the complex number $z=a+bi$. This interpretation also tells us that the inequality $|z_1|<|z_2|$ means that z_1 is closer to the origin (in the complex plane) than z_2 is.

Polar Form

Let's now take a look at the first alternate form for a complex number. If we think of the non-zero complex number $z=a+bi$ as the point (a,b) in the xy-plane we also know that we can represent this point by the polar coordinates (r,θ), where r is the distance of the point from the origin and θ is the angle, in radians, from the positive x-axis to the ray connecting the origin to the point.