Row-Echelon (RE) Form

Row-Reduced Echelon (REE) Form

Examples: Solve the following system using Gauss-Elimination Method

The corresponding system of equations is

$$x + y - 3z = -1$$

$$y - 3z = -2$$

Assigning the free variable z an arbitrary value t,

$y = 3t - 2$,

$$x = -1 - 3t + 2 + 3t = 1$$

Hence, $x = 1, y = 3t - 2, z = t$ is solution of the given system of equations.

Since t is arbitrary real number, The system has **infinitely many solutions.**

Example 5: Consider the following system

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

For what values of λ and μ the system has (i) infinitely many solutions (ii) unique solution and (iii) no solution.

Solution: The Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right]$$

$$R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc|c} -1 & 3 & 4 & 30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right]$$

$$R_1 \to (-1)R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 3 & 2 & -1 & 9 \\ 2 & -1 & 2 & 10 \end{array}\right]$$

$$R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 11 & 11 & 99 \\ 0 & 5 & 10 & 70 \end{array}\right]$$

$$R_2 \to \left(\frac{1}{11}\right)R_2, R_3 \to \left(\frac{1}{5}\right)R_3$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 1 & 2 & 14 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & -30 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Eigenvalues of $4A^{-1} = 4\lambda^{-1}$	$\frac{4}{5}, -2$
Eigenvalues of $A^2 = \lambda^2$	$25, 4$
Eigenvalues of $A^2 - 2A + I = \lambda^2 - 2\lambda + 1$	$16, 9$
Eigenvalues of $A^3 + 2I = \lambda^3 + 2$	$127, -6$

Example 2: Find the eigen values of $A = \begin{vmatrix} 3 & 2 \\ 3 & 8 \end{vmatrix}$

Solution: Given $A = \begin{vmatrix} 3 & 2 \\ 3 & 8 \end{vmatrix}$, then the characteristic equation of matrix A is

$$|A - \lambda I_2| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 3 & 8-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(8-\lambda) - 6 = 0 \Rightarrow \lambda^2 - 11\lambda + 18 = 0 \Rightarrow (\lambda - 9)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 9 \text{ or } \lambda_2 = 2$$

Types of Eigen Values

Eigenvalues are non-repeated, whether matrix is symmetric or non-symmetric

Eigenvalues are repeated and the matrix isnon-symmetric

Eigenvalues are repeated and the matrix issymmetric

Type-1Type-2 Type-3

Example 3: Find the eigen values and eigen vector of the matrix A= $\begin{bmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$	
Solution: The characteristic equation is $\lvert A - \lambda I_n \rvert$=0	$= \left[\begin{array}{ccc\|c} 1 & 8/3 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 \to R_1 - 8/3R_2$

Let A be $n \times n$ matrix and λ be an eigen value for A. If λ occurs $(k \geq 1)$ times then k is called the **Algebraic multiplicity** of λ, and the number of basis vectors is called **Geometric multiplicity**.

Example: Find eigen values and eigen vectors of the matrix. $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Also determine algebraic and geometric multiplicity.

Solution: The characteristic equation is $|A - \lambda I_n| = 0$.

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix}$$

$$= (-2-\lambda)[(1-\lambda)(-\lambda)-(-2)(-6)] - 2[2(-\lambda)-(-1)(-6)] - 3[2(-2)-(-1)(1-\lambda)]$$

$$= (-2-\lambda)[-\lambda+\lambda^2-2] - 2[-2\lambda-6] - 3[-4+1-\lambda]$$

$$= -\lambda^3 - \lambda^2 + 21\lambda + 45$$

$$= -(\lambda^3 + \lambda^2 - 21\lambda - 45)$$

$$\therefore -(\lambda^3 + \lambda^2 - 21\lambda - 45) = 0$$

$$\therefore \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\therefore (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0$$

$$\Rightarrow \lambda_1 = 5, \lambda_2 = -3, \lambda_3 = -3$$

Algebraic Multiplicity of $\lambda = -3$ is 2 and of $\lambda = 5$ is 1.

We solve the following homogeneous system:

$$\therefore [A - \lambda I]X = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case I: When $\lambda_1 = 5$	Case II :When $\lambda_2 = -3, \lambda_3 = -3$

Theorem: Every square matrix can be decomposed as a sum of symmetric and skew-symmetric matrices.

Proof: Let A be $m \times n$ matrix.

Let $B = \frac{1}{2}(A + A^T)$ and $C = = \frac{1}{2}(A - A^T)$ be two matrices.

Obviously, $A = B + C$

Now, $B^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}[(A + A^T)]^T = \frac{1}{2}[A^T + (A^T)^T] = \frac{1}{2}(A^T + A) = B$

As $B^T = B$, B is symmetric.

$$C^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}[(A - A^T)]^T = \frac{1}{2}[A^T - (A^T)^T] = \frac{1}{2}(A^T - A) = -C$$

Therefore, $C^T = -C$, C is skew- symmetric.

Therefore, A is a sum of symmetric and skew-symmetric matrices.

Caley –Hamilton Theorem

Every square matrix satisfies its own characteristic equation i.e. The theorem states that, for a square matrix A of order n, if $|A - \lambda I_n| = 0$.

Example (i): Verify Cayley-Hamilton theorem and hence find the inverse of $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and A^4.

Solution: The characteristic equation for given matrix is

$$|A - \lambda I_2| = 0.$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda - 5 = 0$$

Now, by putting $\lambda = A$, we have

$$A^2 - 4A - 5I = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence, Cayley-Hamilton theorem verified.

Now, by using Cayley-Hamilton theorem, we have

$=$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] R_2 \to -1/3R_2$$

$=$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 + 5R_2 \end{array}$$

$=$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = k, y - z = 0 \;\&\; x - z = 0$

$\Rightarrow z = k, y = k, x = k$

$\therefore (x, y, z) = k(1,1,1); k \in R$

$E_1 = \{k(1,1,1) / k \in R\}$

For $\lambda = 2$

$$\therefore [A - \lambda I | O] = \begin{bmatrix} -1-\lambda & 4 & -2 \\ -3 & 4-\lambda & 0 \\ -3 & 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$=$

$$\left[\begin{array}{ccc|c} -3 & 4 & -2 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right] R_1 \to -1/3R_1$$

$=$

$$\left[\begin{array}{ccc|c} 1 & -4/3 & 2/3 & 0 \\ -3 & 2 & 0 & 0 \\ -3 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$=$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & -2 & 3/2 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] R_2 \to -1/2R_2$$

$=$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1/2 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & -2 & 3/2 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + 2R_2 \end{array}$$

$=$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore z = k, y - 3/4z = 0 \;\&\; x - 1/4z = 0$

$\Rightarrow z = k, y = 3/4k, x = 1/4k$

$\therefore (x, y, z) = k(\frac{1}{4}, \frac{3}{4}, 1); k \in R$

$\therefore (x, y, z) = 4k(1,3,4); k \in R \quad (\because 4k = k')$

$E_1 = \{k'(1,3,4) / k' \in R\}$

$$\therefore P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\therefore P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$