## 1.3 The complex plane

**Exercise 1.** *Prove (3.4) and give necessary and sufficient conditions for equality.*

**Solution.** *Let $z$ and $w$ be complex numbers. Then*

$$\begin{aligned} ||z| - |w|| &= ||z - w + w| - |w|| \\ &\leq ||z - w| + |w| - |w|| \\ &= ||z - w|| \\ &= |z - w| \end{aligned}$$

*Notice that $|z|$ and $|w|$ is the distance from $z$ and $w$, respectively, to the origin while $|z - w|$ is the distance between $z$ and $w$. Considering the construction of the implied triangle, in order to guarantee equality, it is necessary and sufficient that*

$$\begin{aligned} & ||z| - |w|| = |z - w| \\ \iff \quad & (|z| - |w|)^2 = |z - w|^2 \\ \iff \quad & (|z| - |w|)^2 = |z|^2 - 2Re(z\overline{w}) + |w|^2 \\ \iff \quad & |z|^2 - 2|z||w| + |w|^2 = |z|^2 - 2Re(z\overline{w}) + |w|^2 \\ \iff \quad & |z||w| = Re(z\overline{w}) \\ \iff \quad & |z\overline{w}| = Re(z\overline{w}) \end{aligned}$$

*Equivalently, this is $z\overline{w} \geq 0$. Multiplying this by $\frac{w}{w}$, we get $z\overline{w} \cdot \frac{w}{w} = |w|^2 \cdot \frac{z}{w} \geq 0$ if $w \neq 0$. If $t = \frac{z}{w} = \left(\frac{1}{|w|^2}\right) \cdot |w|^2 \cdot \frac{z}{w}$. Then $t \geq 0$ and $z = tw$.*



**Exercise 2.** *Show that equality occurs in (3.3) if and only if $z_k/z_l \geq 0$ for any integers $k$ and $l$, $1 \leq k, l \leq n$, for which $z_l \neq 0$.*

**Solution.** *Not available.*

**Exercise 3.** *Let $a \in \mathbb{R}$ and $c > 0$ be fixed. Describe the set of points $z$ satisfying*

$$|z - a| - |z + a| = 2c$$

*for every possible choice of $a$ and $c$. Now let $a$ be any complex number and, using a rotation of the plane, describe the locus of points satisfying the above equation.*

**Solution.** *Not available.*

## 1.4 Polar representation and roots of complex numbers

**Exercise 1.** *Find the sixth roots of unity.*

**Solution.** *Start with $z^6 = 1$ and $z = r\text{cis}(\theta)$, therefore $r^6\text{cis}(6\theta) = 1$. Hence $r = 1$ and $\theta = \frac{2k\pi}{6}$ with $k \in \{-3, -2, -1, 0, 1, 2\}$. The following table gives a list of principle values of arguments and the corresponding value of the root of the equation $z^6 = 1$.*

| | |
|---|---|
| $\theta_0 = 0$ | $z_0 = 1$ |
| $\theta_1 = \frac{\pi}{3}$ | $z_1 = \text{cis}(\frac{\pi}{3})$ |
| $\theta_2 = \frac{2\pi}{3}$ | $z_2 = \text{cis}(\frac{2\pi}{3})$ |
| $\theta_3 = \pi$ | $z_3 = \text{cis}(\pi) = -1$ |
| $\theta_4 = \frac{-2\pi}{3}$ | $z_4 = \text{cis}(\frac{-2\pi}{3})$ |
| $\theta_5 = \frac{-\pi}{3}$ | $z_5 = \text{cis}(\frac{-\pi}{3})$ |

*It remains to verify that $fg$ is uniformly continuous, since we have already shown that $fg$ is bounded. We have*

$$\begin{aligned}
\rho(f(x)g(x), f(y)g(y)) &= |f(x)g(x) - f(y)g(y)| = \\
&= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\
&\leq |f(x)|\,|g(x) - g(y)| + |f(x)|\,|g(y) - g(y)| \\
&\leq M_1\epsilon_2 + M_2\epsilon_1,
\end{aligned}$$

*whenever $d(x,y) < \min(\delta_1, \delta_2)$. So choosing $\epsilon = M_1\epsilon_2 + M_2\epsilon_1$ and $\delta = \min(\delta_1, \delta_2)$, we have $\forall\epsilon > 0\ \exists\delta > 0$ such that*

$$|f(x)g(x) - f(y)g(y)| < \epsilon$$

*whenever $d(x,y) < \delta$. Thus, $fg$ is uniformly continuous and bounded.*

**Exercise 4.** *Is the composition of two uniformly continuous (Lipschitz) functions again uniformly continuous (Lipschitz)?*

**Solution.** *Not available.*

**Exercise 5.** *Suppose $f : X \to \Omega$ is uniformly continuous; show that if $\{x_n\}$ is a Cauchy sequence in $X$ then $\{f(x_n)\}$ is a Cauchy sequence in $\Omega$. Is this still true if we only assume that $f$ is continuous? (Prove or give a counterexample.)*



**Solution.** *Assume $f : X \to \Omega$ is uniformly continuous, that is for every $\epsilon > 0$ there exists $\delta > 0$ such that $\rho(f(x), f(y)) < \epsilon$ if $d(x,y) < \delta$. If $\{x_n\}$ is a Cauchy sequence in $X$, we have, for every $\epsilon_1 > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon_1$ for all $n, m \geq N$. But then, by the uniform continuity, we have that*

$$\rho(f(x_n), f(x_m)) < \epsilon \qquad \forall n, m \geq N$$

*whenever $d(x_n, x_m) < \delta$ which tells us that $\{f(x_n)\}$ is a Cauchy sequence in $\Omega$.*
*If $f$ is continuous, the statement is not **true**. Here is a counterexample: Let $f(x) = \frac{1}{x}$ which is continuous on $(0,1)$. The sequence $x_n = \frac{1}{n}$ is apparently convergent and therefore a Cauchy sequence in $X$. But $\{f(x_n)\} = \{f(\frac{1}{n})\} = \{n\}$ is obviously not Cauchy. Note that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0,1)$. To see that pick $\epsilon = 1$. Then there is no $\delta > 0$ such that $|f(x) - f(y)| < 1$ whenever $|x - y| < \delta$. Assume there exists such a $\delta$. WLOG assume $\delta < 1$ since the interval $(0,1)$ is considered. Let $y = x + \delta/2$ and set $x = \delta/2$, then*

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{y - x}{xy} = \frac{\delta/2}{\delta/2 \cdot \delta} = \frac{1}{\delta} > 1,$$

*that is no matter what $\delta < 1$ we choose, we always obtain $|f(x) - f(y)| > 1$. Therefore $f(x) = \frac{1}{x}$ cannot be uniformly continuous.*

**Exercise 6.** *Recall the definition of a dense set (1.14). Suppose that $\Omega$ is a complete metric space and that $f : (D,d) \to (\Omega;\rho)$ is uniformly continuous, where $D$ is dense in $(X,d)$. Use Exercise 5 to show that there is a uniformly continuous function $g : X \to \Omega$ with $g(x) = f(x)$ for every $x$ in $D$.*

**Solution.** *Not available.*

**Exercise 7.** *Let $G$ be an open subset of $\mathbb{C}$ and let $P$ be a polygon in $G$ from $a$ to $b$. Use Theorems 5.15 and 5.17 to show that there is a polygon $Q \subset G$ from $a$ to $b$ which is composed of line segments which are parallel to either the real or imaginary axes.*

**Solution.** *Not available.*

**Exercise 8.** *Use Lebesgue's Covering Lemma (4.8) to give another proof of Theorem 5.15.*

**Solution.** *Suppose $f : X \to \Omega$ is continuous and $X$ is compact. To show $f$ is uniformly continuous. Let $\epsilon > 0$. Since $f$ is continuous, we have for all $x \in X$ there is a $\delta_x > 0$ such that $\rho(f(x), f(y)) < \epsilon/2$ whenever $d(x, y) < \delta_x$. In addition,*

$$X = \bigcup_{x \in X} B(x; \delta_x)$$

*is an open cover of $X$. Since $X$ is by assumption compact (it is also sequentially compact as stated in Theorem 4.9 p. 22), we can use Lebesgue's Covering Lemma 4.8 p. 21 to obtain a $\delta > 0$ such that $x \in X$ implies that $B(x, \delta) \subset B(z; \delta_z)$ for some $z \in X$. More precisely, $x, y \in B(z; \delta_z)$ and therefore*

$$\rho(f(x), f(z)) \leq \rho(f(x), f(z)) + \rho(f(z), f(y)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

*and hence $f$ is uniformly continuous on $X$.*

**Exercise 9.** *Prove the following converse to Exercise 2.5. Suppose $(X, d)$ is a compact metric space having the property that for every $\epsilon > 0$ and for any points $a, b$ in $X$, there are points $z_0, z_1, \ldots, z_n$ in $X$ with $z_0 = a$, $z_n = b$, and $d(z_{k-l}, z_k) < \epsilon$ for $1 \leq k \leq n$. Then $(X, d)$ is connected. (Hint: Use Theorem 5.17.)*

**Solution.** *Not available.*

**Exercise 10.** *Let $f$ and $g$ be continuous functions from $(X, d)$ to $(\Omega, p)$ and let $D$ be a dense subset of $X$. Prove that if $f(x) = g(x)$ for $x$ in $D$ then $f = g$. Use this to show that the function $g$ obtained in Exercise 6 is unique.*

**Solution.** *Not available.*



## 2.6 Uniform convergence

**Exercise 1.** *Let $\{f_n\}$ be a sequence of uniformly continuous functions from $(X, d)$ into $(\Omega, \rho)$ and suppose that $f = u - -\lim f_n$ exists. Prove that $f$ is uniformly continuous. If each $f_n$ is a Lipschitz function with constant $M_n$ and $\sup M_n < \infty$, show that $f$ is a Lipschitz function. If $\sup M_n = \infty$, show that $f$ may fail to be Lipschitz.*

**Solution.** *Not available.*

# Chapter 3

# Elementary Properties and Examples of Analytic Functions

## 3.1 Power series

**Exercise 1.** *Prove Proposition 1.5.*

**Solution.** *Not available.*

**Exercise 2.** *Give the details of the proof of Proposition 1.6.*

**Solution.** *Not available.*



**Exercise 3.** *Prove that* $\limsup(a_n + b_n) \le \limsup a_n + \limsup b_n$ *and* $\liminf(a_n + b_n) \ge \liminf a_n + \liminf b_n$ *for bounded sequences of real numbers* $\{a_n\}$ *and* $\{b_n\}$.

**Solution.** *Let* $r > \limsup_{n\to\infty} a_n$ *(we know there are only finitely many by definition) and let* $s > \limsup_{n\to\infty}$ *(same here, there are only finitely many by definition). Then* $r + s > a_n + b_n$ *for all but finitely many* $n$*'s. This however, implies that*

$$r + s \ge \limsup_{n\to\infty}(a_n + b_n).$$

*Since this holds for any* $r > \limsup_{n\to\infty} a_n$ *and* $s > \limsup_{n\to\infty} b_n$*, we have*

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

*Let* $r < \liminf_{n\to\infty} a_n$ *(we know there are only finitely many by definition) and let* $s < \liminf_{n\to\infty}$ *(same here, there are only finitely many by definition). Then* $r + s < a_n + b_n$ *for all but finitely many* $n$*'s. This however, implies that*

$$r + s \le \liminf_{n\to\infty}(a_n + b_n).$$

*Since this holds for any* $r < \liminf_{n\to\infty} a_n$ *and* $s < \liminf_{n\to\infty} b_n$*, we have*

$$\liminf_{n\to\infty}(a_n + b_n) \ge \liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n.$$

**Exercise 4.** *Show that* $\liminf a_n \le \limsup a_n$ *for any sequence in* $\mathbb{R}$.

**Solution.** *Let* $m = \liminf_{n\to\infty} a_n$ *and* $b_n = \inf\{a_n, a_{n+1}, \ldots\}$*. Let* $M = \limsup_{n\to\infty} a_n$*. Take any* $s > M$*. Then, by definition of the* $\limsup_{n\to\infty} a_n = M$*, we obtain that* $a_n < s$ *for infinitely many* $n$*'s which implies that* $b_n < s$ *for all* $n$ *and hence* $\limsup_{n\to\infty} b_n = m < s$*. This holds for all* $s > M$*. But the infimum of all these* $s$*'s is* $M$*. Therefore* $m \le M$ *which is*

$$\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n.$$

**Exercise 5.** *If* $\{a_n\}$ *is a convergent sequence in* $\mathbb{R}$ *and* $a = \lim a_n$*, show that* $a = \liminf a_n = \limsup a_n$*.*

**Solution.** *Suppose that* $\{a_n\}$ *is a convergent sequence in* $\mathbb{R}$ *with limit* $a = \lim_{n\to\infty} a_n$*. Then by definition, we have:* $\forall \epsilon > 0\ \exists N > 0$ *such that* $\forall n \ge N$*, we have* $|a_n - a| \le \epsilon$*, that is* $a - \epsilon \le a_n \le a + \epsilon$*. This means that all but finitely many* $a_n$*'s are* $\le a + \epsilon$ *and* $\ge a - \epsilon$*. This shows that*

$$a - \epsilon \le \underbrace{\liminf_{n\to\infty} a_n}_{=:m} \le a + \epsilon$$

*and*

$$a - \epsilon \le \underbrace{\limsup_{n\to\infty} a_n}_{=:M} \le a + \epsilon.$$

*By the previous Exercise 4, we also have*

$$a - \epsilon \le m \le M \le a + \epsilon.$$

*Hence,*

$$0 \le M - m \le 2\epsilon.$$

*Since* $\epsilon > 0$ *is arbitrary, we obtain* $m = M$ *and further*

$$a - \epsilon \le \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n \le a + \epsilon,$$

*we obtain*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a.$$



**Exercise 6.** *Find the radius of convergence for each of the following power series: (a)* $\sum_{n=0}^{\infty} a^n z^n$*,* $a \in \mathbb{C}$*; (b)* $\sum_{n=0}^{\infty} a^{n^2} z^n$*,* $a \in \mathbb{C}$*; (c)* $\sum_{n=0}^{\infty} k^n z^n$*,* $k$ *an integer* $\ne 0$*; (d)* $\sum_{n=0}^{\infty} z^{n!}$*.*

**Solution.** *a) We have* $\sum_{n=0}^{\infty} a^n z^n = \sum_{k=0}^{\infty} b_k z^n$ *with* $b_k = a^k$*,* $a \in \mathbb{C}$*. We also have,*

$$\limsup_{k\to\infty} |b_k|^{1/k} = \limsup_{k\to\infty} |a^k|^{1/k} = \limsup_{k\to\infty} |a| = |a|.$$

*Therefore,* $R = 1/|a|$*, so*

$$R = \begin{cases} \frac{1}{|a|}, & a \ne 0 \\ \infty, & a = 0 \end{cases}.$$

*b) In this case,* $b_n = a^{n^2}$ *where* $a \in \mathbb{C}$*.*

$$\begin{aligned} R &= \lim_{n\to\infty} \left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{a^{n^2}}{a^{(n+1)^2}}\right| = \lim_{n\to\infty} \left|\frac{a^{n^2}}{a^{n^2+2n+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{a^{2n+1}}\right| \\ &= \lim_{n\to\infty} \frac{1}{|a|^{2n+1}} = \begin{cases} 0, & |a| > 1 \\ 1, & |a| = 1 \\ \infty, & |a| < 1 \end{cases}. \end{aligned}$$

*c) Now, $b_n = k^n$, $k$ is an integer $\neq 0$. We have*

$$R = \limsup_{n\to\infty} |b_n|^{1/n} = \limsup_{n\to\infty} |k^n|^{1/n} = \limsup_{n\to\infty} |k| = |k|.$$

*So*

$$R = \frac{1}{|k|} = \begin{cases} \frac{1}{k}, & k > 0, k \text{ integer} \\ -\frac{1}{k}, & k < 0, k \text{ integer} \end{cases}.$$

*d) We can write $\sum_{n=0}^{\infty} z^{n!} = \sum_{k=0}^{\infty} a_k z^k$ where*

$$a_k = \begin{cases} 0, & k = 0 \\ 2, & k = 1 \\ 1, & k = n!, n \in \mathbb{N}, n > 1 \\ 0, & \text{otherwise} \end{cases}$$

*Thus,*

$$\limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} |1|^{1/k!} = 1.$$

*Therefore $1/R = 1$ which implies $R = 1$.*

**Exercise 7.** *Show that the radius of convergence of the power series*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

*is 1, and discuss convergence for $z = 1, -1$, and i. (Hint: The nth coefficient of this series is not $(-1)^n/n$.)*

**Solution.** *Rewrite the power series in standard form, then*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} = \sum_{n=1}^{\infty} a_k z^k \text{ with } a_k = \begin{cases} \frac{(-1)^n}{n} & \text{if } \exists\, n \in \mathbb{N} \text{ s.t. } k = n(n+1) \\ 0 & \text{else} \end{cases}.$$

*To find the radius of convergence we use the root criterion and therefore need the estimates*

$$1 \leq \sqrt[n(n+1)]{n} \leq \sqrt[n]{n} \qquad \text{for } n \in \mathbb{N}.$$

*The first inequality is immediate from the fact that $n \geq 1$ and hence $n^{\frac{1}{n(n+1)}} \geq 1$. For the second inequality note that*

$$\begin{aligned} & n \leq n^{n+1} \\ \Leftrightarrow \quad & n^{\frac{1}{n(n+1)}} \leq n^{\frac{1}{n}} \\ \Leftrightarrow \quad & \sqrt[n(n+1)]{n} \leq \sqrt[n]{n} \end{aligned}.$$

*Using this one obtains*

$$\sqrt[n(n+1)]{\left|\frac{(-1)^n}{n}\right|} = \frac{1}{\sqrt[n(n+1)]{n}} \leq 1$$

*and*

$$\sqrt[n(n+1)]{\left|\frac{(-1)^n}{n}\right|} = \frac{1}{\sqrt[n(n+1)]{n}} \geq \frac{1}{\sqrt[n]{n}}.$$

*Vague memories of calculus classes tell me that* $\sqrt[n]{n} \to 1$*, thus* $\frac{1}{R} = \limsup \sqrt[n(n+1)]{a_n} = 1$*, i.e.* $R = 1$.

*If* $z = 1$ *the series reduces to* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *which converges with the Leibniz Criterion.*

*If* $z = -1$ *we note that the exponents* $n(n+1)$ *are always even integers and therefore the series is the same as in the previous case of* $z = 1$.

*Now let* $z = i$*. The expression* $i^{n(n+1)}$ *will always be real, so if the series converges at* $z = i$*, it converges to a real number. We also note that formally*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} i^{n(n+1)} = \sum_{n=1}^{\infty} c_n \text{ with } \begin{cases} \frac{1}{n} & \text{if} n \mod 4 \in \{0,1\} \\ -\frac{1}{n} & \text{if } n \mod 4 \in \{2,3\} \end{cases}.$$

*Define the partial sums* $S_k := \sum_{n=0}^{k} c_k$*. We claim that the following chain of inequalities holds*

$$0 \le S_{4k+3} \overset{a)}{<} S_{4k} \overset{b)}{<} S_{4k+4} \overset{c)}{<} S_{4k+2} \overset{d)}{<} S_{4k+1} \le 1.$$

*To verify this, note that*

$$S_{4k+3} - S_{4k} = c_{4k+1} + c_{4k+2} + c_{4k+3} = -\frac{16k^2 + 8k - 1}{(4k+1)(4k+2)(4k+3)} < 0, \text{ hence a)}$$

$$S_{4k+4} - S_{4k} = c_{4k+3} + c_{4k+4} = -\frac{1}{4k+3} + \frac{1}{4k+4} < 0, \text{ hence c)}$$

$$S_{4k+2} - S_{4k+1} = c_{4k+2} < 0, \text{ hence d)}$$

*Relation b) is obvious and so are the upper bound* $c_1 = 1$ *and the non-negativity constraint. We remark that* $\{S_{4k+l}\}_{k \ge 1}, l \in \{0,1,2,3\}$ *describe bounded and monotone subsequences that converge to some point. Now that* $|c_k| \searrow 0$ *the difference between* $S_{4k+l}$ *and* $S_{4k+m}, l, m \in 0,1,2,3$ *tends to zero, i.e. all subsequences converge to the same limit. Therefore the power series converges also in the case of* $z = i$.

## 3.2 Analytic functions

**Exercise 1.** *Show that* $f(z) = |z|^2 = x^2 + y^2$ *has a derivative only at the origin.*

**Solution.** *The derivative of* $f$ *at* $z$ *is given by*

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \quad h \in \mathbb{C}$$

*provided the limit exist. We have*

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{|z+h|^2 - |z|^2}{h} = \frac{(z+h)(\overline{z} + \overline{h}) - z\overline{z}}{h} = \frac{z\overline{z} + h\overline{z} + z\overline{h} + h\overline{h} - z\overline{z}}{h} \\ &= \overline{z} + \overline{h} + z\frac{\overline{h}}{h} =: D. \end{aligned}$$

*If the limit of D exists, it may be found by letting the point* $h = (x, y)$ *approach the origin (0,0) in the complex plane* $\mathbb{C}$ *in any manner.*
*1.) Take the path along the real axes, that is* $y = 0$*. Then* $\overline{h} = h$ *and thus*

$$D = \overline{z} + h + z\frac{h}{h} = \overline{z} + h + z$$

*1. Claim:* $\cos(z)\cos(w) - \sin(z)\sin(w) = \cos(z+w)$.
*Proof:*

$$\begin{aligned}
\cos(z)\cos(w) - \sin(z)\sin(w) &= \frac{e^{iz}+e^{-iz}}{2}\frac{e^{iw}+e^{-iw}}{2} - \frac{e^{iz}-e^{-iz}}{2i}\frac{e^{iw}-e^{-iw}}{2i} \\
&= \frac{1}{4}\left(e^{iz}e^{iw} + e^{-iz}e^{-iw}\right) + \frac{1}{4}\left(e^{iz}e^{iw} + e^{-iz}e^{-iw}\right) \\
&= \frac{1}{2}e^{iz}e^{iw} + \frac{1}{2}e^{-iz}e^{-iw} \\
&= \frac{1}{2}\left(e^{iz}e^{iw} + e^{-iz}e^{-iw}\right) \\
&= \cos(z+w).
\end{aligned}$$

*2. Claim:* $\sin(z)\cos(w) + \cos(z)\sin(w) = \sin(z+w)$.
*Proof:*

$$\begin{aligned}
\sin(z)\cos(w) + \cos(z)\sin(w) &= \frac{e^{iz}-e^{-iz}}{2i}\frac{e^{iw}+e^{-iw}}{2} + \frac{e^{iz}+e^{-iz}}{2}\frac{e^{iw}-e^{-iw}}{2i} \\
&= \frac{1}{4i}\left(e^{iz}e^{iw} - e^{-iz}e^{-iw}\right) + \frac{1}{4i}\left(e^{iz}e^{iw} - e^{-iz}e^{-iw}\right) \\
&= \frac{1}{2i}e^{iz}e^{iw} - \frac{1}{2i}e^{-iz}e^{-iw} \\
&= \frac{1}{2i}\left(e^{iz}e^{iw} - e^{-iz}e^{-iw}\right) \\
&= \sin(z+w).
\end{aligned}$$



**Exercise 8.** *Define* $\tan z = \frac{\sin z}{\cos z}$*; where is this function defined and analytic?*

**Solution.** *Since both* $\sin z$ *and* $\cos z$ *are analytic in the entire complex plane, it follows from the discussion in the text following Definition 2.3 that* $\tan z = \frac{\sin z}{\cos z}$ *is analytic wherever* $\cos z \neq 0$*. Now,* $\cos z = 0$ *implies that* $z$ *is real and equal to an odd multiple of* $\frac{\pi}{2}$*. Thus let*

$$G \equiv \left\{ \frac{(2k+1)\pi}{2} \,\middle|\, k \in \mathbb{Z} \right\}.$$

*Then* $\tan z$ *is defined and analytic on* $\mathbb{C} - G$*. If* $z \in G$*, then* $\cos z = 0$ *so* $\tan z$ *is undefined on the non-extended complex plane.*

**Exercise 9.** *Suppose that* $z_n, z \in G = \mathbb{C} - \{z : z \leq 0\}$ *and* $z_n = r_n e^{i\theta_n}, z = re^{i\theta}$ *where* $-\pi < \theta, \theta_n < \pi$*. Prove that if* $z_n \to z$ *then* $\theta_n \to \theta$ *and* $r_n \to r$*.*

**Solution.** *Not available.*

**Exercise 10.** *Prove the following generalization of Proposition 2.20. Let* $G$ *and* $\Omega$ *be open in* $\mathbb{C}$ *and suppose* $f$ *and* $h$ *are functions defined on* $G$*,* $g : \Omega \to \mathbb{C}$ *and suppose that* $f(G) \subset \Omega$*. Suppose that* $g$ *and* $h$ *are analytic,* $g'(\omega) \neq 0$ *for any* $\omega$*, that* $f$ *is continuous,* $h$ *is one-one, and that they satisfy* $h(z) = g(f(z))$ *for* $z$ *in* $G$*. Show that* $f$ *is analytic. Give a formula for* $f'(z)$*.*

**Solution.** *Not available.*

**Exercise 11.** *Suppose that* $f : G \to \mathbb{C}$ *is a branch of the logarithm and that* $n$ *is an integer. Prove that* $z^n = \exp(nf(z))$ *for all* $z$ *in* $G$*.*

*For the real and imaginary part of w the following equations must hold*

$$\begin{aligned} a &= \tfrac{1}{2}\left(r + \tfrac{1}{r}\right)\cos\theta \\ b &= \tfrac{1}{2}\left(r - \tfrac{1}{r}\right)\sin\theta. \end{aligned} \tag{3.7}$$

*If $f(z) = w$ has imaginary part $b = 0$ then $\sin\theta = 0$ and $|\cos\theta| = 1$. Therefore points of the form $a + ib, a \in [-1, 1], b = 0$ cannot be in the range of $f$. For all other points the equations (3.7) can be solved for $r$ and $\theta$ uniquely (after restricting the argument to $[-\pi, \pi)$).*
*Given any value of $r \in (0, 1)$, the graph of $f(re^{i\theta})$ as a function of $\theta$ looks like an ellipse. In fact from formulas (3.7) we see that $\left(\frac{a}{\frac{1}{2}(r+\frac{1}{r})}\right)^2 + \left(\frac{b}{\frac{1}{2}(1-\frac{1}{r})}\right)^2 = 1$.*
*If we fix the argument $\theta$ and let $r$ vary in $(0, 1)$ it follows from from equation (3.7) that the graph of $f(re^{i\theta})$ is a hyperbola and it degenerates to rays if $z$ is purely real or imaginary. In the case $\theta \in \{(2k + 1)\pi \,|\, k \in \mathbb{Z}\}$ the graph of $f$ in dependence on $r$ is on the imaginary axis and for $\theta \in \{2k\pi \,|\, k \in \mathbb{Z}\}$ the graph of $f(re^{i\theta})$ is either $(-\infty, -1)$ or $(1, \infty)$. If $\cos\theta \neq 0$ and $\sin\theta \neq 0$ then $\left(\frac{a}{\cos\theta}\right)^2 - \left(\frac{b}{\sin\theta}\right)^2 = 1$.*

**Exercise 14.** *Suppose that one circle is contained inside another and that they are tangent at the point $a$. Let $G$ be the region between the two circles and map $G$ conformally onto the open unit disk. (Hint: first try $(z - a)^{-1}$.)*

**Solution.** *Using the hint, define the Möbius transformation $T(z) = (z - a)^{-1}$ which sends the region $G$ between two lines. Afterward applying a rotation followed by a translation, it is possible to send this region to any region bounded by two parallel lines we want. Hence, choose $S(z) = cz + d$ where $|c| = 1$ such that*

$$S(T(G)) = \left\{x + iy : 0 < y < \frac{\pi}{2}\right\}.$$

*Applying the exponential function to this region yields the right half plane*

$$\exp(S(T(G))) = \{x + iy : x > 0\}.$$

*Finally, the Möbius transformation*

$$R(z) = \frac{z-1}{z+1}$$

*maps the right half plane onto the unit disk (see page 53). Hence the function $f$ defined by $R(\exp(S(T(z))))$ maps $G$ onto $D$ and is the desired conformal mapping ($f$ is a composition of conformal mappings). Doing some simplifications, we obtain*

$$f(z) = \frac{e^{\frac{c}{z-a}+d} - 1}{e^{\frac{c}{z-a}+d} + 1}$$

*where the constants $c$ and $d$ will depend on the circle location.*

**Exercise 15.** *Can you map the open unit disk conformally onto $\{z : 0 < |z| < 1\}$?*

**Solution.** *Not available.*

**Exercise 16.** *Map $G = \mathbb{C} - \{z : -1 \leq z \leq 1\}$ onto the open unit disk by an analytic function $f$. Can $f$ be one-one?*

**Solution.** *Not available.*

**Exercise 17.** *Let $G$ be a region and suppose that $f : G \to \mathbb{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that $f$ is constant.*

**Exercise 24.** *Let $T$ be a Möbius transformation, $T \neq$ the identity. Show that a Möbius transformation $S$ commutes with $T$ if $S$ and $T$ have the same fixed points. (Hint: Use Exercises 21 and 22.)*

**Solution.** *Let $T$ and $S$ have the same fixed points. To show $TS = ST$, $T \neq id$.*

$\star$ : *Suppose $T$ and $S$ have two fixed points, say $z_1$ and $z_2$. Let $M$ be a Möbius transformation with $M(z_1) = 0$ and $M(z_2) = \infty$. Then*

$$MSM^{-1}(0) = MSM^{-1}Mz_1 = MSz_1 = Mz_1 = 0$$

*and*

$$MSM^{-1}(\infty) = MSM^{-1}Mz_2 = MSz_2 = Mz_2 = \infty.$$

*Thus $MSM^{-1}$ is a dilation by exercise 22 a) since $MSM^{-1}$ has 0 and $\infty$ as its only fixed points. Similar, we obtain $MTM^{-1}(0) = 0$ and $MTM^{-1}(\infty) = \infty$ and therefore is also a dilation. It is easy to check that dilations commute (define $C(z) = az$, $a > 0$ and $D(z) = bz$, $b > 0$, then $CD(z) = abz = baz = DC(z)$), thus*

$$\begin{aligned} & (MTM^{-1})(MSM^{-1}) = (MSM^{-1})(MTM^{-1}) \\ \Rightarrow \quad & MTSM^{-1} = MSTM^{-1} \\ \Rightarrow \quad & TS = ST. \end{aligned}$$

$\star$ : *Suppose $T$ and $S$ have one fixed points, say $z$. Let $M$ be a Möbius transformation with $M(z) = \infty$. Then*

$$MSM^{-1}(\infty) = MSM^{-1}Mz = MSz = Mz = \infty.$$

*Thus $MSM^{-1}$ is a translation by exercise 22 b) since $MSM^{-1}$ has $\infty$ as its only fixed point. Similar, we obtain $MTM^{-1}(\infty) = \infty$ and therefore is also a translation. It is easy to check that translations commute (define $C(z) = z + 1$ and $D(z) = z + b$, $b > 0$, then $CD(z) = z + a + b = z + b + a = DC(z)$), thus*

$$\begin{aligned} & (MTM^{-1})(MSM^{-1}) = (MSM^{-1})(MTM^{-1}) \\ \Rightarrow \quad & MTSM^{-1} = MSTM^{-1} \\ \Rightarrow \quad & TS = ST. \end{aligned}$$



**Exercise 25.** *Find all the abelian subgroups of the group of Möbius transformations.*

**Solution.** *Not available.*

**Exercise 26.** *26. (a) Let $GL_2(\mathbb{C})$ = all invertible $2 \times 2$ matrices with entries in $\mathbb{C}$ and let $\mathcal{M}$ be the group of Möbius transformations. Define $\varphi : GL_2(\mathbb{C}) \to \mathcal{M}$ by $\varphi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{az+b}{cz+d}$. Show that $\varphi$ is a group homomorphism of $GL_2(\mathbb{C})$ onto $\mathcal{M}$. Find the kernel of $\varphi$.*
*(b) Let $SL_2(\mathbb{C})$ be the subgroup of $GL_2(\mathbb{C})$ consisting of all matrices of determinant 1. Show that the image of $SL_2(\mathbb{C})$ under $\varphi$ is all of $\mathcal{M}$. What part of the kernel of $\varphi$ is in $SL_2(\mathbb{C})$?*

**Solution.** *a) We have to check that if*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}$$

*then $\varphi(AB) = \varphi(A) \circ \varphi(B)$. A simple calculation shows that this is true. To find the kernel of the group homomorphism we have to find all $z$ such that $\frac{az+b}{cz+d} = z$. This is equivalent to $az + b = cz^2 + dz$ and by comparing coefficients we obtain $b = c = 0$ and $a = d$. Therefore, the kernel is given by $N = ker(\varphi) = \{\lambda I : \lambda \in \mathbb{C}^\times\}$. Note that the kernel is a normal subgroup of $GL_2(\mathbb{C})$.*
*b) Restricting $\varphi$ to $SL_2(\mathbb{C})$ still yields a surjective map since for any matrix $A \in GL_2(\mathbb{C})$ both $A$ and the modification $M = \frac{1}{\sqrt{\det A}}A$ have the same image and the modification matrix $M$ has by construction determinant 1. The kernel of the restriction is simply $N \cap SL_2(\mathbb{C}) = \{\pm I\}$.*

$0 \le t \le 1\}$. *So* $\gamma_1'(t) = i$ *and* $\gamma_2'(t) = -1$. *Therefore,*

$$\begin{aligned}
\int_\gamma f &= \int_{\gamma_1} f + \int_{\gamma_2} f \\
&= \int_{\gamma_1} |z|^2 dz + \int_{\gamma_2} |z|^2 dz \\
&= \int_0^1 (t^2+1)(i)dt + \int_0^1 ((1-t)^2+1)(-1)dt \\
&= i\int_0^1 (t^2+1)dt - \int_0^1 (t^2-2t+2)dt \\
&= i\left[\frac{t^3}{3}+t\right]_0^1 - \left[\frac{t^3}{3}-t^2+2t\right]_0^1 \\
&= \frac{4}{3}i - \left(\frac{1}{3}-1+2\right) \\
&= \frac{4}{3}i - \frac{4}{3} \\
&= \frac{4}{3}(-1+i)
\end{aligned}$$

**Exercise 9.** *Define* $\gamma : [0, 2\pi] \to \mathbb{C}$ *by* $\gamma(t) = \exp(int)$ *where* $n$ *is some integer (positive, negative, or zero). Show that* $\int_\gamma \frac{1}{z}\, dz = 2\pi in$.

**Solution.** *Clearly,* $\gamma(t) = e^{int}$ *is continuous and smooth on* $[0, 2\pi]$. *Thus*

$$\int_\gamma z^{-1}\, dz = \int_0^{2\pi} e^{-int} ine^{int}\, dt = \int_0^{2\pi} in\, dt = in(2\pi - 0) = 2\pi in.$$



**Exercise 10.** *Define* $\gamma(t) = e^{it}$ *for* $0 \le t \le 2\pi$ *and find* $\int_\gamma z^n\, dz$ *for every integer* $n$.

**Solution.** *Clearly,* $\gamma(t) = e^{int}$ *is continuous and smooth on* $[0, 2\pi]$ *(It is the unit circle).*
*Case 1:* $n = -1$

$$\int_\gamma z^{-1}\, dz = \int_0^{2\pi} e^{-it} ie^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i.$$

*Case 2:* $n \neq -1$

$$\begin{aligned}
\int_\gamma z^n\, dz &= \int_0^{2\pi} e^{int} ie^{it}\, dt = i\int_0^{2\pi} e^{i(n+1)t}\, dt = t\left[\frac{e^{i(n+1)t}}{i(n+1)}\right]_0^{2\pi} \\
&= \frac{1}{n+1}\left[e^{i(n+1)t}\right]_0^{2\pi} = \frac{1}{n+1}\left[e^{i(n+1)2\pi} - 1\right] \\
&= \frac{1}{n+1}[\cos((n+1)2\pi) - i\sin((n+1)2\pi)1] = \frac{1}{n+1}[1 - i\cdot 0 - 1] \\
&= 0.
\end{aligned}$$

**Exercise 11.** *Let* $\gamma$ *be the closed polygon* $[1-i, 1+i, -1+i, -1-i, 1-i]$. *Find* $\int_\gamma \frac{1}{z}\, dz$.

$0 \le t \le 2\pi$. *Clearly $f(z)$ is analytic on $\mathbb{C}$ and $\overline{B}(0; 2 < r < \infty) \subset \mathbb{C}$. Then,*

$$f(-2i) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w+2i}dw \iff 1 = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w+2i}dw$$
$$\iff \int_\gamma \frac{1}{z+2i}dz = 2\pi i$$

*if $2 < r < \infty$. Similarly,*

$$f(2i) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-2i}dw \iff \int_\gamma \frac{1}{z-2i}dz = 2\pi i.$$

*It follows that,*

$$\begin{aligned}\int_\gamma \frac{z^2+1}{z(z^2+4)}dz &= \frac{1}{4}\int_\gamma \frac{1}{z}dz + \frac{3}{8}\int_\gamma \frac{1}{z-2i}dz + \frac{3}{8}\int_\gamma \frac{1}{z+2i}dz \\ &= \frac{1}{4}(2\pi i) + \frac{3}{8}(2\pi i) + \frac{3}{8}(2\pi i) \\ &= 2\pi i\end{aligned}$$

**Exercise 11.** *Find the domain of analyticity of*

$$f(z) = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right);$$

*also, show that $\tan f(z) = z$ (i.e., $f$ is a branch of $\arctan z$). Show that*

$$f(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{2k+1}, \quad \textit{for} \quad |z| < 1$$

*(Hint: see Exercise III. 3.19.)*

**Solution.** *Not available.*

**Exercise 12.** *Show that*

$$\sec z = 1 + \sum_{k=1}^{\infty}\frac{E_{2k}}{(2k)!}z^{2k}$$

*for some constants $E_2, E_4, \cdots$. These numbers are called Euler's constants. What is the radius of convergence of this series? Use the fact that $1 = \cos z \sec z$ to show that*

$$E_{2n} - \binom{2n}{2n-2}E_{2n-2} + \binom{2n}{2n-4}E_{2n-4} + \cdots + (-1)^{n-1}\binom{2n}{2}E_2 + (-1)^n = 0.$$

*Evaluate $E_2, E_4, E_6, E_8$. ($E_{10} = 50521$ and $E_{12} = 2702765$).*

**Solution.** *It is easily seen that*

$$\sec z = 1 + \sum_{k=1}^{\infty}\frac{E_{2k}}{(2k)!}z^{2k}, \quad (\textit{for some } E_2, E_4, \ldots),$$

**Solution.** *Assume $\gamma$ is a closed and rectifiable curve in $G$ and $\gamma \sim \sigma_1$. Since $\sigma_1(t) \equiv a$ and $\sigma_2(t) \equiv b$, we surely have that $\sigma_1$ and $\sigma_2$ are closed and rectifiable curves in $G$. If we can show that $\sigma_1 \sim \sigma_2$, then we get $\gamma \sim \sigma_2$, since $\sim$ is an equivalence relation, that is:*

$$\gamma \sim \sigma_1 \quad \text{and} \quad \sigma_1 \sim \sigma_2 \Rightarrow \gamma \sim \sigma_2.$$

*So, now we show $\sigma_1 \sim \sigma_2$, that is, we have to find a continuous function $\Gamma : [0,1] \times [0,1] \to G$ such that*

$$\Gamma(s,0) = \sigma_1(s) = a \quad \text{and} \quad \Gamma(s,1) = \sigma_2(s) = b, \quad \text{for} \quad 0 \le s \le 1$$

*and $\Gamma(0,t) = \Gamma(1,t)$ for $0 \le t \le 1$. Let $\Gamma(s,t) = tb + (1-t)a$. Clearly $\Gamma$ is a continuous function. (it is constant with respect to $s$ and a line with respect to $t$). In addition, it satisfies $\Gamma(s,0) = a$ $\forall 0 \le s \le 1$ and $\Gamma(s,1) = b$ $\forall 0 \le s \le 1$ and $\Gamma(0,t) = tb + (1-t)a = \Gamma(1,t)$ $\forall 0 \le t \le 1$. So $\sigma_1 \sim \sigma_2$.*

**Exercise 2.** *Show that if we remove the requirement "$\Gamma(0,t) = \Gamma(1,t)$ for all $t$" from Definition6.1 then the curve $\gamma_0(t) = e^{2\pi i t}$, $0 \le t \le 1$, is homotopic to the constant curve $\gamma_1(t) \equiv 1$ in the region $G = \mathbb{C} - \{0\}$.*

**Solution.** *Not available.*

**Exercise 3.** *3. Let $C$ = all rectifiable curves in $G$ joining $a$ to $b$ and show that Definition 6.11 gives an equivalence relation on $C$.*

**Solution.** *Not available.*

**Exercise 4.** *Let $G = \mathbb{C} - \{0\}$ and show that every closed curve in $G$ is homotopic to a closed curve whose trace is contained in $\{z : |z| = 1\}$.*

**Solution.** *Not available.*

**Exercise 5.** *Evaluate the integral $\int_\gamma \frac{dz}{z^2+1}$ where $\gamma(\theta) = 2|\cos 2\theta| e^{i\theta}$ for $0 \le \theta \le 2\pi$.*

**Solution.** *A sketch shows that the two zeros of $z^2+1$ (they are $\pm i$) are inside of the closed and rectifiable curve $\gamma$ (The region looks like a clover with four leaves ). Using partial fraction decompositions gives*

$$\begin{aligned}
\int_\gamma \frac{dz}{z^2+1} &= \frac{1}{2i}\int_\gamma \frac{1}{z-i}\,dz - \frac{1}{2i}\int_\gamma \frac{1}{z+i}\,dz \\
&= \pi\frac{1}{2\pi i}\int_\gamma \frac{1}{z-i}\,dz - \pi\frac{1}{2i}\int_\gamma \frac{1}{z+i}\,dz \\
&\underset{\text{Def 4.2 p. 81}}{=} \pi\left(n(\gamma; i) - n(\gamma; -i)\right) \\
&= \pi \cdot 0 = 0,
\end{aligned}$$

*since $n(\gamma; i) = n(\gamma; -i)$ since $i$ and $-i$ are contained in the region generated by $\gamma$. Hence*

$$\int_\gamma \frac{dz}{z^2+1} = 0.$$

**Exercise 6.** *Let $\gamma(\theta) = \theta e^{i\theta}$ for $0 \le \theta \le 2\pi$ and $\gamma(\theta) = 4\pi - \theta$ for $2\pi \le \theta \le 4\pi$. Evaluate $\int_\gamma \frac{dz}{z^2+\pi^2}$.*

**Solution.** *A sketch reveals that the zero $-\pi i$ is inside the region and the zero $\pi i$ is outside the region. Using partial fraction decomposition yields*

$$\begin{aligned}
\int_\gamma \frac{dz}{z^2+\pi^2} &= \frac{1}{2\pi i}\int_\gamma \frac{1}{z-i\pi}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{1}{z+i\pi}\,dz \\
&\underset{\text{Def 4.2 p. 81}}{=} n(\gamma; i\pi) - n(\gamma; -i\pi) = 0 - 1 = -1,
\end{aligned}$$

*since $i\pi$ is not contained in the region generated by $\gamma$, so $n(\gamma; i\pi) = 0$ and $-i\pi$ is contained in the region, so $n(\gamma; -i\pi) = 1$. Hence,*

$$\int_\gamma \frac{dz}{z^2+\pi^2} = -1.$$

**Exercise 7.** *Let $f(z) = [(z-\frac{1}{2}-i)\cdot(z-1-\frac{3}{2}i)\cdot(z-1-\frac{i}{2})\cdot(z-\frac{3}{2}-i)]^{-1}$ and let $\gamma$ be the polygon $[0, 2, 2+2i, 2i, 0]$. Find $\int_\gamma f$.*

**Solution.** *Not available.*

**Exercise 8.** *Let $G = \mathbb{C} - \{a, b\}$, $a \neq b$, and let $\gamma$ be the curve in the figure below.*
*(a) Show that $n(\gamma; a) = n(\gamma; b) = 0$.*
*(b) Convince yourself that $\gamma$ is not homotopic to zero. (Notice that the word is "convince" and not "prove". Can you prove it?) Notice that this example shows that it is possible to have a closed curve $\gamma$ in a region such that $n(\gamma; z) = 0$ for all $z$ not in $G$ without $\gamma$ being homotopic to zero. That is, the converse to Corollary 6.10 is false.*

**Solution.** *Let $\gamma$ be the path depicted on p. 96. We can write it as a sum of 6 paths. Two of them will be closed and have $a$ and $b$ in their unbounded component. Therefore, two integrals will be zero and we will have another 2 pair of non-closed paths. The first pair begins at the leftmost crossing pair and goes around $a$ in opposite direction. The second pair begins at the middle crossing pair and goes around $b$ in opposite direction. They also meet at the rightmost crossing point. If we integrate over the path around $a$ is equivalent to evaluate $\int_{\gamma_1-\gamma_2} \frac{1}{z-a}\,dz$ where $\gamma_1(t) = a + re^{it}$, $0 \leq t \leq \pi$ and $\gamma_2(t) = a + re^{it}$, $\pi \leq t \leq 2\pi$ for some $r > 0$. It is easily seen that we have*

$$\int_{\gamma_1} \frac{1}{z-a}\,dz = \int_{\gamma_2} \frac{1}{z-a}\,dz = \pi i$$

*and therefore $n(\gamma; a) = 0$ and similarly we obtain $n(\gamma; b) = 0$.*



**Exercise 9.** *Let $G$ be a region and let $\gamma_0$ and $\gamma_1$ be two closed smooth curves in $G$. Suppose $\gamma_0 \sim \gamma_1$, and $\Gamma$ satisfies (6.2). Also suppose that $\gamma_t(s) = \Gamma(s, t)$ is smooth for each $t$. If $w \in \mathbb{C} - G$ define $h(t) = n(\gamma_t; w)$ and show that $h : [0, 1] \to \mathbb{Z}$ is continuous.*

**Solution.** *Not available.*

**Exercise 10.** *Find all possible values of $\int_\gamma \frac{dz}{1+z^2}$, where $\gamma$ is any closed rectifiable curve in $\mathbb{C}$ not passing through $\pm i$.*

**Solution.** *Let $\gamma$ be a closed rectifiable curve in $\mathbb{C}$ not passing through $\pm i$. Using partial fraction decomposition and the definition of the winding number we obtain*

$$\int_\gamma \frac{dz}{1+z^2} = \frac{1}{2i}\left(\int_\gamma \frac{1}{z-1}\,dz - \int_\gamma \frac{1}{z+i}\,dz\right) = \frac{1}{2i}\left(2\pi i n(\gamma; i) - 2\pi i n(\gamma; -i)\right) = \pi\left(n(\gamma; i) - n(\gamma; -i)\right).$$

**Exercise 11.** *Evaluate $\int_\gamma \frac{e^z - e^{-z}}{z^4}\,dz$ where $\gamma$ is one of the curves depicted below. (Justify your answer.)*

**Solution.** *Using Corollary 5.9 p. 86 we have $\forall a \in G - \{\gamma\}$*

$$\int_\gamma \frac{f(z)}{(z-a)^{k+1}}\,dz = 2\pi i f^{(k)}(a) n(\gamma; a) \frac{1}{k!},$$

*If $z = a$, then clearly $f^{(n)}(a) = 0$ for $m - (n - k) > 0$ and $f^{(n)}(a) \neq 0$ for $m - (n - k) = 0$. If $k = 0$ and $m = n$, then $f^{(m)}(a) = f^{(n)}(a) = m!g(a) \neq 0$ since $g(a) \neq 0$ by assumption. Hence, $f^{(n)}(a) = 0$ for $n = 1, 2, \ldots, m-1$ and $f^{(n)}(a) \neq 0$ for $n = m$.*

**Exercise 4.** *Suppose that $f : G \to \mathbb{C}$ is analytic and one-one; show that $f'(z) \neq 0$ for any $z$ in $G$.*

**Solution.** *Assume $f'(a) = 0$ for some $a \in G$. Then $f(a) = \alpha$ where $\alpha$ is a constant. Define*

$$F(z) = f(z) - \alpha.$$

*Then $F(a) = f(a) - \alpha = \alpha - \alpha = 0$ and $F'(a) = f'(a) = 0$ by assumption. Hence, $F$ has $a$ as a root of multiplicity $m \geq 2$.*
*Then, we can use Theorem 7.4 p. 98 to argue that there is an $\epsilon > 0$ and $\delta > 0$ such that for $0 < |\xi - a| < \delta$, the equation $f(z) = \xi$ has exactly $m$ simple roots in $B(a; \epsilon)$, since $f$ is analytic in $B(a; R)$ and $\alpha = f(a)$. Also $f(z) - \alpha$ has a zero of order $m \geq 2$ at $z = a$.*
*Since $f(z) = \xi$ has exactly $m$ (our case $m \geq 2$) simple roots in $B(a; \epsilon)$, we can find at least two distinct points $z_1, z_2 \in B(a; \epsilon) \subset G$ such that*

$$f(z_1) = \xi = f(z_2)$$

*contradicting that $f$ is $1 - 1$. So $f'(z) \neq 0$ for any $z$ in $G$.*

**Exercise 5.** *Let $X$ and $\Omega$ be metric spaces and suppose that $f : X \to \Omega$ is one-one and onto. Show that $f$ is an open map iff $f$ is a closed map. (A function $f$ is a closed map if it takes closed sets onto closed sets.)*

**Solution.** *Not available.*

**Exercise 6.** *Let $P : \mathbb{C} \to \mathbb{R}$ be defined by $P(z) = \operatorname{Re} z$; show that $P$ is an open map but is not a closed map. (Hint: Consider the set $F = \{z : \operatorname{Im} z = (\operatorname{Re} z)^{-1}$ and $\operatorname{Re} z \neq 0\}$.)*

**Solution.** *Not available.*

**Exercise 7.** *Use Theorem 7.2 to give another proof of the Fundamental Theorem of Algebra.*

**Solution.** *Not available.*

## 4.8 Goursat's Theorem

No exercises are assigned in this section.

# Chapter 5

# Singularities

## 5.1 Classification of singularities

**Exercise 1.** *Each of the following functions $f$ has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ so that $f$ is analytic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity determine $f(\{z : 0 < |z| < \delta\})$ for arbitrarily small values of $\delta$.*

*a)*

$$f(z) = \frac{\sin z}{z};$$

*b)*

$$f(z) = \frac{\cos z}{z};$$

*c)*

$$f(z) = \frac{\cos z - 1}{z};$$

*d)*

$$f(z) = \exp(z^{-1});$$

*e)*

$$f(z) = \frac{\log(z + 1)}{z^2};$$

*f)*

$$f(z) = \frac{\cos(z^{-1})}{z^{-1}};$$

*g)*

$$f(z) = \frac{z^2 + 1}{z(z - 1)};$$

*h)*

$$f(z) = (1 - e^z)^{-1};$$

*i)*

$$f(z) = z \sin \frac{1}{z};$$

# Chapter 6

# The Maximum Modulus Theorem

## 6.1 The Maximum Principle

**Exercise 1.** *Prove the following Minimum Principle. If $f$ is a non-constant analytic function on a bounded open set $G$ and is continuous on $G^-$, then either $f$ has a zero in $G$ or $|f|$ assumes its minimum value on $\partial G$. (See Exercise IV. 3.6.)*

**Solution.** *Since $f \in C(\overline{G})$ we have $|f| \in C(\overline{G})$. Hence $\exists\, a \in \overline{G}$ such that $|f(a)| \le |f(z)|\ \forall z \in \overline{G}$. If $a \in \partial G$, then $|f|$ assumes its minimum value on $\partial G$ and we are done. Otherwise, if $a \notin \partial G$, then $a \in G$ and we can write $G = \bigcup A_i$ where $A_i$ are the components of $G$, that is $a \in A_i$ for some $i$. But each $A_i$ is a region, so we can use Exercise IV 3.6 which yields either $f(a) = 0$ or $f$ is constant. But $f$ was assumed to be non-constant, so $f$ has to have a zero in $G$. Therefore, either $f$ has a zero in $G$ or $|f|$ assumes its minimum value on $\partial G$.*



**Exercise 2.** *Let $G$ be a bounded region and suppose $f$ is continuous on $G^-$ and analytic on $G$. Show that if there is a constant $c \ge 0$ such that $|f(z)| = c$ for all $z$ on the boundary of $G$ then either $f$ is a constant function or $f$ has a zero in $G$.*

**Solution.** *Assume there is a constant $c \ge 0$ such that $|f(z)| = c$ for all $z \in \partial G$. According to the Maximum Modulus Theorem (Version 2), we get*

$$\max\{|f(z)| : z \in \overline{G}\} = \max\{|f(z)| : z \in \partial G\} = c.$$

*So*

$$|f(z)| \le c, \qquad \forall z \in \overline{G}. \tag{6.1}$$

*Since $f \in C(G)$ which implies $|f| \in C(\overline{G})$ and hence there exists an $a \in G$ such that*

$$|f(a)| \le |f(z)| \underset{(6.1)}{\le} c,$$

*then by Exercise IV 3.6 either $f$ is constant of $f$ has a zero in $G$.*

**Exercise 3.** *(a) Let $f$ be entire and non-constant. For any positive real number $c$ show that the closure of $\{z : |f(z)| < c\}$ is the set $\{z : |f(z)| \le c\}$.*
*(b) Let $p$ be a polynomial and show that each component of $\{z : |p(z)| < c\}$ contains a zero of $p$. (Hint: Use*

*Exercise 2.)*
*(c) If $p$ is a polynomial and $c > 0$ show that $\{z : |p(z)| = c\}$ is the union of a finite number of closed paths. Discuss the behavior of these paths as $c \to \infty$.*

**Solution.** *Not available.*

**Exercise 4.** *Let $0 < r < R$ and put $A = \{z : r \leq |z| \leq R\}$. Show that there is a positive number $\epsilon > 0$ such that for each polynomial $p$,*

$$\sup\{|p(z) - z^{-1}| : z \in A\} \geq \epsilon$$

*This says that $z^{-1}$ is not the uniform limit of polynomials on $A$.*

**Solution.** *Not available.*

**Exercise 5.** *Let $f$ be analytic on $\overline{B}(0; R)$ with $|f(z)| \leq M$ for $|z| \leq R$ and $|f(0)| = a > 0$. Show that the number of zeros of $f$ in $B(0; \frac{1}{3}R)$ is less than or equal to $\frac{1}{\log 2} \log\left(\frac{M}{a}\right)$. Hint: If $z_1, \ldots, z_n$ are the zeros of $f$ in $B(0; \frac{1}{3}R)$; consider the function*

$$g(z) = f(z)\left[\prod_{k=1}^{n}\left(1 - \frac{z}{z_k}\right)\right]^{-1},$$

*and note that $g(0) = f(0)$. (Notation: $\prod_{k=1}^{n} a_k = a_1 a_2 \ldots a_n$.)*

**Solution.** *Not available.*

**Exercise 6.** *Suppose that both $f$ and $g$ are analytic on $B(0; R)$ with $|f(z)| = |g(z)|$ for $|z| = R$. Show that if neither $f$ nor $g$ vanishes in $B(0; R)$ then there is a constant $\lambda$, $|\lambda| = 1$, such that $f = \lambda g$.*

**Solution.** *Not available.*

**Exercise 7.** *Let $f$ be analytic in the disk $B(0; R)$ and for $0 \leq r < R$ define $A(r) = \max\{Re\, f(z) : |z| = r\}$. Show that unless $f$ is a constant, $A(r)$ is a strictly increasing function of $r$.*



**Solution.** *Not available.*

**Exercise 8.** *Suppose $G$ is a region, $f : G \to \mathbb{C}$ is analytic, and $M$ is a constant such that whenever $z$ is on $\partial_\infty G$ and $\{z_n\}$ is a sequence in $G$ with $z = \lim z_n$ we have $\limsup |f(z_n)| \leq M$. Show that $|f(z)| \leq M$, for each $z$ in $G$.*

**Solution.** *We need to show*

$$\limsup_{z \to a} |f(z)| \leq M \qquad \forall u \in \partial_\infty G.$$

*Then we can use the Maximum Modulus Theorem (Version 3). Instead of showing that*

$$\limsup |f(z_n)| \leq M \Rightarrow \limsup_{z \to a} |f(z)| \leq M,$$

*we show the contrapositive, that is*

$$\limsup_{z \to a} |f(z)| > M \Rightarrow \limsup |f(z_n)| > M.$$

*So assume $\limsup_{z \to a} |f(z)| > M$. But this implies $\limsup_{z \to a} |f(z_n)| > M$ since $z_n \to z$ as $z \to \infty$, that is $z_n$ gets arbitrarily close to $z$ and since $f$ is analytic, that is continuous, we have $f(z_n)$ gets arbitrarily close to $f(z)$.*

**Exercise 6.** *Prove Hardy's Theorem: If $f$ is analytic on $\bar{B}(0;R)$ and not constant then*

$$I(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta$$

*is strictly increasing and $\log I(r)$ is a convex function of $\log r$. Hint: If $0 < r_1 < r < r_2$ find a continuous function $\varphi : [0, 2\pi] \to \mathbb{C}$ such that $\varphi(\theta) f(re^{i\theta}) = |f(re^{i\theta})|$ and consider the function $F(z) = \int_0^{2\pi} f(ze^{i\theta})\varphi(\theta)\, d\theta$. (Note that $r$ is fixed, so $\varphi$ may depend on $r$.)*

**Solution.** *Not available.*

**Exercise 7.** *Let $f$ be analytic in $ann(0; R_1, R_2)$ and not identically zero; define*

$$I_2(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta.$$

*Show that $\log I_2(r)$ is a convex function of $\log r$, $R_1 < r < R_2$.*

**Solution.** *Not available.*

## 6.4 The Phragmén-Lindelöf Theorem



**Exercise 1.** *In the statement of the Phragmén-Lindelöf Theorem, the requirement that $G$ be simply connected is not necessary. Extend Theorem 4.1 to regions $G$ with the property that for each $z$ in $\partial_\infty G$ there is a sphere $V$ in $\mathbb{C}_\infty$ centered at $z$ such that $V \cap G$ is simply connected. Give some examples of regions that are not simply connected but have this property and some which don't.*

**Solution.** *Not available.*

**Exercise 2.** *In Theorem 4.1 suppose there are bounded analytic functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ on $G$ that never vanish and $\partial_\infty G = A \cup B_1 \cup \ldots \cup B_n$ such that condition (a) is satisfied and condition (b) is also satisfied for each $\varphi_k$ and $B_k$. Prove that $|f(z)| \leq M$ for all $z$ in $G$.*

**Solution.** *We have $\varphi_1, \ldots, \varphi_n \in A(G)$, bounded and $\varphi_k \not\equiv 0$ by assumption. Hence, $|\varphi_k(z)| \leq \kappa_k \ \forall\ z$ in $G$ for each $k = 1, \ldots, n$. Further a) for every $a$ in $A$, $\limsup_{z\to a} |f(z)| \leq M$ and b) for every $b$ in $B_k$ and $\eta_k > 0$, $\limsup_{z\to b} |f(z)||\varphi_k(z)|^{\eta_k} \leq M\ \forall\ k = 1, \ldots, n$.*
*Also because $G$ is simply connected, there is an analytic branch of $\log\varphi_k(z)$ on $G$ for each $k = 1, \ldots, n$ (Corollary IV 6.17). Hence, $g_k(z) = \exp\{\eta_k \log\varphi_k(z)\}$ is an analytic branch of $\varphi_k(z)^{\eta_k}$ for $\eta_k > 0$ and $|g_k(z)| = |\varphi_k(z)|^{\eta_k}\ \forall\ k = 1, \ldots, n$. Define $F : G \to \mathbb{C}$ by*

$$F(z) = f(z)\prod_{k=1}^n g_k(z)\kappa_k^{-\eta_k};$$

*then $F$ is analytic on $G$ and*

$$|F(z)| \leq |f(z)|\prod_{k=1}^n |\varphi_k(z)|^{\eta_k}\kappa_k^{-\eta_k} \underset{|\varphi_k(z)|\leq\kappa_k\ \forall k}{\leq} |f(z)|\prod_{k=1}^n \kappa_k^{\eta_k}\kappa_k^{-\eta_k}|\varphi_k(z)| = |f(z)| \quad \forall z \in G.$$

*Hence,*

$$\limsup_{z\to a} |f(z)| \underset{(a),(b)}{=} \begin{cases} M, & a \in A \\ M\kappa_k^{-\eta_k}, & a \in B_k \quad \forall k \end{cases}.$$

*by definition. Then, clearly* $\exists\delta > 0$ *such that* $\lim_{r\to 0^+}\sup\{|f(z)| : z \in G \cap B(a;r)\} > M + \delta$. *Let* $\{z_n\}$ *be a sequence with* $\lim_{n\to\infty} = a$ *and let* $r_n = 2|z_n - a|$. *Obviously, we have* $\lim_{n\to\infty} r_n = 0$. *Next, consider the sequence* $\alpha_n = \sup\{|f(z)| : z \in F \cap B(a;r_n)\}$. *Then* $\lim_{n\to\infty}\alpha_n > M + \delta$. *In addition,* $\{\alpha_n\}$ *is a nonincreasing sequence, therefore* $\lim_{n\to\infty}\alpha_n > M + \delta$ *implies* $\alpha_n > M + \delta\ \forall n$. *Thus,* $\exists\, y_n \in G \cap B(a;r_n)$ *such that* $|f(y_n)| > M + \delta$. *Taking* $\limsup$ *in this inequality yields*

$$\limsup_{n\to\infty} |f(y_n)| \geq M + \delta,$$

*that is*

$$\limsup_{n\to\infty} |f(y_n)| \geq M$$

*where* $\lim_{n\to\infty} y_n = a$, *which gives "not (6.5)". Hence (6.5) implies (6.6).*

**Exercise 5.** *Let* $f : G \to \mathbb{C}$ *be analytic and suppose that* $G$ *is bounded. Fix* $z_0$ *in* $\partial G$ *and suppose that* $\limsup_{z\to w}|f(z)| \leq M$ *for* $w$ *in* $\partial G$, $w \neq z_0$. *Show that if* $\lim_{z\to z_0}|z - z_0|^{\epsilon}|f(z)| = 0$ *for every* $\epsilon > 0$ *then* $|f(z)| \leq M$ *for every* $z$ *in* $\partial G$. *(Hint: If* $a \notin G$, *consider* $\varphi(z) = (z - z_0)(z - a)^{-1}$.*)*

**Solution.** *Not available.*

**Exercise 6.** *Let* $G = \{z : Re\, z > 0\}$ *and let* $f : G \to \mathbb{C}$ *be an analytic function with* $\limsup_{z\to w}|f(z)| \leq M$ *for* $w$ *in* $\partial G$, *and also suppose that for every* $\epsilon > 0$,

$$\lim_{r\to\infty}\sup\{\exp(-\epsilon\sqrt{r})|f(re^{i\theta})| : |\theta| < \frac{1}{2}\pi\} = 0.$$

*Show that* $|f(z)| \leq M$ *for all* $z$ *in* $G$.

**Solution.** *Not available.*

**Exercise 7.** *Let* $G = \{z : Re\, z > 0\}$ *and let* $f : G \to \mathbb{C}$ *be analytic such that* $f(1) = 0$ *and such that* $\limsup_{z\to w}|f(z)| \leq M$ *for* $w$ *in* $\partial G$. *Also, suppose that for every* $\delta$, $0 < \delta < 1$, *there is a constant* $P$ *such that*

$$|f(z)| \leq P\exp(|z|^{1-\delta}).$$

*Prove that*

$$|f(z)| \leq M\left[\frac{(1-x)^2 + y^2}{(1+x)^2 + y^2}\right]^{\frac{1}{2}}.$$

*Hint: Consider* $f(z)\left(\frac{1+z}{1-z}\right)$.

**Solution.** *Not available.*

*We also have*

$$|h(z)| \leq 1 \qquad \forall z \in D$$

*since $h : D \to D$. Thus, the hypothesis of Schwarz's Lemma are satisfied, and hence, we get*

$$|h'(0)| \leq 1.$$

*We have*

$$\begin{aligned} h'(z) &= \left[g(f(g^{-1}(z)))\right]' = g'(f(g^{-1}(z)))f'(g^{-1}(z))\left(g^{-1}(z)\right)' \\ &= g'(f(g^{-1}(z)))f'(g^{-1}(z))\frac{1}{g'(g^{-1}(z))} \end{aligned}$$

*where the last step follows from Proposition 2.20 provided $g'(g^{-1}(z)) \neq 0$. So*

$$\begin{aligned} h'(0) &= g'(f(g^{-1}(0)))f'(g^{-1}(0))\frac{1}{g'(g^{-1}(0))} \\ &= g'(f(a))f'(a)\frac{1}{g'(a)} = g'(a)f'(a)\frac{1}{g'(a)} = f'(a) \end{aligned}$$

*($g'(a) > 0$ by assumption). Therefore*

$$|h'(0)| = |f'(a)| \leq 1.$$

*Thus, we have shown that*

$$|f'(a)| \leq 1.$$

**Exercise 6.** *Let $G_1$ and $G_2$ be simply connected regions neither of which is the whole plane. Let $f$ be a one-one analytic mapping of $G_1$ onto $G_2$. Let $a \in G_1$ and put $\alpha = f(a)$. Prove that for any one-one analytic map $h$ of $G_1$ into $G_2$ with $h(a) = \alpha$ it follows that $|h'(a)| \leq |f'(a)|$. Suppose $h$ is not assumed to be one-one; what can be said?*



**Solution.** *Define the function $F(z) = f^{-1}(h(z))$. This is well-defined since $f : G_1 \xrightarrow[onto]{1-1} G_2$ and $h : G_1 \xrightarrow{1-1} G_2$. Clearly $F(z)$ is analytic since $f$ and $h$ are analytic, $F(z)$ is one-one and $F : G_1 \to G_1$ by construction. We have*

$$F(a) = f^{-1}(h(a)) = f^{-1}(\alpha) = a.$$

*Thus by Exercise 5, we get*

$$|F'(a)| \leq 1.$$

*We have*

$$F'(z) = \left(f^{-1}(h(z))\right)' = f'(h(z))h'(z) = \frac{1}{f'(f^{-1}(h(z)))}h'(z)$$

*where the last step follows by Proposition 2.20 provided $f'(f^{-1}(h(z))) \neq 0$. So*

$$F'(a) = \frac{1}{f'(f^{-1}(h(a)))}h'(a) = \frac{h'(a)}{f'(f^{-1}(\alpha))} = \frac{h'(a)}{f'(a)}$$

*which is well-defines since $f'(a) \neq 0$ by assumption ($f$ is one-one). Since*

$$|F'(a)| = \frac{|h'(a)|}{|f'(a)|} \leq 1$$

*iff $A$ has a limit point in $G$.*
*(d) Let $a \in G$ and $\mathcal{J} = \mathcal{J}(\{a\})$. Show that $\mathcal{J}$ is a maximal ideal.*
*(e) Show that every maximal ideal in $H(G)$ is a prime ideal.*
*(f) Give an example of an ideal which is not a prime ideal.*

**Solution.** *Not available.*

**Exercise 12.** *Find an entire function $f$ such that $f(n + in) = 0$ for every integer $n$ (positive, negative or zero). Give the most elementary example possible (i.e., choose the $p_n$ to be as small as possible).*

**Solution.** *Not available.*

**Exercise 13.** *Find an entire function $f$ such that $f(m + in) = 0$ for all possible integers $m, n$. Find the most elementary solution possible.*

**Solution.** *Not available.*

## 7.6 Factorization of the sine function

**Exercise 1.** *Show that $\cos \pi z = \prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n-1)^2}\right]$.*

**Solution.** *We know by the double-angle identity of sine $\sin(2z) = 2\sin(z)\cos(z)$ (this is proved easily by using the definition) or $\sin(2\pi z) = 2\sin(\pi z)\cos(\pi z)$. Since, we know $\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$, we obtain*

$$\begin{aligned}
&\sin(2\pi z) = 2\sin(\pi z)\cos(\pi z) \\
\iff\; & 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{n^2}\right) = 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\cos(\pi z) \\
\iff\; & 2\pi z \prod_{m=1}^{\infty}\left(1 - \frac{4z^2}{(2m)^2}\right)\prod_{m=1}^{\infty}\left(1 - \frac{4z^2}{(2m-1)^2}\right) = 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\cos(\pi z)
\end{aligned}$$

*where the last statement follows by splitting the product into a product of the even and odd terms (rearrangement of the terms is allowed). Hence*

$$\begin{aligned}
& 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n)^2}\right)\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right) = 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\cos(\pi z) \\
\iff\; & 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right) = 2\pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)\cos(\pi z).
\end{aligned}$$

*Thus,*

$$\cos(\pi z) = \prod_{n=1}^{\infty}\left(1 - \frac{4z^2}{(2n-1)^2}\right).$$

**Exercise 2.** *Find a factorization for $\sinh z$ and $\cosh z$.*

*Thus,*

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t} t^{z-1}\, dt \tag{7.12}$$

*Claim 1:* $\Gamma(z)$ *given by* (7.12) *is the analytic continuation of* (7.11)*, that is* $\Gamma(z)$ *given by* (7.12) *is defined for all* $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$.
*Proof of Claim 1: We know from the book that* $\Psi(z)$ *is analytic for* $Re\,(z) > 0$.
*Claim 2:* $\Psi(z)$ *is analytic for* $Re\,(z) \leq 0$. *Thus* $\Psi(z)$ *is analytic on* $\mathbb{C}$.
*Proof of Claim 2: Assume* $Re\,(z) \leq 0$. *Then*

$$|t^{z-1}| = t^{Re(z)-1} \underset{t\in[1,\infty), Re(z)\leq 0}{\leq} t^{-1}.$$

*But since* $e^{-\frac{1}{2}t} t^{Re\,(z)-1} \to 0$ *as* $t \to \infty$, *there exists a constant* $C > 0$ *such that* $t^{Re(z)-1} \leq Ce^{\frac{1}{2}}$ *when* $t \geq 1$. *Hence, we have*

$$|e^{-t}t^{z-1}| \leq |e^{-t}|\cdot|t^{z-1}| = e^{-t}t^{Re(z)-1} \leq e^{-t}Ce^{\frac{1}{2}t} = Ce^{-\frac{1}{2}t}$$

*and therefore* $Ce^{-\frac{1}{2}t}$ *is integrable on* $(1, \infty)$. *By Fubini's Theorem for any* $\{\gamma\} \subset G = \{z \mid Re(z) \leq 0\}$,

$$\int_{\gamma}\int_1^{\infty} e^{-t}t^{z-1}\, dt\, dz = \int_1^{\infty}\int_{\gamma} e^{-t}t^{z-1}\, dz\, dt = 0$$

*which implies*

$$\int_1^{\infty} e^{-t}t^{z-1}\, dt \in H(G).$$

*In summary,*

$$\Psi(z) = \int_1^{\infty} e^{-t}t^{z-1}\, dt \in H(\mathbb{C}).$$

*Thus, Claim 2 is proved.*
*It remains to show that*

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)}$$

*is analytic on* $\mathbb{C} - \{0, -1, -2, \ldots\}$. *Note that* $\Phi(z)$ *is uniformly and absolutely convergent as a series in any closed domain which contains none of the points* $0, -1, -2, \ldots$ *and thus provides the analytic continuation of* $\Phi(z)$. *Since*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t}t^{z-1}\, dt$$

*is analytic and we know (Theorem 7.15 p. 180)*

$$\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}\, dt$$

*for* $Re(z) > 0$, *we get*

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)} + \int_1^{\infty} e^{-t}t^{z-1}\, dt$$

*is the analytic continuation of* (7.11) *for* $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$.

## 8.3 Mittag-Leffler's Theorem

**Exercise 1.** *Let $G$ be a region and let $\{a_n\}$ and $\{b_m\}$ be two sequences of distinct points in $G$ without limit points in $G$ such that an $a_n \neq b_m$ for all $n, m$. Let $S_n(z)$ be a singular part at $a_n$ and let $p_m$ be a positive integer. Show that there is a meromorphic function $f$ on $G$ whose only poles and zeros are $\{a_n\}$ and $\{b_m\}$ respectively, the singular part at $z = a_n$ is $S_n(z)$, and $z = b_m$ is a zero of multiplicity $p_m$.*

**Solution.** *Let $G$ be a region and let $\{b_m\}$ be a sequence of distinct points in $G$ with no limit point in $G$; and let $\{p_m\}$ be a sequence of integers. By Theorem 5.15 p.170 there is an analytic function $g$ defined on $G$ whose only zeros are at the points $b_m$; furthermore, $b_m$ is a zero of $g$ of multiplicity $p_m$.*
*Since $g \in H(g)$ and $\{a_n\} \in G$, $g$ has a Taylor series in a neighborhood $B(a_n; R_n)$ of each $a_n$, that is*

$$g_n(z) = \sum_{k=0}^{\infty} \alpha_k (z - a_n)^k \in B(a_n; R_n)$$

*where $\alpha_k = \frac{1}{k!} g^{(k)}(a_n)$. Goal: Try to use this series to create a singular part $r_n(z)$ at $a_n$ such that $r_n(z) g_n(z) = s_n(z)$ or*

$$r_n(z) \sum_{k=0}^{\infty} \alpha_k (z - a_n)^k = \sum_{j=1}^{m_n} \frac{A_{jn}}{(z - a_n)^j} \iff \sum_{j=1}^{m_n} \frac{A_{jn}}{(z - a_n)^j} = \sum_{k=0}^{\infty} \alpha_k r_n(z)(z - a_n)^k.$$

*Claim:*

$$r_n(z) = \sum_{j=1}^{m_n} B_{jk}(z - a_n)^{-j-k} \tag{8.1}$$

*works.*
*Proof of the claim:*

$$\begin{aligned} \sum_{k=0}^{\infty} \alpha_k r_n(z)(z - a_n)^k &= \sum_{k=0}^{\infty} \alpha_k \sum_{j=1}^{m_n} B_{jk}(z - a_n)^{-j-k}(z - a_n)^k \\ &= \sum_{k=0}^{\infty} \alpha_k \sum_{j=1}^{m_n} B_{jk}(z - a_n)^{-j} \\ &= \sum_{j=1}^{m_n} (z - a_n)^{-j} \sum_{k=0}^{\infty} \alpha_k B_{jk} = \sum_{j=1}^{m_n} \frac{A_{jn}}{(z - a_n)^j} \end{aligned}$$

*where the last step follows by choosing*

$$\sum_{k=0}^{\infty} \alpha_k B_{jk} = A_{jn}.$$

*Since $G$ is a region, $G$ is open. Let $\{a_n\}$ be a sequence of distinct points without a limit point in $G$ and such that $a_n \neq b_m$ for all $n, m$. Let $\{r_n(z)\}$ be the sequence of rational functions given by*

$$r_n(z) = \sum_{j=1}^{m_n} \frac{B_{jk}}{(z - a_n)^{j+k}}$$

*(see (8.1)). By Mittag-Leffler's Theorem, there is a meromorphic function $h$ on $G$ whose poles are exactly the points $\{a_n\}$ and such that the singular part of $h$ at $a_n$ is $r_n(z)$.*
*Set $f = g \cdot h$. Then by construction $f$ is the meromorphic function on $G$ whose only poles and zeros are $\{a_n\}$ and $\{b_m\}$ respectively, the singular part at $z = a_n$ is $S_n(z)$, and $z = b_m$ is a zero of multiplicity $p_m$. (Note that the zeros do not cancel the poles since by assumption $a_n \neq b_m$ $\forall$ $n, m$).*

**Solution.** *Not available.*

**Exercise 4.** *Let $G$ be a simply connected region and let $\Gamma$ be its closure in $\mathbb{C}_\infty$; $\partial_\infty G = \Gamma - G$. Suppose there is a homeomorphism $\varphi$ of $\Gamma$ onto $D^-(D = \{z : |z| < 1\})$ such that $\varphi$ is analytic on $G$.*
*(a) Show that $\varphi(G) = D$ and $\varphi(\partial_\infty G) = \partial D$.*
*(b) Show that if $f : \partial_\infty G \to \mathbb{R}$ is a continuous function then there is a continuous function $u : \Gamma \to \mathbb{R}$ such that $u(z) = f(z)$ for $z$ in $\partial_\infty G$ and $u$ is harmonic in $G$.*
*(c) Suppose that the function $f$ in part (b) is not assumed to be continuous at $\infty$. Show that there is a continuous function $u : G^- \to \mathbb{R}$ such that $u(z) = f(z)$ for $z$ in $\partial G$ and $u$ is harmonic in $G$ (see Exercise 2).*

**Solution.** *Not available.*

**Exercise 5.** *Let $G$ be an open set, $a \in G$, and $G_0 = G - \{a\}$. Suppose that $u$ is a harmonic function on $G_0$ such that $\lim_{z\to a} u(z)$ exists and is equal to $A$. Show that if $U : G \to \mathbb{R}$ is defined by $U(z) = u(z)$ for $z \neq a$ and $U(a) = A$ then $U$ is harmonic on $G$.*

**Solution.** *Not available.*

**Exercise 6.** *Let $f : \{z : Re\ z = 0\} \to \mathbb{R}$ be a bounded continuous function and define $u : \{z : Re\ z > 0\} \to \mathbb{R}$ by*

$$u(x + iy) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x f(it)}{x^2 + (y - t)^2}\, dt.$$

*Show that $u$ is a bounded harmonic function on the right half plane such that for $c$ in $\mathbb{R}$, $f(ic) = \lim_{z\to ic} u(z)$.*

**Solution.** *Not available.*

**Exercise 7.** *Let $D = \{z : |z| < 1\}$ and suppose $f : \partial D \to \mathbb{R}$ is continuous except for a jump discontinuity at $z = 1$. Define $u : D \to \mathbb{R}$ by (2.5). Show that $u$ is harmonic. Let $v$ be a harmonic conjugate of $u$. What can you say about the behavior of $v(r)$ as $r \to 1^-$? What about $v(re^{i\theta})$ as $r \to 1^-$ and $\theta \to 0$?*



**Solution.** *Not available.*

## 10.3 Subharmonic and superharmonic functions

**Exercise 1.** *Which of the following functions are subharmonic? superharmonic? harmonic? neither subharmonic nor superharmonic? (a) $\varphi(x, y) = x^2 + y^2$; (b) $\varphi(x, y) = x^2 - y^2$; (c) $\varphi(x, y) = x^2 + y$; (d) $\varphi(x, y) = x^2 - y$; (e) $\varphi(x, y) = x + y^2$; (f) $\varphi(x, y) = x - y^2$.*

**Solution.** *Note that $\int_{-\pi}^{\pi} \sin\theta\, d\theta = 0$, $\int_{-\pi}^{\pi} \cos\theta\, d\theta = 0$, $\int_{-\pi}^{\pi} \sin^2\theta\, d\theta = \pi$ and $\int_{-\pi}^{\pi} \cos^2\theta\, d\theta = \pi$. For all $a = (\alpha, \beta) \in \mathbb{C}$ and any $r > 0$ we have*
*a)*

$$\begin{aligned}
\varphi(a + re^{i\theta}) &= \varphi(\alpha + r\cos\theta, \beta + r\sin\theta) = (\alpha + r\cos\theta)^2 + (\beta + r\sin\theta)^2 \\
&= \alpha^2 + 2\alpha r\cos\theta + r^2\cos^2\theta + \beta^2 + 2\beta r\sin\theta + r^2\sin^2\theta \\
&= \alpha^2 + \beta^2 + 2\alpha r\cos\theta + 2\beta r\sin\theta + r^2.
\end{aligned}$$

*Thus*

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(a + re^{i\theta})\, d\theta &= \alpha^2 + \beta^2 + \frac{\alpha r}{\pi}\underbrace{\int_{-\pi}^{\pi} \cos\theta\, d\theta}_{=0} + \frac{\beta r}{\pi}\underbrace{\int_{-\pi}^{\pi} \sin\theta\, d\theta}_{=0} + r^2 \\
&= \alpha^2 + \beta^2 + r^2 \geq \alpha^2 + \beta^2 = \varphi(a).
\end{aligned}$$

*Thus*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(a+re^{i\theta})\,d\theta = \alpha+\beta^2+\frac{r}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos\theta\,d\theta}_{=0}+\frac{\beta r}{\pi}\underbrace{\int_{-\pi}^{\pi}\sin\theta\,d\theta}_{=0}+\frac{r^2}{2\pi}\underbrace{\int_{-\pi}^{\pi}\sin^2\theta\,d\theta}_{=\pi}$$
$$= \alpha+\beta^2+\frac{r^2}{2}\geq\alpha+\beta^2=\varphi(a).$$

*Hence,* $\varphi\in$ *Subhar(G).*
*f)*

$$\varphi(a+re^{i\theta}) = \varphi(\alpha+r\cos\theta,\beta+r\sin\theta)=(\alpha+r\cos\theta)-(\beta+r\sin\theta)^2$$
$$= \alpha+r\cos\theta-\beta^2-2\beta r\sin\theta-r^2\sin^2\theta.$$

*Thus*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(a+re^{i\theta})\,d\theta = \alpha-\beta^2+\frac{r}{2\pi}\underbrace{\int_{-\pi}^{\pi}\cos\theta\,d\theta}_{=0}-\frac{\beta r}{\pi}\underbrace{\int_{-\pi}^{\pi}\sin\theta\,d\theta}_{=0}-\frac{r^2}{2\pi}\underbrace{\int_{-\pi}^{\pi}\sin^2\theta\,d\theta}_{=\pi}$$
$$= \alpha-\beta^2-\frac{r^2}{2}\leq\alpha-\beta^2=\varphi(a).$$

*Hence,* $\varphi\in$ *Superhar(G).*



**Exercise 2.** *Let Subhar(G) and Superhar(G) denote, respectively, the sets of subharmonic and superharmonic functions on G.*
*(a) Show that Subhar(G) and Superhar(G) are closed subsets of* $\mathbb{C}(G;\mathbb{R})$.
*(b) Does a version of Harnack's Theorem hold for subharmonic and superharmonic functions?*

**Solution.** *Not available.*

**Exercise 3.** *(This exercise is difficult.) If G is a region and if* $f:\partial_\infty G\to\mathbb{R}$ *is a continuous function let* $u_f$ *be the Perron Function associated with* $f$*. This defines a map* $T:(\partial_\infty G;\mathbb{R})\to Har(G)$ *by* $T(f)=u_f$*. Prove:*
*(a)* $T$ *is linear (i.e.,* $T(a_1f_1+a_2f_2)=a_1T(f_1)+a_2T(f_2)$*).*
*(b)* $T$ *is positive (i.e., if* $f(a)\geq 0$ *for all a in* $\partial_\infty G$ *then* $T(f)(z)\geq 0$ *for all z in G).*
*(c)* $T$ *is continuous. Moreover, if* $\{f_n\}$ *is a sequence in* $C(\partial_\infty G;\mathbb{R})$ *such that* $f_n\to f$ *uniformly then* $T(f_n)\to T(f)$ *uniformly on* $G$.
*(d) If the Dirichlet Problem can be solved for G then* $T$ *is one-one. Is the converse true?*

**Solution.** *Not available.*

**Exercise 4.** *In the hypothesis of Theorem 3.11, suppose only that* $f$ *is a bounded function on* $\partial_\infty G$*; prove that the conclusion remains valid. (This is useful if G is an unbounded region and g is a bounded continuous function on* $\partial G$*. If we define* $f:\partial_\infty G\to\mathbb{R}$ *by* $f(z)=g(z)$ *for z in* $\partial G$ *and* $f(\infty)=0$ *then the conclusion of Theorem 3.11 remains valid. Of course there is no reason to expect that the harmonic function will have predictable behavior near* $\infty$ *— we could have assigned any value to* $f(\infty)$*. However, the behavior near points of* $\partial G$ *can be studied with hope of success.)*

**Solution.** *Not available.*

**Exercise 5.** *Show that the requirement that* $G_1$ *is bounded in Corollary 3.5 is necessary.*

## 11.3 Hadamard Factorization Theorem

**Exercise 1.** *Let $f$ be analytic in a region $G$ and suppose that $f$ is not identically zero. Let $G_0 = G - \{z : f(z) = 0\}$ and define $h : G_0 \to \mathbb{R}$ by $h(z) = \log|f(z)|$. Show that $\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y} = \frac{f'}{f}$ on $G_0$.*

**Solution.** *Let $f$ be analytic in a region $G$ and suppose that $f$ is not identically zero. Let $G_0 = G - \{z : f(z) = 0\}$, then $h : G_0 \to \mathbb{R}$ given by $h(z) = \log|f(z)|$ is well defined as well as $\frac{f'}{f}$ is well defined on $G_0$. Let $f = u(x,y) + iv(x,y) = u + iv$. Since $f$ is analytic, the Cauchy-Riemann (C-R) equations $u_x = v_y$ and $u_y = -v_x$ are satisfied. We have by p. 41 Equation 2.22 and 2.23*

$$f' = u_x + iv_x \quad \text{and} \quad f' = -iu_y + v_y$$

*and thus*

$$2f' = u_x + iv_x - iu_y + v_y$$

*implies*

$$f' = \frac{1}{2}\left(u_x + iv_x - iu_y + v_y\right) \underset{(C\text{-}R)}{=} \frac{1}{2}\left(u_x - iu_y - iu_y + u_x\right) = \frac{1}{2}\left(2u_x - 2iu_y\right) = u_x - iu_y.$$

*Therefore,*

$$\frac{f'}{f} = \frac{u_x - iu_y}{u + iv}. \tag{11.6}$$

*Next, we calculate $\frac{\partial h}{\partial x} = h_x$ and $\frac{\partial h}{\partial y} = h_y$ where $h(z) = \log|f(z)| = \frac{1}{2}\log(u^2 + v^2)$. Using the chain rule, we get*

$$h_x = h_u u_x + h_v v_x = \frac{u}{u^2 + v^2}u_x + \frac{v}{u^2 + v^2}v_x$$

*and*

$$h_y = h_u u_y + h_v v_y = \frac{u}{u^2 + v^2}u_y + \frac{v}{u^2 + v^2}v_y$$

*and hence*

$$\begin{aligned}
\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y} &= h_x - ih_y \\
&= \frac{u}{u^2 + v^2}u_x + \frac{v}{u^2 + v^2}v_x - i\frac{u}{u^2 + v^2}u_y - i\frac{v}{u^2 + v^2}v_y \\
&= \frac{u}{u^2 + v^2}(u_x - iu_y) + \frac{v}{u^2 + v^2}(v_x - iv_y) \\
&\underset{(C\text{-}R)}{=} \frac{u}{u^2 + v^2}(u_x - iu_y) + \frac{v}{u^2 + v^2}(-u_y - iu_x) \\
&= \frac{u}{u^2 + v^2}(u_x - iu_y) - \frac{iv}{u^2 + v^2}(u_x - iu_y) \\
&= \frac{u - iv}{u^2 + v^2}(u_x - iu_y) \\
&= \frac{u - iv}{(u - iv)(u + iv)}(u_x - iu_y) \\
&= \frac{u_x - iu_y}{u + iv}.
\end{aligned}$$