Practical class №6. Linear vector operations.

Problems

1. In parallelogram $ABCD$:

a) point out among vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$ equal vectors, opposite vectors, collinear vectors, complanar vectors;

b) express the vector $\overrightarrow{AC}$ using vectors: $\overrightarrow{AB}$ and $\overrightarrow{BC}$; $\overrightarrow{CD}$ and $\overrightarrow{AD}$; $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

2. The points $A(-1;\ 0;\ 2)$, $B(1;\ -2;\ 3)$ are given. Find

a) vector $\overrightarrow{AB}$; b) $\left|\overrightarrow{AB}\right|$.

3. The points $A(-2;\ 2)$, $B(2;\ 6)$ are given and point C is the midpoint of AB. Build vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, find their components and check , that $\overrightarrow{AB} = -2\overrightarrow{BC}$.

4. The vector $\vec{a} = 3\vec{i} - 4\vec{k}$ is given. Find the unit vector for vector $\vec{a}$ and the projection of $\vec{a}$ on Oy.

5. The points $A(-2;\ p;\ 3)$, $B(-1;\ 0;\ 2)$ and $C(q;\ 1;\ 1)$ are given. Which values of p and q are vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ equal at?

6. The vector $\overrightarrow{BC}(-1;\ -2;\ 3)$ and point $B(-3;\ 2;\ -2)$ are given. Find the coordinates of the point C.

7. Find components of the vector $3\vec{a} + 2\vec{b}$, if $\vec{a} = 5\vec{i} - \vec{j} + \vec{k}$, $\vec{b}(-2;\ 1;\ 0)$.

8. Find components of vector $\vec{p}$, which is collinear and co-directed to the vector $\vec{q}(-8;\ 16;\ 4)$ and $|\vec{p}| = \sqrt{21}$.

9. Which values of m and n are vectors $\vec{a}(1;\ m;\ 2)$ and $\vec{b}(0{,}5n + 1;\ 3;\ 1)$ collinear at?

10. Verify that the points $A(2;\ 1;\ 0)$, $B(0;\ 4;\ -3)$, $C(-2;\ 3;\ -5)$ and $D(2;\ -3;\ 1)$ are vertices of a trapezoid. Find the lengths of their bases.

11. In parallelogram $ABCD$ vectors $\overrightarrow{AB}(-4;\ -4;\ 2)$, $\overrightarrow{CB}(-3;\ -6;\ 1)$ and point $A(3;\ 8;\ -5)$ are given. Find the coordinates of the intersection point of its diagonals.

12. The vectors $\overrightarrow{AB}(\alpha;\ \beta;\ -6)$ and $\overrightarrow{AC}(4;\ 2;\ -3)$ are given. Which values of α and β do the points A, B and C lie on same line?

13. The vectors $\vec{a}(-3;\ 2;\ 0)$, $\vec{b}(2;\ 0;\ -3)$ are given. Find $\left|\ 2\vec{a} - \vec{b}\ \right|$.

using following formulae:

$$\cos\alpha = \frac{a_x}{|\vec{a}|}; \qquad \cos\beta = \frac{a_y}{|\vec{a}|}; \qquad \cos\gamma = \frac{a_z}{|\vec{a}|}.$$

Then unit vector for vector $\vec{a}$ has component $\vec{a}^0 = (\cos\alpha;\ \cos\beta;\ \cos\gamma)$.

Property of direction cosines is $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Dot product application: we can find the work done by constant force $\vec{F}$ as $W = \vec{F}\cdot\vec{S}$, where $\vec{S}$ is displacement.

Problems

1. Find $\vec{a}\cdot\vec{b}$, if $|\vec{a}| = 3$, $|\vec{b}| = 4$, $(\hat{\vec{a}};\vec{b}) = 60^0$.

2. Find $\vec{a}\cdot\vec{b}$, if $\vec{a}(2;-4;4)$, $\vec{b} = -3\vec{i} + 2\vec{j} - 6\vec{k}$.

3. Find $\overrightarrow{AB}\cdot\overrightarrow{BC}$, if $A(2;-1;3)$, $B(0;-1;2)$, $C(-1;-2;3)$.

4. Find values of m at which vectors $\vec{a}(1;2m+1;-2)$ and $\vec{b}(m;\ 1;2m)$ are perpendicular.

5. Can the vector $\vec{a}$ form angles with coordinate axes:
$\alpha = 45^0; \beta = 60^0; \gamma = 120^0$?

6. Find the direction cosines of the vector $\vec{a}(6;2;3)$.

7. Find the cosine of the angle between vectors $\vec{a}(2;-1;3)$ and $\vec{b}(-1;\ 3;\ 2)$.

8. Find $Proj_{\vec{a}}\vec{b}$, if $\vec{a}(-2;\ 2;\ 1)$, $\vec{b}(-6;\ 6;\ 2)$.

9. Find the work of the force $\vec{F}(2;-1;4)$, if the point of its application moves from $B(3;5;-2)$ to $C(2;1;0)$.

10. Find dot product of the vectors $(\vec{a}-4\vec{b})\cdot(\vec{a}+\vec{b})$, if $|\vec{a}| = 2\sqrt{2}$, $|\vec{b}| = 0{,}5$, $(\hat{\vec{a}};\vec{b}) = 135^0$.

11. The vectors $\vec{a}(p;\ 1;\ 1)$ and $\vec{b}(-1;-p;\ 2)$ are given. Which values of p is vector $\vec{a}-\vec{b}$ perpendicular to $\vec{a}$ at?

12. The points $A(-3;\ 2;\ -1)$ and $B(-1;\ 2;\ 1)$ are given. Find the angle between $\overrightarrow{AB}$ and the positive direction of the axis Ox.

13. The points $A(2;\ 2;\ 4)$, $B(3;\ 1;\ 0)$, $C(1;\ 0;\ 2)$ are the vertices of the triangle. Find the inner corners of this triangle.

14. Find the angle between the diagonals of the parallelogram constructed on the vectors $\vec{a} = 2\vec{i} + \vec{j}$ and $\vec{b} = -2\vec{j} + \vec{k}$.

15. The force $\vec{F}(3;5;-2)$ moves the material point from the point $B(4;2;3)$ to the point C, lying on the axis Oy. ◁ Find coordinates of the point C, when the work of the force is equal to 4.

16. The vector $\vec{a}(-1;9,2)$ and point $A(4;0;-3)$ are given. Find the magnitude of the vector $\overrightarrow{AB}$, which is perpendicular to vector $\vec{a}$, if the point B lies on the axis Oz.

Material for individual work

17. Find dot product $(\bar{a}+2\bar{b})\cdot(2\bar{a}-\bar{b})$, if $|\vec{a}|=2\sqrt{2}$, $|\vec{b}|=3$, $(\hat{\vec{a}};\vec{b})=135^0$.

18. Can vector $\vec{a}$ form angles with coordinate axes:
$\alpha=30^0;\beta=60^0;\gamma=150^0$?

19. Vector $\vec{a}$ forms with the positive direction of the axis Oz angle 120^0. Find component a_z, if $|\vec{a}|=6$.

20. Calculate the angle between vectors $\vec{a}$ and $\vec{b}$, if: $\vec{a}(2;-2;0)$, $\vec{b}(3;0;-3)$.

21. What value of α are vectors $\vec{a}=\alpha\vec{i}+3\vec{j}+4\vec{k}$ and $\vec{b}(-4;\alpha;-7)$ perpendicular at?

22. The vector $\vec{a}(2;-3)$ and the point $A(4;5)$ are given. Find the magnitude of the vector $\overrightarrow{AB}$, if the point B lies on the axis Oy and dot product $\vec{a}\cdot\overrightarrow{AB}=2$.

23. The force $\vec{F}(2;-3;1)$ moves the material point from $B(3;1;-2)$ to the point $C(4;-1;3)$. Find the work of force $\vec{F}$.

24. Find $Proj_{\overrightarrow{AC}}\overrightarrow{AB}$, if $A(1;-2;3)$, $B(-2;4;5)$, $C(-1;0;4)$.

25. What values of m is the angle between vectors $\vec{a}(2;m;-4)$ and $\vec{b}(m;1;1)$ obtuse at?

26. Verify that the quadrangle $ABCD$ is square, if: $A(-3;5;6)$, $B(-1;8;12)$, $C(5;10;9)$, $D(3;7;3)$.

27. What values of x and y is vector $\vec{a}(3;y;-1)$ perpendicular to the vector $\vec{b}=x\vec{i}-5\vec{j}+2\vec{k}$, and the vector $\vec{b}$ is perpendicular to the vector $\vec{c}(1;0,4y;-0,5)$?

28. The vector $\vec{a}(-4;2)$ and point $A(3;5)$ are given. Find dot product $\vec{a}\cdot\overrightarrow{AB}$ if the point B lies on the axis Ox, and vectors $\overrightarrow{AB}$ and $\vec{a}$ are collinear.

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 4 & -5 & -2\\ 2 & -1 & 2\end{vmatrix}=\vec{i}\begin{vmatrix}-5 & -2\\ -1 & 2\end{vmatrix}-\vec{j}\begin{vmatrix}4 & -2\\ 2 & 2\end{vmatrix}+\vec{k}\begin{vmatrix}4 & -5\\ 2 & -1\end{vmatrix}=$$

$=-12\vec{i}-12\vec{j}+6\vec{k}$. Using geometric application of across vector product, we obtain $S=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|=\sqrt{144+144+36}=\sqrt{324}=18$. Then $\overrightarrow{AD}$: $\left|\overrightarrow{AD}\right|=\sqrt{4+1+4}=3$ and the height length of the parallelogram, drawn from the vertex B is $\left|\overrightarrow{BH}\right|=\dfrac{\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|}{\left|\overrightarrow{AD}\right|}=\dfrac{18}{3}=6$. ▷

Problem 2. Find the volume of a triangular pyramid with vertices at the points $A(0;\,0;\,1)$, $B(2;\,3;\,5)$, $C(6;\,2;\,3)$, $D(3;\,7;\,2)$.

◁ Find components of the vectors $\overrightarrow{AB}(2;\,3;\,4)$, $\overrightarrow{AC}(6;\,2;\,2)$, $\overrightarrow{AD}(3;\,7;\,1)$ and calculate scalar triple product of the vectors:

$$\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}=\begin{vmatrix}2 & 3 & 4\\ 6 & 2 & 2\\ 3 & 7 & 1\end{vmatrix}=2\cdot\begin{vmatrix}2 & 2\\ 7 & 1\end{vmatrix}-3\cdot\begin{vmatrix}6 & 2\\ 3 & 1\end{vmatrix}+4\cdot\begin{vmatrix}6 & 2\\ 3 & 7\end{vmatrix}=$$

$=2(2-14)-3(6-6)+4(42-6)=-24-0+144=120$.

Then $V_{nup}=\dfrac{1}{6}\cdot\left|\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right|=\dfrac{1}{6}\cdot 120=20$ (cubic units)▷

Problem 3. The vectors $\vec{a}=(0;\,2;\,1)$, $\vec{b}=(2;\,-3;\,-2)$, $\vec{c}=(-2;\,4;\,3)$ are given. Are these vectors coplanar? In the case of their non-planarity to find triple (right-hand or left-hand) they form, and calculate the volume of parallelepiped formed by these vectors.

◁ Calculate scalar triple product:

$$\vec{a}\ \vec{b}\ \vec{c}=\begin{vmatrix}0 & 2 & 1\\ 2 & -3 & -2\\ -2 & 4 & 3\end{vmatrix}=-2.$$

Since the scalar triple product is non-zero, the vectors are non-planar and the triple is left-hand (since the scalar triple product is negative). Then $V=|-2|=2$ (cubic units)▷

Answers

1. 12. **2.** $-3\vec{i}+3\vec{j}-3\vec{k}$. **3.** $11\vec{i}+21\vec{j}+13\vec{k}$. **4.** -1. **5.** No. **6.** Left-hand.

7. $12\sqrt{3}$. **8.** $9\sqrt{2}$. **9.** $S=\sqrt{24}$, $h=2\sqrt{3}$. **10.** $\left|\overrightarrow{M}\right|=2\sqrt{5}$; $\cos\alpha=\dfrac{2}{\sqrt{5}}$;

19. Equations of two sides of a rectangle $2x-3y+5=0,\ 3x+2y-7=0$ and one of its vertices $A(2;-3)$ are given. Find the equations of the other two sides.

20. Find the intersection point of triangle heights with vertices: $A(-6;\ 2),\ B(2;\ -2),\ C(2;\ 4)$.

21. Find the distance from the point $A(2;-1)$ to the line $3x-4y+5=0$.

22. Shade the half-plane $2x-3y-6>0$ on the coordinate plane .

The solution of typical problems

Problem 1. The vertices of the triangle ΔABC: $A(1;\ 4)$, $B(-2;1)$, $C(2;\ 1)$ are given. Find the standard equation of the line: 1) the side AB; 2) the height BH (pic. 2.6).

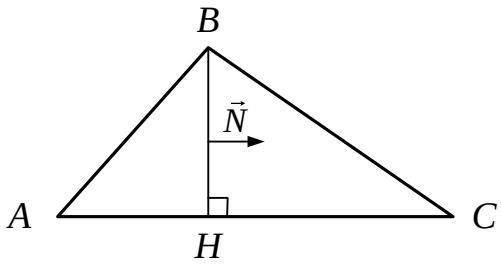

Рис. 2.6

◁ 1. The vector $\vec{a}=\overrightarrow{AB}=(-3;-3)$ is the direction vector of the straight line AB. Since $A\in AB$, then symmetric equation of the line AB has form: $\frac{x-1}{-3}=\frac{y-4}{-3}$. Then $x-1=y-4\Rightarrow x-y+3=0$ is the standard equation of the line AB.

2. The vector $\overrightarrow{AC}(1;-3)$ is the normal vector to the line BH. Then the equation of the line passing through the point $B(-2;\ 1)$ is $BH:\ 1\cdot(x+2)-3(y-1)=0$ or $x-3y+5=0$ is the standard equation of the line BH. ▷

Problem 2. Find slope of the straight line $2x-3y+6=0$.

◁ Express y from this equation. Then: $y=\frac{2}{3}x+2\Rightarrow m=\frac{2}{3}$. ▷

Problem 3. Find the intersection point A of the straight lines $14x-9y-24=0,\ 7x-2y-17=0.$

◁ Coordinates of the intersection point we find from the system of equations: $\begin{cases}7x-2y-17=0,\\14x-9y-24=0.\end{cases}$

Solution of this system is $A(3;\ 2)$. ▷

Problem 4. What values of c are lines $L_1:(c+1)x+(3-c)y+16=0$ и $L_2:(c-3)x+(2c-3)y+6=0$ perpendicular at?