20. $\begin{vmatrix} x & x+1 & 1 \\ -4 & x+1 & 1 \\ 1 & 1 & 1 \end{vmatrix} < 0$. **21.** $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 3 \\ \log_{\frac{1}{2}} x & 1 & -5 \end{vmatrix} \le 0.$

22. Calculate the determinant $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{vmatrix}$.

The solution of typical problems

Problem 1. Calculate the determinant $\Delta = \begin{vmatrix} 3 & 2 & 5 \\ 5 & 12 & 17 \\ 11 & 9 & 19 \end{vmatrix}$, using decomposition by the elements of the first row. ◁

$$\Delta = 3\begin{vmatrix} 12 & 17 \\ 9 & 19 \end{vmatrix} - 2\begin{vmatrix} 5 & 17 \\ 11 & 19 \end{vmatrix} + 5\begin{vmatrix} 5 & 12 \\ 11 & 9 \end{vmatrix} = -26. \triangleright$$

Problem 2. Calculate the determinant, transforming it to a triangular form: $\begin{vmatrix} 1 & -2 & 5 \\ 1 & -1 & 7 \\ 1 & 3 & 3 \end{vmatrix}$.

◁ Let S_i *i*-th row of the determinant. Then, using the 2 determinants property, we obtain a triangular form of the determinant whose value is equal to the product of the main diagonal elements:

$$\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 1 & -1 & 7 \\ 1 & 3 & 3 \end{vmatrix} \begin{matrix} \\ S_2 - S_1 \\ = \\ S_3 - S_1 \end{matrix} \begin{vmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 5 & -2 \end{vmatrix} \begin{matrix} \\ = \\ S_3 - 5S_2 \end{matrix} \begin{vmatrix} 1 & -2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -12 \end{vmatrix} = -12. \triangleright$$

Answers

1. 10. **2.** 1. **3.** $7a$. **4.** 0. **5.** 25. **6.** $10a+3$. **7.** -50. **8.** -45. **9.** -144. **10.** 6. **11.** 4. **12.** $M_{13}=8$, $A_{13}=8$, $M_{32}=23$, $A_{32}=-23$. **13.** -73. **14.** -5. **15.** $\{0;2\}$. **16.** $[-1;1]$. **17.** 3,5. **18.** $\{2;3\}$. **19.** 0,125. **20.** $(-4;0)$. **21.** $[0,5;\ +\infty)$. **22.** 160.

Practical class № 2. Matrix operations. Rank of matrix

◁ The product BA exists, since matrix B has the size 3×2, and matrix A has the size 2×2, that is, the number of columns in the matrix B is the same as the number of rows in the matrix A.

$$BA=\begin{pmatrix}4 & 7\\ 1 & 8\\ 3 & 1\end{pmatrix}\begin{pmatrix}2 & 5\\ 1 & 0\end{pmatrix}=\begin{pmatrix}4\cdot 2+7\cdot 1 & 4\cdot 5+7\cdot 0\\ 1\cdot 2+8\cdot 1 & 1\cdot 5+8\cdot 0\\ 3\cdot 2+1\cdot 1 & 3\cdot 5+1\cdot 0\end{pmatrix}=\begin{pmatrix}15 & 20\\ 10 & 5\\ 7 & 15\end{pmatrix}. \quad ▷$$

Problem 3. Find $f(A)$, if $A=\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$, $f(x)=x^2+2x-3$.

◁ Let $f(A)$ is matrix $f(A)=A^2+2A-3E$, where E is identity matrix of the same order as matrix A.

Then $A^2=A\cdot A=\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$,

$$2A-3E=2\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}-3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}2 & -2\\ 2 & -2\end{pmatrix}-\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}=\begin{pmatrix}-1 & -2\\ 2 & -5\end{pmatrix},$$

$$f(A)=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}+\begin{pmatrix}-1 & -2\\ 2 & -5\end{pmatrix}=\begin{pmatrix}-1 & -2\\ 2 & -5\end{pmatrix}. ▷$$

Problem. 4. Find the rank of the matrix $A=\begin{pmatrix}1 & 2 & 3 & 4\\ 0 & 2 & 1 & 2\\ 0 & 4 & 2 & 4\end{pmatrix}$.

◁ Any minor of the 3rd order of this matrix is zero, since the second and third rows are proportional. Highest order of nonzero minor $\begin{vmatrix}1 & 2\\ 0 & 2\end{vmatrix}\neq 0$ is equal to 2.

Therefore, $rangA=2$. ▷

Problem 5. Find the rank of the matrix $A=\begin{pmatrix}1 & 2 & 3 & 4\\ 5 & 6 & 7 & 8\\ -1 & -2 & -3 & -4\end{pmatrix}$.

This equation is rewritten as $BXA = C$, where $C = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$. Then

$B^{-1}BXAA^{-1} = B^{-1}CA^{-1}$ or $X = B^{-1}CA^{-1}$.

Since $B^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$, and A^{-1} is found in problem 1, then

$$X = \frac{1}{10}\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -1 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -1 & -7 & -8 \\ 3 & 1 & 4 \end{pmatrix} \cdot$$

$$\begin{pmatrix} -2 & -1 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -4 & -6 & 3 \\ -8 & -2 & 11 \end{pmatrix} = \begin{pmatrix} -0{,}4 & -0{,}6 & 0{,}3 \\ -0{,}8 & -0{,}2 & 1{,}1 \end{pmatrix}. \ \triangleright$$

Answers

1. $-3{,}5$. **2.** 4. **3.** $\begin{pmatrix} -2 & 1 \\ -1{,}5 & 0{,}5 \end{pmatrix}$. **4.** $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$. **5.** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. **6.** $\begin{pmatrix} -3 & 2 & 3 \\ 0 & 1 & 2 \\ -4 & 3 & 5 \end{pmatrix}$.

7. $\begin{pmatrix} -1 & -\frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{1}{3} \\ 1 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. **8.** $\begin{pmatrix} -11 & -11 \\ 15 & 15 \end{pmatrix}$. **9.** $\begin{pmatrix} -2 & -3 \\ 3 & 6 \end{pmatrix}$. **10.** $\begin{pmatrix} 1 & -5 & -10 \\ 1{,}5 & -4 & -7{,}5 \end{pmatrix}$.

11. $\begin{pmatrix} 2{,}5 \\ -6 \\ -6{,}5 \end{pmatrix}$. **12.** $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$. **13.** $\begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & \frac{1}{7} \end{pmatrix}$. **14.** $\begin{pmatrix} 0{,}2 & 0 & 0{,}2 \\ 0{,}2 & 0{,}2 & -0{,}2 \\ -0{,}4 & 0{,}2 & 0{,}2 \end{pmatrix}$. **15.** $\begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix}$.

16. $\begin{pmatrix} -2 & 4 \\ -1 & -1 \\ -1 & 6 \end{pmatrix}$. **17.** $-0{,}25$. **18.** $\begin{pmatrix} 0{,}1 & 0{,}3 \\ 0{,}3 & -0{,}1 \end{pmatrix}$. **19.** -8. **20.** $\begin{pmatrix} 13 & 8 \\ 5 & 3 \end{pmatrix}$.

22. $\begin{pmatrix} a & b \\ \frac{1-3a}{2} & \frac{-1-3b}{2} \end{pmatrix}$, $a \in \mathbb{R}$, $b \in \mathbb{R}$.