Solution. By definition, $\mathbf{b}$ is in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$ if there exists scalars x_1 and x_2 such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b},$$

that is, if $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. From our previous discussion on the linear combination problem, we must consider the augmented matrix $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{b}\end{bmatrix}$. Using row reduction, the augmented matrix is consistent and there is only one solution (see Example 3.4). Therefore, yes, $\mathbf{b} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and the linear combination is unique. $\square$

Example 3.9. Is the vector $\mathbf{b} = (1, 0, 1)$ in the span of the vectors $\mathbf{v}_1 = (1, 0, 2), \mathbf{v}_2 = (0, 1, 0), \mathbf{v}_3 = (2, 1, 4)$?

Solution. From Example 3.5, we have that

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b}\end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The last row is inconsistent and therefore $\mathbf{b}$ is not in $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. $\square$

Example 3.10. Is the vector $\mathbf{b} = (8, 8, 12)$ in the span of the vectors $\mathbf{v}_1 = (2, 1, 3), \mathbf{v}_2 = (4, 2, 6), \mathbf{v}_3 = (6, 4, 9)$?

Solution. From Example 3.6, we have that

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b}\end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The system is consistent and therefore $\mathbf{b} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In this case, the solution set contains $d = 1$ free parameters and therefore, it is possible to write $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in infinitely many ways. $\square$

Example 3.11. Answer the following with True or False, and explain your answer.

(a) The vector $\mathbf{b} = (1, 2, 3)$ is in the span of the set of vectors

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -7 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix} \right\}.$$

(b) The solution set of the linear system whose augmented matrix is $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b}\end{bmatrix}$ is the same as the solution set of the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}$.

(c) Suppose that the augmented matrix $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{b}\end{bmatrix}$ has an inconsistent row. Then either $\mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ or $\mathbf{b} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(d) The span of the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ (at least one of which is nonzero) contains only the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and the zero vector $\mathbf{0}$.

After this lecture you should know the following:

- what a vector is
- what a linear combination of vectors is
- what the linear combination problem is
- the relationship between the linear combination problem and the problem of solving linear systems of equations
- how to solve the linear combination problem
- what the span of a set of vectors is
- the relationship between what it means for a vector $\mathbf{b}$ to be in the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ and the problem of writing $\mathbf{b}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$
- the geometric interpretation of the span of a set of vectors

(a)

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 3 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 2 \\ -4 \\ -3 \\ 8 \end{bmatrix}$$

(b)

$$\mathbf{A} = \begin{bmatrix} 3 & 3 & -2 \\ 4 & -4 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(c)

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 4 & 1 & -2 \\ 3 & -3 & 3 \\ 0 & -2 & -3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

Solution. We compute:

(a)

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ -3 \\ 8 \end{bmatrix}$$
$$= \begin{bmatrix} (1)(2) + (-1)(-4) + (3)(-3) + (0)(8) \end{bmatrix} = \begin{bmatrix} -3 \end{bmatrix}$$

(b)

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 3 & 3 & -2 \\ 4 & -4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} (3)(1) + (3)(0) + (-2)(-1) \\ (4)(1) + (-4)(0) + (-1)(-1) \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Solution. After row reducing we obtain

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 & 1 & 4 \\ 3 & 7 & 7 & 3 & 13 \\ 2 & 5 & 5 & 2 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here $n = 5$, and $r = 2$, and therefore the number of parameters in the solution set is $d = n - r = 3$. The second row of $\texttt{rref}(\mathbf{A})$ gives the equation

$$x_2 + x_3 + x_5 = 0.$$

Setting $x_5 = t_1$ and $x_3 = t_2$ as free parameters we obtain that

$$x_2 = -x_3 - x_5 = -t_2 - t_1.$$

From the first row we obtain the equation

$$x_1 + x_4 + 2x_5 = 0$$

The unknown x_5 has already been assigned, so we must now choose either x_1 or x_4 to be a parameter. Choosing $x_4 = t_3$ we obtain that

$$x_1 = -x_4 - 2x_5 = -t_3 - 2t_1$$

In summary, the general solution can be written as

$$\mathbf{x} = \begin{bmatrix} -t_3 - 2t_1 \\ -t_2 - t_1 \\ t_2 \\ t_3 \\ t_1 \end{bmatrix} = t_1 \underbrace{\begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_1} + t_2 \underbrace{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{v}_2} + t_3 \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_3} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3$$

where t_1, t_2, t_3 are arbitrary parameters. In other words, any solution $\mathbf{x}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$\mathbf{x} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

□

The form of the general solution in Example 5.3 holds in general and is summarized in the following theorem.

Theorem 5.4: Consider the homogenous linear system $\mathbf{A}\mathbf{x} = \mathbf{0}$, where $\mathbf{A} \in M_{m\times n}$ and $\mathbf{0} \in \mathbb{R}^m$. Let r be the rank of $\mathbf{A}$.

1. If $r = n$ then the only solution to the system is the trivial solution $\mathbf{x} = \mathbf{0}$.

2. Otherwise, if $r < n$ and we set $d = n - r$, then there exist vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$ such that any solution $\mathbf{x}$ of the linear system can be written as

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_d.$$