spans the solution set of the system $\mathbf{Ax} = \mathbf{0}$. Choosing for instance $t = 2$ we obtain the solution

$$\mathbf{x} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}.$$

Therefore,

$$4\mathbf{v}_1 - 2\mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{0}$$

is a non-trivial linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ that gives the zero vector $\mathbf{0}$. And, for instance,

$$\mathbf{v}_3 = -2\mathbf{v}_1 + \mathbf{v}_2$$

that is, $\mathbf{v}_3 \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. □

Below we record some simple observations on the linear independence of simple sets:

- A set consisting of a single non-zero vector $\{\mathbf{v}_1\}$ is linearly independent. Indeed, if $\mathbf{v}_1$ is non-zero then

  $$t\mathbf{v}_1 = \mathbf{0}$$

  is true if and only if $t = 0$.

- A set consisting of two non-zero vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent if and only if neither of the vectors is a multiple of the other. For example, if $\mathbf{v}_2 = t\mathbf{v}_1$ then

  $$t\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$$

  is a non-trivial linear combination of $\mathbf{v}_1, \mathbf{v}_2$ giving the zero vector $\mathbf{0}$.

- Any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ containing the zero vector, say that $\mathbf{v}_p = \mathbf{0}$, is linearly dependent. For example, the linear combination

  $$0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_{p-1} + 2\mathbf{v}_p = \mathbf{0}$$

  is a non-trivial linear combination giving the zero vector $\mathbf{0}$.

6.2 The maximum size of a linearly independent set

The next theorem puts a constraint on the maximum size of a linearly independent set in $\mathbb{R}^n$.

Theorem 6.7: Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set of vectors in $\mathbb{R}^n$. If $p > n$ then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly dependent. Equivalently, if the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ are linearly independent then $p \leq n$.

Lecture 7

Introduction to Linear Mappings

7.1 Vector mappings

By a **vector mapping** we mean simply a function

$$\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^m.$$

The **domain** of T is $\mathbb{R}^n$ and the **co-domain** of T is $\mathbb{R}^m$. The case $n = m$ is allowed of course. In engineering or physics, the domain is sometimes called the **input space** and the co-domain is called the **output space**. Using this terminology, the points $\mathbf{x}$ in the domain are called the **inputs** and the points $\mathsf{T}(\mathbf{x})$ produced by the mapping are called the **outputs**.

Definition 7.1: The vector $\mathbf{b} \in \mathbb{R}^m$ is in the **range** of T, or in the **image** of T, if there exists some $\mathbf{x} \in \mathbb{R}^n$ such that $\mathsf{T}(\mathbf{x}) = \mathbf{b}$.

In other words, $\mathbf{b}$ is in the range of T if there is an input $\mathbf{x}$ in the domain of T that outputs $\mathbf{b} = \mathsf{T}(\mathbf{x})$. In general, not every point in the co-domain of T is in the range of T. For example, consider the vector mapping $\mathsf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$\mathsf{T}(\mathbf{x}) = \begin{bmatrix} x_1^2 \sin(x_2) - \cos(x_1^2 - 1) \\ x_1^2 + x_2^2 + 1 \end{bmatrix}.$$

The vector $\mathbf{b} = (3, -1)$ is not in the range of T because the second component of $\mathsf{T}(\mathbf{x})$ is positive. On the other hand, $\mathbf{b} = (-1, 2)$ is in the range of T because

$$\mathsf{T}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1^2 \sin(0) - \cos(1^2 - 1) \\ 1^2 + 0^2 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \mathbf{b}.$$

Hence, a corresponding input for this particular $\mathbf{b}$ is $\mathbf{x} = (1, 0)$. In Figure 7.1 we illustrate the general setup of how the domain, co-domain, and range of a mapping are related. A crucial idea is that the range of T may not equal the co-domain.

If we scale $\mathbf{v}$ by any $c > 0$ then performing the same computation as above we obtain that $\mathsf{T}_\theta(c\mathbf{v}) = c\mathsf{T}(\mathbf{v})$. Therefore, T_θ is a matrix mapping with corresponding matrix

$$\mathbf{A} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Thus, T_θ is a linear mapping. □

Example 7.11. (Projections) Let $\mathsf{T} : \mathbb{R}^3 \to \mathbb{R}^2$ be the vector mapping

$$\mathsf{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}.$$

Show that T is a linear mapping and describe the range of T.

Solution. First notice that

$$\mathsf{T}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thus, T is a matrix mapping corresponding to the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, T is a linear mapping. Geometrically, T takes the vector $\mathbf{x}$ and projects it to the (x_1, x_2) plane, see Figure 7.2. What is the range of T? The range of T consists of all vectors in $\mathbb{R}^3$ of the form

$$\mathbf{b} = \begin{bmatrix} t \\ s \\ 0 \end{bmatrix}$$

where the numbers t and s are arbitrary. For each $\mathbf{b}$ in the range of T, there are infinitely many $\mathbf{x}$'s such that $\mathsf{T}(\mathbf{x}) = \mathbf{b}$. □

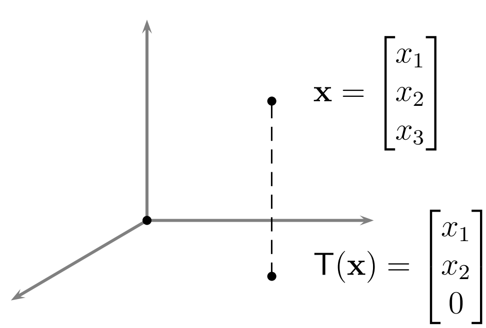

Figure 7.2: Projection onto the (x_1, x_2) plane

Theorem 8.11: Let $\mathsf{T}_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix mapping, where $\mathbf{A} = [\mathbf{v}_1 \; \mathbf{v}_2 \; \cdots \; \mathbf{v}_n] \in M_{m\times n}$. The following statements are equivalent:

1. $\mathsf{T}_{\mathbf{A}}$ is one-to-one.
2. The rank of $\mathbf{A}$ is $r = \text{rank}(\mathbf{A}) = n$.
3. The columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

Example 8.12. Let $\mathsf{T}_{\mathbf{A}} : \mathbb{R}^4 \to \mathbb{R}^3$ be the matrix mapping with matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -2 & 6 & 4 \\ -1 & 0 & -2 & -1 \\ 2 & -2 & 0 & 2 \end{bmatrix}.$$

Is $\mathsf{T}_{\mathbf{A}}$ one-to-one?

Solution. By Theorem 8.11, $\mathsf{T}_{\mathbf{A}}$ is one-to-one if and only if the columns of $\mathbf{A}$ are linearly independent. The columns of $\mathbf{A}$ lie in $\mathbb{R}^3$ and there are $n = 4$ columns. From Lecture 6, we know then that the columns are not linearly independent. Therefore, $\mathsf{T}_{\mathbf{A}}$ is not one-to-one. Alternatively, $\mathbf{A}$ will have rank at most $r = 3$ (why?). Therefore, the solution set to $\mathbf{Ax} = \mathbf{0}$ will have at least one parameter, and thus there exists infinitely many solutions to $\mathbf{Ax} = \mathbf{0}$. Intuitively, because $\mathbb{R}^4$ is "larger" than $\mathbb{R}^3$, the linear mapping $\mathsf{T}_{\mathbf{A}}$ will have to project $\mathbb{R}^4$ onto $\mathbb{R}^3$ and thus infinitely many vectors in $\mathbb{R}^4$ will be mapped to the same vector in $\mathbb{R}^3$. $\square$

Example 8.13. Let $\mathsf{T}_{\mathbf{A}} : \mathbb{R}^2 \to \mathbb{R}^3$ be the matrix mapping with matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 2 & 0 \end{bmatrix}$$

Is $\mathsf{T}_{\mathbf{A}}$ one-to-one?

Solution. By inspection, we see that the columns of $\mathbf{A}$ are linearly independent. Therefore, $\mathsf{T}_{\mathbf{A}}$ is one-to-one. Alternatively, one can compute that

$$\texttt{rref}(\mathbf{A}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Therefore, $r = \text{rank}(\mathbf{A}) = 2$, which is equal to the number columns of $\mathbf{A}$. $\square$