Example 9.3. Given $\mathbf{A}$ and $\mathbf{B}$ below, find $3\mathbf{A} - 2\mathbf{B}$.

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & -3 & 9 \\ 4 & -6 & 7 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 & 0 & -11 \\ 3 & -5 & 1 \\ -1 & -9 & 0 \end{bmatrix}$$

Solution. We compute:

$$3\mathbf{A} - 2\mathbf{B} = \begin{bmatrix} 3 & -6 & 15 \\ 0 & -9 & 27 \\ 12 & -18 & 21 \end{bmatrix} - \begin{bmatrix} 10 & 0 & -22 \\ 6 & -10 & 2 \\ -2 & -18 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -6 & 37 \\ -6 & 1 & 25 \\ 14 & 0 & 21 \end{bmatrix}$$

□

Below are some basic algebraic properties of matrix addition/scalar multiplication.

Theorem 9.4: Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be matrices of the same size and let α, β be scalars. Then

(a) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$

(b) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

(c) $\mathbf{A} + \mathbf{0} = \mathbf{A}$

(d) $\alpha(\mathbf{A} + \mathbf{B}) = \alpha\mathbf{A} + \alpha\mathbf{B}$

(e) $(\alpha + \beta)\mathbf{A} = \alpha\mathbf{A} + \beta\mathbf{A}$

(f) $\alpha(\beta\mathbf{A}) = (\alpha\beta)\mathbf{A}$

9.2 Matrix Multiplication

Let $\mathsf{T}_{\mathbf{B}} : \mathbb{R}^p \to \mathbb{R}^n$ and let $\mathsf{T}_{\mathbf{A}} : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings. If $\mathbf{x} \in \mathbb{R}^p$ then $\mathsf{T}_{\mathbf{B}}(\mathbf{x}) \in \mathbb{R}^n$ and thus we can apply $\mathsf{T}_{\mathbf{A}}$ to $\mathsf{T}_{\mathbf{B}}(\mathbf{x})$. The resulting vector $\mathsf{T}_{\mathbf{A}}(\mathsf{T}_{\mathbf{B}}(\mathbf{x}))$ is in $\mathbb{R}^m$. Hence, each $\mathbf{x} \in \mathbb{R}^p$ can be mapped to a point in $\mathbb{R}^m$, and because $\mathsf{T}_{\mathbf{B}}$ and $\mathsf{T}_{\mathbf{A}}$ are linear mappings the resulting mapping is also linear. This resulting mapping is called the **composition** of $\mathsf{T}_{\mathbf{A}}$ and $\mathsf{T}_{\mathbf{B}}$, and is usually denoted by $\mathsf{T}_{\mathbf{A}} \circ \mathsf{T}_{\mathbf{B}} : \mathbb{R}^p \to \mathbb{R}^m$ (see Figure 9.1). Hence,

$$(\mathsf{T}_{\mathbf{A}} \circ \mathsf{T}_{\mathbf{B}})(\mathbf{x}) = \mathsf{T}_{\mathbf{A}}(\mathsf{T}_{\mathbf{B}}(\mathbf{x})).$$

Because $(\mathsf{T}_{\mathbf{A}} \circ \mathsf{T}_{\mathbf{B}}) : \mathbb{R}^p \to \mathbb{R}^m$ is a linear mapping it has an associated standard matrix, which we denote for now by $\mathbf{C}$. From Lecture 8, to compute the standard matrix of any linear mapping, we must compute the images of the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_p$ under the linear mapping. Now, for any $\mathbf{x} \in \mathbb{R}^p$,

$$\mathsf{T}_{\mathbf{A}}(\mathsf{T}_{\mathbf{B}}(\mathbf{x})) = \mathsf{T}_{\mathbf{A}}(\mathbf{B}\mathbf{x}) = \mathbf{A}(\mathbf{B}\mathbf{x}).$$

Applying this to $\mathbf{x} = \mathbf{e}_i$ for all $i = 1, 2, \ldots, p$, we obtain the standard matrix of $\mathsf{T}_{\mathbf{A}} \circ \mathsf{T}_{\mathbf{B}}$:

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}(\mathbf{B}\mathbf{e}_1) & \mathbf{A}(\mathbf{B}\mathbf{e}_2) & \cdots & \mathbf{A}(\mathbf{B}\mathbf{e}_p) \end{bmatrix}.$$

Solution. First $\mathbf{AB} = [\mathbf{Ab}_1 \;\; \mathbf{Ab}_2 \;\; \mathbf{Ab}_3 \;\; \mathbf{Ab}_4]$:

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -3 \end{bmatrix} \begin{bmatrix} -4 & 2 & 4 & -4 \\ -1 & -5 & -3 & 3 \\ -4 & -4 & -3 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & & & \\ 7 & & & \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & & \\ 7 & 9 & & \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & 4 & \\ 7 & 9 & 10 & \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & 4 & 4 \\ 7 & 9 & 10 & 2 \end{bmatrix}
\end{aligned}$$

On the other hand, $\mathbf{BA}$ is not defined! $\mathbf{B}$ has 4 columns and $\mathbf{A}$ has 2 rows. $\square$

Example 9.7. For $\mathbf{A}$ and $\mathbf{B}$ below compute $\mathbf{AB}$ and $\mathbf{BA}$.

$$\mathbf{A} = \begin{bmatrix} -4 & 4 & 3 \\ 3 & -3 & -1 \\ -2 & -1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & -1 & 0 \\ -3 & 0 & -2 \\ -2 & 1 & -2 \end{bmatrix}$$

Solution. First $\mathbf{AB} = [\mathbf{Ab}_1 \;\; \mathbf{Ab}_2 \;\; \mathbf{Ab}_3]$:

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} -4 & 4 & 3 \\ 3 & -3 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ -3 & 0 & -2 \\ -2 & 1 & -2 \end{bmatrix} \\
&= \begin{bmatrix} -14 & & \\ 8 & & \\ 3 & & \end{bmatrix} \\
&= \begin{bmatrix} -14 & 7 & \\ 8 & -4 & \\ 3 & 3 & \end{bmatrix} \\
&= \begin{bmatrix} -14 & 7 & -14 \\ 8 & -4 & 8 \\ 3 & 3 & 0 \end{bmatrix}
\end{aligned}$$

Contracting $\mathbf{e}_2$ by a factor of $k = 3$ results in $(0, \frac{1}{3})$ and then rotation by θ results in

$$\begin{bmatrix} -\frac{1}{3}\sin(\theta) \\ \frac{1}{3}\cos(\theta) \end{bmatrix} = \mathsf{T}(\mathbf{e}_2).$$

Therefore,

$$\mathbf{A} = \begin{bmatrix} \mathsf{T}(\mathbf{e}_1) & \mathsf{T}(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\cos(\theta) & -\frac{1}{3}\sin(\theta) \\ \frac{1}{3}\sin(\theta) & \frac{1}{3}\cos(\theta) \end{bmatrix}$$

On the other hand, the standard matrix corresponding to a contraction by a factor $k = \frac{1}{3}$ is

$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Therefore,

$$\underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\text{rotation}} \underbrace{\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}}_{\text{contraction}} = \begin{bmatrix} \frac{1}{3}\cos(\theta) & -\frac{1}{3}\sin(\theta) \\ \frac{1}{3}\sin(\theta) & \frac{1}{3}\cos(\theta) \end{bmatrix} = \mathbf{A}$$

$\square$

After this lecture you should know the following:

- know how to add and multiply matrices
- that matrix multiplication corresponds to composition of linear mappings
- the algebraic properties of matrix multiplication (Theorem 9.8)
- how to compute the transpose of a matrix
- the properties of matrix transposition (Theorem 9.12)

Example 11.2. Compute the determinant of $\mathbf{A}$.

(i) $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 8 & 2 \end{bmatrix}$

(ii) $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$

(iii) $\mathbf{A} = \begin{bmatrix} -110 & 0 \\ 568 & 0 \end{bmatrix}$

Solution. For (i):

$$\det(\mathbf{A}) = \begin{vmatrix} 3 & -1 \\ 8 & 2 \end{vmatrix} = (3)(2) - (8)(-1) = 14$$

For (ii):

$$\det(\mathbf{A}) = \begin{vmatrix} 3 & 1 \\ -6 & -2 \end{vmatrix} = (3)(-2) - (-6)(1) = 0$$

For (iii):

$$\det(\mathbf{A}) = \begin{vmatrix} -110 & 0 \\ 568 & 0 \end{vmatrix} = (-110)(0) - (568)(0) = 0$$

□

As in the 2×2 case, the solution of a 3×3 linear system $\mathbf{Ax} = \mathbf{b}$ can be shown to be

$$x_1 = \frac{\text{Numerator}_1}{D}, \quad x_2 = \frac{\text{Numerator}_2}{D}, \quad x_3 = \frac{\text{Numerator}_3}{D}$$

where

$$D = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Notice that the terms of D in the parenthesis are determinants of 2×2 submatrices of $\mathbf{A}$:

$$D = a_{11}(\underbrace{a_{22}a_{33} - a_{23}a_{32}}_{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}) - a_{12}(\underbrace{a_{21}a_{33} - a_{23}a_{31}}_{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}) + a_{13}(\underbrace{a_{21}a_{32} - a_{22}a_{31}}_{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}).$$

Let

$$\mathbf{A}_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{A}_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \quad \text{and } \mathbf{A}_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

Then we can write

$$D = a_{11}\det(\mathbf{A}_{11}) - a_{12}\det(\mathbf{A}_{12}) + a_{13}\det(\mathbf{A}_{13}).$$

The matrix $\mathbf{A}_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ is obtained from $\mathbf{A}$ by deleting the 1st row and the 1st column:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} \\ a_{31} & \mathbf{a_{32}} & \mathbf{a_{33}} \end{bmatrix} \longrightarrow \mathbf{A}_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}.$$