since then rows k and j are equal. The jth row of $\mathbf{B}$ is $\mathbf{b}_j = \mathbf{a}_j + \beta\mathbf{a}_k$. Therefore, expanding $\det\mathbf{B}$ along the jth row:

$$\begin{aligned}\det\mathbf{B} &= (\mathbf{a}_j + \beta\mathbf{a}_k)\cdot\mathbf{c}_j^T \\ &= \mathbf{a}_j\cdot\mathbf{c}_j^T + \beta\left(\mathbf{a}_k\cdot\mathbf{c}_j^T\right) \\ &= \det\mathbf{A}.\end{aligned}$$

$\square$

Example 12.5. Suppose that $\mathbf{A}$ is a 4×4 matrix and suppose that $\det\mathbf{A} = 11$. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by interchanging rows 2 and 4, what is $\det\mathbf{B}$?

Solution. Interchanging (or swapping) rows changes the sign of the determinant. Therefore,

$$\det\mathbf{B} = -11.$$

$\square$

Example 12.6. Suppose that $\mathbf{A}$ is a 4×4 matrix and suppose that $\det\mathbf{A} = 11$. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ denote the rows of $\mathbf{A}$. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by replacing row $\mathbf{a}_3$ by $3\mathbf{a}_1 + \mathbf{a}_3$, what is $\det\mathbf{B}$?

Solution. This is a Type 3 elementary row operation which preserves the value of the determinant. Therefore,

$$\det\mathbf{B} = 11.$$

$\square$

Example 12.7. Suppose that $\mathbf{A}$ is a 4×4 matrix and suppose that $\det\mathbf{A} = 11$. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ denote the rows of $\mathbf{A}$. If $\mathbf{B}$ is obtained from $\mathbf{A}$ by replacing row $\mathbf{a}_3$ by $3\mathbf{a}_1 + 7\mathbf{a}_3$, what is $\det\mathbf{B}$?

Solution. This is not quite a Type 3 elementary row operation because $\mathbf{a}_3$ is multiplied by 7. The third row of $\mathbf{B}$ is $\mathbf{b}_3 = 3\mathbf{a}_1 + 7\mathbf{a}_3$. Therefore, expanding $\det\mathbf{B}$ along the third row

$$\begin{aligned}\det\mathbf{B} &= (3\mathbf{a}_1 + 7\mathbf{a}_3)\cdot\mathbf{c}_3^T \\ &= 3\mathbf{a}_1\cdot\mathbf{c}_3^T + 7\mathbf{a}_3\cdot\mathbf{c}_3^T \\ &= 7(\mathbf{a}_3\cdot\mathbf{c}_3^T) \\ &= 7\det\mathbf{A} \\ &= 77\end{aligned}$$

$\square$

(6) The scalar multiple of $\mathbf{v}$ by α, denoted $\alpha\mathbf{v}$, is in V. (closure under scalar multiplication)

(7) $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$

(8) $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$

(9) $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$

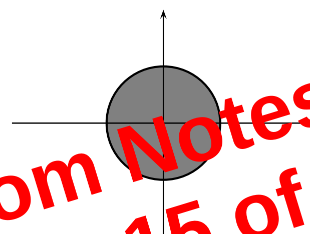
(10) $1\mathbf{v} = \mathbf{v}$

It can be shown that $0 \cdot \mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v}$ in V. To better understand the definition of a vector space, we first consider a few elementary examples.

Example 14.2. Let V be the unit disc in $\mathbb{R}^2$:

$$\mathsf{V} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

Is V a vector space?

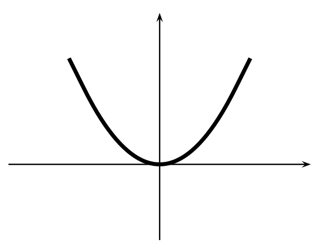


Solution. The circle is not closed under scalar multiplication. For example, take $\mathbf{u} = (1, 0) \in \mathsf{V}$ and multiply by say $\alpha = 2$. Then $\alpha\mathbf{u} = (2, 0)$ is not in V. Therefore, property (6) of the definition of a vector space fails, and consequently the unit disc is not a vector space. $\square$

Example 14.3. Let V be the graph of the quadratic function $f(x) = x^2$:

$$\mathsf{V} = \left\{(x, y) \in \mathbb{R}^2 \mid y = x^2\right\}.$$

Is V a vector space?



Solution. The set V is not closed under scalar multiplication. For example, $\mathbf{u} = (1, 1)$ is a point in V but $2\mathbf{u} = (2, 2)$ is not. You may also notice that V is not closed under addition either. For example, both $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (2, 4)$ are in V but $\mathbf{u} + \mathbf{v} = (3, 5)$ and $(3, 5)$ is not a point on the parabola V. Therefore, the graph of $f(x) = x^2$ is not a vector space. $\square$

Example 14.15. A square matrix $\mathbf{A}$ is said to be **symmetric** if $\mathbf{A}^T = \mathbf{A}$. For example, here is a 3×3 symmetric matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 5 \\ -3 & 5 & 7 \end{bmatrix}$$

Verify for yourself that we do indeed have that $\mathbf{A}^T = \mathbf{A}$. Let W be the set of all symmetric $n \times n$ matrices. Is W a subspace of $\mathsf{V} = M_{n\times n}$?

Example 14.16. For any vector space V, there are two trivial subspaces in V, namely, V itself is a subspace of V and the set consisting of the zero vector $\mathsf{W} = \{\mathbf{0}\}$ is a subspace of V.

There is one particular way to generate a subspace of any given vector space V using the span of a set of vectors. Recall that we defined the span of a set of vectors in $\mathbb{R}^n$ but we can define the same notion on a general vector space V.

Definition 14.17: Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors in V. The **span** of $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = \Big\{t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_p\mathbf{v}_p \;|\; t_1, t_2, \ldots, t_p \in \mathbb{R}\Big\}.$$

We now show that the span of a set of vectors in V is a subspace of V.

Theorem 14.18: If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are vectors in V then $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

Solution. Let $\mathbf{u} = t_1\mathbf{v}_1 + \cdots + t_p\mathbf{v}_p$ and $\mathbf{w} = s_1\mathbf{v}_1 + \cdots + s_p\mathbf{v}_p$ be two vectors in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$. Then

$$\mathbf{u} + \mathbf{w} = (t_1\mathbf{v}_1 + \cdots + t_p\mathbf{v}_p) + (s_1\mathbf{v}_1 + \cdots + s_p\mathbf{v}_p) = (t_1 + s_1)\mathbf{v}_1 + \cdots + (t_p + s_p)\mathbf{v}_p.$$

Therefore $\mathbf{u} + \mathbf{w}$ is also in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_p$. Now consider $\alpha\mathbf{u}$:

$$\alpha\mathbf{u} = \alpha(t_1\mathbf{v}_1 + \cdots + t_p\mathbf{v}_p) = (\alpha t_1)\mathbf{v}_1 + \cdots + (\alpha t_p)\mathbf{v}_p.$$

Therefore, $\alpha\mathbf{u}$ is in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_p$. Lastly, since $0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p = \mathbf{0}$ then the zero vector $\mathbf{0}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$. Therefore, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subspace of V. $\square$

Given a general subspace W of V, if $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p$ are vectors in W such that

$$\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\} = \mathsf{W}$$

then we say that $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p\}$ is a **spanning set** of W. Hence, every vector in W can be written as a linear combination of the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_p$.

After this lecture you should know the following:

Lecture 15

Linear Maps

Before we begin this Lecture, we review subspaces. Recall that W is a **subspace** of a vector space V if W is a subset of V and

1. the zero vector $\mathbf{0}$ in V is also in W,
2. for any vectors $\mathbf{u}, \mathbf{v}$ in W the sum $\mathbf{u} + \mathbf{v}$ is also in W, and
3. for any vector $\mathbf{u}$ in W and any scalar α the vector $\alpha\mathbf{u}$ is also in W.

In the previous lecture we gave several examples of subspaces. For example, we showed that a line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$ and we gave examples of subspaces of $\mathbb{P}_n[t]$ and $M_{n\times m}$. We also showed that if $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in a vector space V then

$$\mathsf{W} = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$$

is a subspace of V.

15.1 Linear Maps on Vector Spaces

In Lecture 7, we defined what it meant for a vector mapping $\mathsf{T} : \mathbb{R}^n \to \mathbb{R}^m$ to be a **linear mapping**. We now want to introduce linear mappings on general vector spaces; you will notice that the definition is essentially the same but the key point to remember is that the underlying spaces are not $\mathbb{R}^n$ but a general vector space.

Definition 15.1: Let $\mathsf{T} : \mathsf{V} \to \mathsf{U}$ be a mapping of vector spaces. Then T is called a **linear mapping** if

- for any $\mathbf{u}, \mathbf{v}$ in V it holds that $\mathsf{T}(\mathbf{u} + \mathbf{v}) = \mathsf{T}(\mathbf{u}) + \mathsf{T}(\mathbf{v})$, and
- for any scalar α and $\mathbf{u}$ in V is holds that $\mathsf{T}(\alpha\mathbf{v}) = \alpha\mathsf{T}(\mathbf{v})$.

Example 15.2. Let $\mathsf{V} = M_{n\times n}$ be the vector space of $n \times n$ matrices and let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be the mapping

$$\mathsf{T}(\mathbf{A}) = \mathbf{A} + \mathbf{A}^T.$$