Example 15.11. Let $V = \mathbb{R}^4$ and consider the following subset of V:

$$\mathsf{W} = \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 - 3x_2 + x_3 - 7x_4 = 0 \}$$

Is W a subspace of V?

Solution. The set W is the null space of the matrix 1×4 matrix A given by

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 & -7 \end{bmatrix}.$$

Hence, $W = \text{Null}(\mathbf{A})$ and consequently W is a subspace.

From our previous remarks, the null space of a matrix $\mathbf{A} \in M_{m \times n}$ is just the solution set of the homogeneous system Ax = 0. Therefore, one way to explicitly describe the null space of A is to solve the system Ax = 0 and write the general solution in parametric vector form. From our previous work on solving linear systems, if the rref(A) has r leading 1's then the number of parameters in the solution set is d = n - r. Therefore, after performing back substitution, we will obtain vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$ such that the general solution in parametric $\mathbf{x} = t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \dots + t_d \mathbf{v}_d$ **e.co.uk** *v* numbers. Therefore, **e.sale**.co.uk vector form can be written as

where t_1, t_2, \ldots, t_d are arbitrary numbers.

Hence, the vectors
$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$$
 for \mathbf{C} manning set for Null(\mathbf{A}).

Example 15.12. Find a spanning set for the null space of the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution. The null space of A is the solution set of the homogeneous system Ax = 0. Performing elementary row operations one obtains

$$\mathbf{A} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $r = \operatorname{rank}(\mathbf{A})$ and since n = 5 we will have d = 3 vectors in a spanning set for Null(A). Letting $x_5 = t_1$, and $x_4 = t_2$, then from the 2nd row we obtain

$$x_3 = -2t_2 + 2t_1$$

Letting $x_2 = t_3$, then from the 1st row we obtain

$$x_1 = 2t_3 + t_2 - 3t_1.$$

Therefore, the only solution to $\mathbf{A}\mathbf{x} = \mathbf{0}$ is the trivial solution. Therefore, \mathcal{B} is linearly independent. Moreover, for any $\mathbf{b} \in \mathbb{R}^3$, the augmented matrix $\begin{bmatrix} \mathbf{A} & \mathbf{b} \end{bmatrix}$ is consistent. Therefore, the columns of \mathbf{A} span all of \mathbb{R}^3 :

$$\operatorname{Col}(\mathbf{A}) = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3.$$

Therefore, \mathcal{B} is a basis for \mathbb{R}^3 .

Hence, rank(A) not a basil of

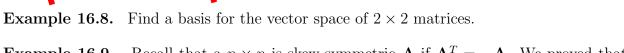
Example 16.7. In $V = \mathbb{R}^4$, consider the vectors

$$\mathbf{v}_{1} = \begin{bmatrix} 1\\3\\0\\-2 \end{bmatrix}, \ \mathbf{v}_{2} = \begin{bmatrix} 2\\-1\\-2\\1 \end{bmatrix}, \ \mathbf{v}_{3} = \begin{bmatrix} -1\\4\\2\\-3 \end{bmatrix}$$

Let $W = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Is $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for W?

frOf thus B is line

Solution. By definition, \mathcal{B} is a spanning set for W, so we need only determine if \mathcal{B} is linearly independent. Form the matrix, $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ and row reduce to obtain $\mathbf{A} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} \\ \mathbf{b} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} \\ \mathbf{b} \\ \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \\ \mathbf{c} \\ \mathbf{c} \end{bmatrix}$



dependent. Notice $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3$. Therefore, \mathcal{B} is

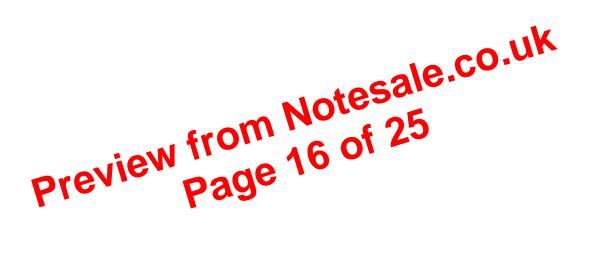
Example 16.9. Recall that a $n \times n$ is skew-symmetric **A** if $\mathbf{A}^T = -\mathbf{A}$. We proved that the set of $n \times n$ matrices is a subspace. Find a basis for the set of 3×3 skew-symmetric matrices.

16.3 Dimension of a Vector Space

The following theorem will lead to the definition of the dimension of a vector space.

Theorem 16.10: Let V be a vector space. Then all bases of V have the same number of vectors.

Proof: We will prove the theorem for the case that $V = \mathbb{R}^n$. We already know that the standard unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis of \mathbb{R}^n . Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p\}$ be nonzero vectors in \mathbb{R}^n and suppose first that p > n. In Lecture 6, Theorem 6.7, we proved that any set of vectors in \mathbb{R}^n containing more than n vectors is automatically linearly dependent. The reason is that the RREF of $\mathbf{A} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix}$ will contain at most r = n leading ones,



On the other hand, the inverse matrix \mathbf{P}^{-1} maps standard coordinates in \mathbb{R}^3 to \mathcal{B} -coordinates. One can verify that

$$\mathbf{P}^{-1} = \begin{bmatrix} 4 & 3 & 6 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Therefore, the \mathcal{B} coordinates of \mathbf{v} are

$$[\mathbf{v}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{v} = \begin{bmatrix} 4 & 3 & 6 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

When V is an abstract vector space, e.g. $\mathbb{P}_n[t]$ or $M_{n \times n}$, the notion of a coordinate mapping is similar as the case when $V = \mathbb{R}^n$. If V is an *n*-dimensional vector space and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V, we define the coordinate mapping $\mathcal{P} : \mathsf{V} \to \mathbb{R}^n$ relative

$$\mathcal{P}(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}.$$

 $\mathcal{P}(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}.$ Example 18.8. Let $\mathsf{V} = M_{2\times 2}$ and let $\mathcal{B} = \{\mathbf{A}, \mathbf{C}, \mathbf{S}, \mathbf{A}_4\}$ be the standard basis for $M_{2\times 2}.$ What is $\mathcal{P}: M_{2\times 2} \to \mathbb{R}^4$?
Solution. Recall, $\mathbf{P}(\mathbf{v}) = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{C}\} = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{C}\} = \{\mathbf{A}_1, \mathbf{A}_$ Then for any $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ we have $\mathcal{P}\left(\begin{bmatrix}a_{11} & a_{12}\\a_{21} & a_{22}\end{bmatrix}\right) = \begin{vmatrix}a_{11}\\a_{12}\\a_{21}\end{vmatrix}.$

18.3 Matrix Representation of a Linear Map

Let V and W be vector spaces and let $T: V \to W$ be a linear mapping. Then by definition of a linear mapping, $T(\mathbf{v} + \mathbf{u}) = T(\mathbf{v}) + T(\mathbf{u})$ and $T(\alpha \mathbf{v}) = \alpha T(\mathbf{v})$ for every $\mathbf{v}, \mathbf{u} \in V$ and $\alpha \in \mathbb{R}$. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be a basis of V and let $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\}$ be a basis of W. Then for any $\mathbf{v} \in V$ there exists scalars c_1, c_2, \ldots, c_n such that

$$\mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n$$