Example 15.11. Let $\mathsf{V} = \mathbb{R}^4$ and consider the following subset of V:

$$\mathsf{W} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 2x_1 - 3x_2 + x_3 - 7x_4 = 0\}.$$

Is W a subspace of V?

Solution. The set W is the null space of the matrix 1×4 matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 1 & -7 \end{bmatrix}.$$

Hence, $\mathsf{W} = \text{Null}(\mathbf{A})$ and consequently W is a subspace. □

From our previous remarks, the null space of a matrix $\mathbf{A} \in M_{m\times n}$ is just the solution set of the homogeneous system $\mathbf{Ax} = \mathbf{0}$. Therefore, one way to explicitly describe the null space of $\mathbf{A}$ is to solve the system $\mathbf{Ax} = \mathbf{0}$ and write the general solution in parametric vector form. From our previous work on solving linear systems, if the $\texttt{rref}(\mathbf{A})$ has r leading 1's then the number of parameters in the solution set is $d = n - r$. Therefore, after performing back substitution, we will obtain vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$ such that the general solution in parametric vector form can be written as

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_d\mathbf{v}_d$$

where $t_1, t_2, \ldots, t_d$ are arbitrary numbers. Therefore,

$$\text{Null}(\mathbf{A}) = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}.$$

Hence, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$ form a spanning set for $\text{Null}(\mathbf{A})$.

Example 15.12. Find a spanning set for the null space of the matrix

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution. The null space of $\mathbf{A}$ is the solution set of the homogeneous system $\mathbf{Ax} = \mathbf{0}$. Performing elementary row operations one obtains

$$\mathbf{A} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly $r = \text{rank}(\mathbf{A})$ and since $n = 5$ we will have $d = 3$ vectors in a spanning set for $\text{Null}(\mathbf{A})$. Letting $x_5 = t_1$, and $x_4 = t_2$, then from the 2nd row we obtain

$$x_3 = -2t_2 + 2t_1.$$

Letting $x_2 = t_3$, then from the 1st row we obtain

$$x_1 = 2t_3 + t_2 - 3t_1.$$

Therefore, the only solution to $\mathbf{Ax} = \mathbf{0}$ is the trivial solution. Therefore, $\mathcal{B}$ is linearly independent. Moreover, for any $\mathbf{b} \in \mathbb{R}^3$, the augmented matrix $\begin{bmatrix}\mathbf{A} & \mathbf{b}\end{bmatrix}$ is consistent. Therefore, the columns of $\mathbf{A}$ span all of $\mathbb{R}^3$:

$$\mathrm{Col}(\mathbf{A}) = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3.$$

Therefore, $\mathcal{B}$ is a basis for $\mathbb{R}^3$. □

Example 16.7. In $\mathsf{V} = \mathbb{R}^4$, consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -1 \\ 4 \\ 2 \\ -3 \end{bmatrix}.$$

Let $\mathsf{W} = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Is $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for W?

Solution. By definition, $\mathcal{B}$ is a spanning set for W, so we need only determine if $\mathcal{B}$ is linearly independent. Form the matrix, $\mathbf{A} = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$ and row reduce to obtain

$$\mathbf{A} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $\mathrm{rank}(\mathbf{A}) = 2$ and thus $\mathcal{B}$ is linearly dependent. Notice $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_3$. Therefore, $\mathcal{B}$ is not a basis of W. □

Example 16.8. Find a basis for the vector space of 2×2 matrices.

Example 16.9. Recall that a $n \times n$ is skew-symmetric $\mathbf{A}$ if $\mathbf{A}^T = -\mathbf{A}$. We proved that the set of $n \times n$ matrices is a subspace. Find a basis for the set of 3×3 skew-symmetric matrices.

16.3 Dimension of a Vector Space

The following theorem will lead to the definition of the dimension of a vector space.

Theorem 16.10: Let V be a vector space. Then all bases of V have the same number of vectors.

Proof: We will prove the theorem for the case that $\mathsf{V} = \mathbb{R}^n$. We already know that the standard unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis of $\mathbb{R}^n$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be nonzero vectors in $\mathbb{R}^n$ and suppose first that $p > n$. In Lecture 6, Theorem 6.7, we proved that any set of vectors in $\mathbb{R}^n$ containing more than n vectors is automatically linearly dependent. The reason is that the RREF of $\mathbf{A} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p\end{bmatrix}$ will contain at most $r = n$ leading ones,

On the other hand, the inverse matrix $\mathbf{P}^{-1}$ maps standard coordinates in $\mathbb{R}^3$ to $\mathcal{B}$-coordinates. One can verify that

$$\mathbf{P}^{-1} = \begin{bmatrix} 4 & 3 & 6 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Therefore, the $\mathcal{B}$ coordinates of $\mathbf{v}$ are

$$[\mathbf{v}]_{\mathcal{B}} = \mathbf{P}^{-1}\mathbf{v} = \begin{bmatrix} 4 & 3 & 6 \\ -1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

□

When V is an abstract vector space, e.g. $\mathbb{P}_n[t]$ or $M_{n\times n}$, the notion of a coordinate mapping is similar as the case when $\mathsf{V} = \mathbb{R}^n$. If V is an n-dimensional vector space and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, we define the coordinate mapping $\mathcal{P} : \mathsf{V} \to \mathbb{R}^n$ relative to $\mathcal{B}$ as the mapping

$$\mathcal{P}(\mathbf{v}) = [\mathbf{v}]_{\mathcal{B}}.$$

Example 18.8. Let $\mathsf{V} = M_{2\times 2}$ and let $\mathcal{B} = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ be the standard basis for $M_{2\times 2}$. What is $\mathcal{P} : M_{2\times 2} \to \mathbb{R}^4$?

Solution. Recall,

$$\mathcal{B} = \{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then for any $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ we have

$$\mathcal{P}\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right) = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}.$$

□

18.3 Matrix Representation of a Linear Map

Let V and W be vector spaces and let $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ be a linear mapping. Then by definition of a linear mapping, $\mathsf{T}(\mathbf{v} + \mathbf{u}) = \mathsf{T}(\mathbf{v}) + \mathsf{T}(\mathbf{u})$ and $\mathsf{T}(\alpha\mathbf{v}) = \alpha\mathsf{T}(\mathbf{v})$ for every $\mathbf{v}, \mathbf{u} \in \mathsf{V}$ and $\alpha \in \mathbb{R}$. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of V and let $\gamma = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ be a basis of W. Then for any $\mathbf{v} \in \mathsf{V}$ there exists scalars $c_1, c_2, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$