Example 20.3. Let $\mathbf{u} = (2, -5, -1)$ and let $\mathbf{v} = (3, 2, -3)$. Compute $\mathbf{u} \bullet \mathbf{v}$, $\mathbf{v} \bullet \mathbf{u}$, $\mathbf{u} \bullet \mathbf{u}$, and $\mathbf{v} \bullet \mathbf{v}$.

Solution. By definition:

$$\begin{aligned}
\mathbf{u} \bullet \mathbf{v} &= (2)(3) + (-5)(2) + (1)(-3) = -1 \\
\mathbf{v} \bullet \mathbf{u} &= (3)(2) + (2)(-5) + (-3)(1) = -1 \\
\mathbf{u} \bullet \mathbf{u} &= (2)(2) + (-5)(-5) + (-1)(-1) = 30 \\
\mathbf{v} \bullet \mathbf{v} &= (3)(3) + (2)(2) + (-3)(-3) = 22.
\end{aligned}$$

□

We now define the length or norm of a vector in $\mathbb{R}^n$.

Definition 20.4: The **length** or **norm** of a vector $\mathbf{u} \in \mathbb{R}^n$ is defined as

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}.$$

A vector $\mathbf{u} \in \mathbb{R}^n$ with norm 1 will be called a **unit vector**:

$$\|\mathbf{u}\| = 1.$$

Below is an important property of the inner product.

Theorem 20.5: Let $\mathbf{u} \in \mathbb{R}^n$ and let α be a scalar. Then

$$\|\alpha\mathbf{u}\| = |\alpha|\|\mathbf{u}\|.$$

Proof. We have

$$\begin{aligned}
\|\alpha\mathbf{u}\| &= \sqrt{(\alpha\mathbf{u}) \bullet (\alpha\mathbf{u})} \\
&= \sqrt{\alpha^2(\mathbf{u} \bullet \mathbf{u})} \\
&= |\alpha|\sqrt{\mathbf{u} \bullet \mathbf{u}} \\
&= |\alpha|\|\mathbf{u}\|.
\end{aligned}$$

□

By Theorem 20.5, any non-zero vector $\mathbf{u} \in \mathbb{R}^n$ can be scaled to obtain a new unit vector in the same direction as $\mathbf{u}$. Indeed, suppose that $\mathbf{u}$ is non-zero so that $\|\mathbf{u}\| \neq 0$. Define the new vector

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u}$$

Notice that $\alpha = \frac{1}{\|\mathbf{u}\|}$ is just a scalar and thus $\mathbf{v}$ is a scalar multiple of $\mathbf{u}$. Then by Theorem 20.5 we have that

$$\|\mathbf{v}\| = \|\alpha\mathbf{u}\| = |\alpha| \cdot \|\mathbf{u}\| = \frac{1}{\|\mathbf{u}\|} \cdot \|\mathbf{u}\| = 1$$

and therefore $\mathbf{v}$ is a unit vector, see Figure 20.1. The process of taking a non-zero vector $\mathbf{u}$ and creating the new vector $\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u}$ is sometimes called **normalization** of $\mathbf{u}$.

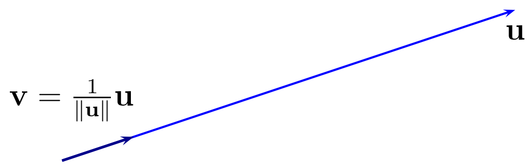

Figure 20.1: Normalizing a non-zero vector.

Example 20.6. Let $\mathbf{u} = (2, 3, 6)$. Compute $\|\mathbf{u}\|$ and find the unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$.

Solution. By definition,

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \bullet \mathbf{u}} = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7.$$

Then the unit vector that is in the same direction as $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 3/7 \\ 6/7 \end{bmatrix}$$

Verify that $\|\mathbf{v}\| = 1$:

$$\|\mathbf{v}\| = \sqrt{(2/7)^2 + (3/7)^2 + (6/7)^2} = \sqrt{4/49 + 9/49 + 36/49} = \sqrt{49/49} = \sqrt{1} = 1.$$

□

Now that we have the definition of the length of a vector, we can define the notion of distance between two vectors.

Definition 20.7: Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. The **distance between u and v** is the length of the vector $\mathbf{u} - \mathbf{v}$. We will denote the distance between $\mathbf{u}$ and $\mathbf{v}$ by $\mathrm{d}(\mathbf{u}, \mathbf{v})$. In other words,

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Example 20.8. Find the distance between $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 7 \\ -9 \end{bmatrix}$.

Solution. We compute:

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(3-7)^2 + (-2+9)^2} = \sqrt{65}.$$

□

Theorem 20.12: Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Then the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is linearly independent. In particular, if $p = n$ then the set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is basis for $\mathbb{R}^n$.

Solution. Suppose that there are scalars $c_1, c_2, \ldots, c_p$ such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p = \mathbf{0}.$$

Take the inner product of $\mathbf{u}_1$ with both sides of the above equation:

$$c_1(\mathbf{u}_1 \bullet \mathbf{u}_1) + c_2(\mathbf{u}_2 \bullet \mathbf{u}_1) + \cdots + c_p(\mathbf{u}_p \bullet \mathbf{u}_1) = \mathbf{0} \bullet \mathbf{u}_1.$$

Since the set is orthogonal, the left-hand side of the last equation simplifies to $c_1(\mathbf{u}_1 \bullet \mathbf{u}_1)$. The right-hand side simplifies to $\mathbf{0}$. Hence,

$$c_1(\mathbf{u}_1 \bullet \mathbf{u}_1) = \mathbf{0}.$$

But $\mathbf{u}_1 \bullet \mathbf{u}_1 = \|\mathbf{u}_1\|^2$ is not zero and therefore the only way that $c_1(\mathbf{u}_1 \bullet \mathbf{u}_2) = \mathbf{0}$ is if $c_1 = 0$. Repeat the above steps using $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_p$ and conclude that $c_2 = 0, c_3 = 0, \ldots, c_p = 0$. Therefore, $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is linearly independent. If $p = n$ then the set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is automatically a basis for $\mathbb{R}^n$. □

Example 20.13. Is the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ an orthogonal set?

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$$

Solution. Compute

$$\begin{aligned}
\mathbf{u}_1 \bullet \mathbf{u}_2 &= (1)(0) + (-2)(1) + (1)(2) = 0 \\
\mathbf{u}_1 \bullet \mathbf{u}_3 &= (1)(-5) + (-2)(-2) + (1)(1) = 0 \\
\mathbf{u}_2 \bullet \mathbf{u}_3 &= (0)(-5) + (1)(-2) + (2)(1) = 0
\end{aligned}$$

Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. By Theorem 20.12, the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. To verify linear independence, we computed that $\det(\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}) = 30$, which is non-zero. □

Lecture 21

Eigenvalues and Eigenvectors

21.1 Eigenvectors and Eigenvalues

An $n \times n$ matrix $\mathbf{A}$ can be thought of as the linear mapping that takes any arbitrary vector $\mathbf{x} \in \mathbb{R}^n$ and outputs a new vector $\mathbf{A}\mathbf{x}$. In some cases, the new output vector $\mathbf{A}\mathbf{x}$ is simply a scalar multiple of the input vector $\mathbf{x}$, that is, there exists a scalar λ such that $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. This case is so important that we make the following definition.

Definition 21.1: Let $\mathbf{A}$ be a $n \times n$ matrix and let $\mathbf{v}$ be a non-zero vector. If $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ then we call the vector $\mathbf{v}$ an **eigenvector** of $\mathbf{A}$ and we call the scalar λ an **eigenvalue** of $\mathbf{A}$ corresponding to $\mathbf{v}$.

Hence, an eigenvector $\mathbf{v}$ of $\mathbf{A}$ is simply scaled by a scalar λ under multiplication by $\mathbf{A}$. Eigenvectors are by definition nonzero vectors because $\mathbf{A}\mathbf{0}$ is clearly a scalar multiple of $\mathbf{0}$ and then it is not clear what that the corresponding eigenvalue should be.

Example 21.2. Determine if the given vectors $\mathbf{v}$ and $\mathbf{u}$ are eigenvectors of $\mathbf{A}$? If yes, find the eigenvalue of $\mathbf{A}$ associated to the eigenvector.

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Solution. Compute

$$\begin{aligned} \mathbf{A}\mathbf{v} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} -6 \\ 0 \\ 2 \end{bmatrix} \\ &= 2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \\ &= 2\mathbf{v} \end{aligned}$$