Example 22.10. Find the eigenvalues of $\mathbf{A}$ and a basis for each eigenspace:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

For each eigenvalue of $\mathbf{A}$, find its algebraic and geometric multiplicity. Does $\mathbb{R}^3$ have a basis of eigenvectors of $\mathbf{A}$?

Solution. One computes

$$p(\lambda) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$$

and therefore the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 1$ and $\lambda_2 = -2$. The algebraic multiplicity of λ_1 is $k_1 = 1$ and that of λ_2 is $k_2 = 2$. For $\lambda_1 = 1$ we compute

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix}$$

and then one finds that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

is a basis for the λ_1-eigenspace. Therefore, the geometric multiplicity of λ_1 is $g_1 =$. For $\lambda_2 = -2$ we compute

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 4 & 4 & 3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, since $\operatorname{rank}(\mathbf{A} - \lambda_2 \mathbf{I}) = 2$, the geometric multiplicity of $\lambda_2 = -2$ is $g_2 = 1$, which is less than the algebraic multiplicity $k_2 = 2$. An eigenvector corresponding to $\lambda_2 = -2$ is

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, for the repeated eigenvalue $\lambda_2 = -2$, we are able to find only one linearly independent eigenvector. Therefore, it is not possible to construct a basis for $\mathbb{R}^3$ consisting of eigenvectors of $\mathbf{A}$. $\square$

Hence, in the previous example, there does not exist a basis of $\mathbb{R}^3$ of eigenvectors of $\mathbf{A}$ because for one of the eigenvalues (namely λ_2) the geometric multiplicity was less than the algebraic multiplicity:

$$g_2 < d_2.$$

In the next lecture, we will elaborate on this situation further.

Example 22.11. Find the algebraic and geometric multiplicities of each eigenvalue of the matrix

$$\mathbf{A} = \begin{bmatrix} -7 & 1 & 0 \\ 0 & -7 & 1 \\ 0 & 0 & -7 \end{bmatrix}.$$

(a) Find the characteristic polynomial and the eigenvalues of $\mathbf{A}$.
(b) Find the geometric and algebraic multiplicity of each eigenvalue of $\mathbf{A}$.

We now introduce a very special type of a triangular matrix, namely, a diagonal matrix.

Definition 23.3: A matrix $\mathbf{D}$ whose off-diagonal entries are all zero is called a diagonal matrix.

For example, here is 3×3 diagonal matrix

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -8 \end{bmatrix}.$$

and here is a 5×5 diagonal matrix

$$\mathbf{D} = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{7}{2} & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{11} \end{bmatrix}.$$

A diagonal matrix is clearly also a triangular matrix and therefore the eigenvalues of a diagonal matrix $\mathbf{D}$ are simply the diagonal entries of $\mathbf{D}$. Moreover, the powers of a diagonal matrix are easy to compute. For example, if $\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ then

$$\mathbf{D}^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

and similarly for any integer $k = 1, 2, 3, \ldots$, we have that

$$\mathbf{D}^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}.$$

23.2 Diagonalization

Recall that two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be **similar** if there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{A} = \mathbf{P}\mathbf{B}\mathbf{P}^{-1}.$$

A very simple type of matrix is a diagonal matrix since many computations with diagonal matrices are trivial. The problem of diagonalization is thus concerned with answering the question of whether a given matrix is similar to a diagonal matrix. Below is the formal definition.

all matrices are diagonalizable. As it turns out, any **symmetric A** is diagonalizable and moreover (and perhaps more importantly) there exists an **orthogonal** eigenvector matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$. The full statement is below.

Theorem 24.3: If $\mathbf{A}$ is a symmetric matrix then $\mathbf{A}$ is diagonalizable. In fact, there is an orthonormal basis of $\mathbb{R}^n$ of eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of $\mathbf{A}$. In other words, the matrix $\mathbf{P} = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]$ is orthogonal, $\mathbf{P}^T\mathbf{P} = \mathbf{I}$, and $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^T$.

The proof of the theorem is not hard but we will omit it. The punchline of Theorem 24.3 is that, for the case of a symmetric matrix, we will never encounter the situation where the geometric multiplicity is strictly less than the algebraic multiplicity. Moreover, we are guaranteed to find an orthogonal matrix that diagonalizes a given symmetric matrix.

Example 24.4. Find an orthogonal matrix $\mathbf{P}$ that diagonalizes the symmetric matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Solution. The characteristic polynomial of $\mathbf{A}$ is

$$p(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^3 - 4\lambda^2 + 3\lambda = \lambda(\lambda - 1)(\lambda - 3)$$

The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 3$. Eigenvectors of $\mathbf{A}$ associated to $\lambda_1, \lambda_2, \lambda_3$ are

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

As expected by Theorem 24.2, the eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ form an orthogonal set:

$$\mathbf{u}_1^T\mathbf{u}_2 = 0,\ \mathbf{u}_1^T\mathbf{u}_3 = 0,\ \mathbf{u}_2^T\mathbf{u}_3 = 0.$$

To find an orthogonal matrix $\mathbf{P}$ that diagonalizes $\mathbf{A}$ we must normalize the eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to obtain an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. To that end, first compute $\mathbf{u}_1^T\mathbf{u}_1 = 3$, $\mathbf{u}_2^T\mathbf{u}_2 = 2$, and $\mathbf{u}_3^T\mathbf{u}_3 = 6$. Then let $\mathbf{v}_1 = \frac{1}{\sqrt{3}}\mathbf{u}_1$, let $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\mathbf{u}_2$, and let $\mathbf{v}_3 = \frac{1}{\sqrt{6}}\mathbf{u}_3$. Therefore, an orthogonal matrix that diagonalizes $\mathbf{A}$ is

$$\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

You can easily verify that $\mathbf{P}^T\mathbf{P} = \mathbf{I}$, and that

$$\mathbf{A} = \mathbf{P}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{P}^T$$

□

In both models, the web is defined as a **directed graph**, where the nodes represent webpages and the directed arcs represent hyperlinks, see Figure 25.1.

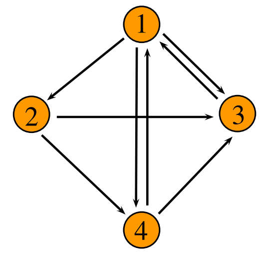


Figure 25.1: A tiny web represented as a directed graph.

25.2 A Description of the PageRank Algorithm

In the PageRank algorithm, each inlink is viewed as a recommendation (or vote). In general, pages with many inlinks are more important than pages with few inlinks. However, the *quality* of the inlink (vote) is important. The vote of each page should be divided by the total number of recommendations made by the page. The **PageRank** of page i, denoted x_i, is the sum of all the weighted PageRanks of all the pages pointing to i:

$$x_i = \sum_{j \to i} \frac{x_j}{N_j}$$

where

(1) N_j is the number of outlinks from page j

(2) $j \to i$ means page j links to page i

Example 25.1. Find the PageRank of each page for the network in Figure 25.1.

From the previous example, we see that the PageRank of each page can be found by solving an eigenvalue/eigenvector problem. However, when dealing with large networks such as the internet, the size of the problem is in the billions (8.1 billion in 2006) and directly solving the equations is not possible. Instead, an iterative method called the **power method** is used. One starts with an initial guess, say $\mathbf{x}_0 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Then one updates the guess by computing

$$\mathbf{x}_1 = \mathbf{H}\mathbf{x}_0.$$

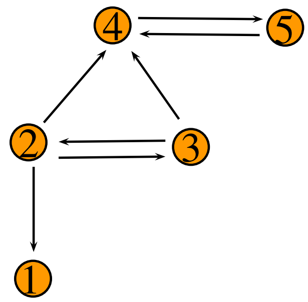
In other words, we have a discrete dynamical system

$$\mathbf{x}_{k+1} = \mathbf{H}\mathbf{x}_k.$$

A natural question is under what conditions will the the limiting value of the sequence

$$\lim_{k\to\infty} \mathbf{x}_k = \lim_{k\to\infty} (\mathbf{H}^k \mathbf{x}_0) = \mathbf{q}$$

converge to an equilibrium of $\mathbf{H}$? Also, if $\lim_{k\to\infty} \mathbf{x}_k$ exists, will it be a positive vector? And lastly, can $\mathbf{x}_0 \neq \mathbf{0}$ be chosen arbitrarily? To see what situations may occur, consider the network displayed in Figure 25.2. Starting with $\mathbf{x}_0 = (\frac{1}{5}, \ldots, \frac{1}{5})$ we obtain that for $k \geq 39$, the vectors $\mathbf{x}_k = \mathbf{H}^k \mathbf{x}_0$ cycle between $(0, 0, 0, 0.28, 0.40)$ and $(0, 0, 0, 0.40, 0.28)$. Therefore, the sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ does not converge. The reason for this is that nodes 4 and 5 form a **cycle**.



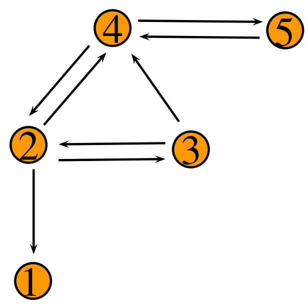
$$\mathbf{H} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure 25.2: Cycles present in the network

Now consider the network displayed in Figure 25.3. If we remove the cycle we are still left with a **dangling node**, namely node 1 (e.g. pdf file, image file). Starting with $\mathbf{x}_0 = (\frac{1}{5}, \ldots, \frac{1}{5})$ results in

$$\lim_{k\to\infty} \mathbf{x}_k = \mathbf{0}.$$

Therefore, in this case the sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to a non-positive vector, which for the purposes of ranking pages would be an undesirable situation.



$$\mathbf{H} = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Figure 25.3: Dangling node present in the network

To avoid the presence of dangling nodes and cycles, Brin and Page used the notion of a **random surfer** to adjust $\mathbf{H}$. To deal with a dangling node, Brin and Page replaced the associated zero-column with the vector $\frac{1}{n}\mathbf{1} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. The justification for this adjustment is that if a random surfer reaches a dangling node, the surfer will "teleport" to any page in the web with equal probability. The new updated hyperlink matrix $\mathbf{H}^*$ may still not have the desired properties. To deal with cycles, a surfer may abandon the hyperlink structure of the web by ocassionally moving to a random page by typing its address in the