Limit

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1 Reminisce

THE SQUEEZE (SANDWICH) THEOREM

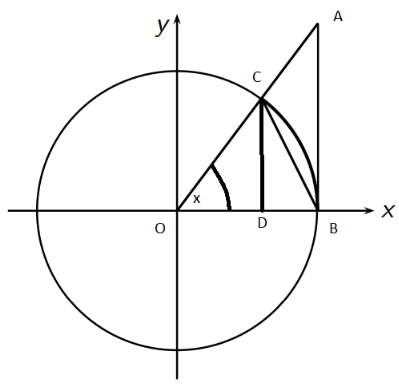
Let g(x) and h(x) be functions such that $g(x) \leq f(x) \leq h(x)$ on a punctured neighborhood of a.

If $\lim_{x \longrightarrow a} g(x) = L$ and $\lim_{x \longrightarrow a} h(x) = L$ ($L \in R$),then

$\lim_{x \longrightarrow a} f(x) = L$

A punctured neighborhood of an element $x \in X$ is a set which is the relative complement $U/\{X\}$

2 An important limit



Reference to the figure.

The line segment AB is a tangent to the circle of radius a at the point B such that OAB is a right angled triangle.

Also

$\angle BOC = x$ Radians.

Where $0 < x < \frac{\pi}{2}$

OB=OC=radii of the circle=a

CD is an altitude on OB.

Now, in the right angled triangle OAB.

$\frac{AB}{OB} = \frac{AB}{a} = \tan x$

$AB = a \tan x$

Similarly

Area of the circle of radius a with 2π Radians=πa^2

Area of the sector of a circle of radius a with 1 Radians=$\frac{\pi a^2}{2\pi} = \frac{a^2}{2}$

Area of the sector of a circle of radius a with x Radians=$\frac{a^2}{2}x$

Also, in the right triangle OCD

$\frac{CD}{OC} = \frac{CD}{a} = \sin x$

$CD = a \sin x$

Area of the triangle OAB= $\frac{1}{2}(OB)(AB) = \frac{1}{2}a(a \tan x) = \frac{1}{2}a^2 \tan x$

Area of the triangle OCB= $\frac{1}{2}(OB)(CD) = \frac{1}{2}a(a \sin x) = \frac{1}{2}a^2 \sin x$

Reference to the figure

Area of the triangle OCB $\leq$ Area of the sector of the circle $O\overset{\frown}{C}B \leq$

Area of the triangle OAB

$\frac{1}{2}a^2 \sin x \leq \frac{a^2}{2}x \leq \frac{1}{2}a^2 \tan x, \qquad 0 < x < \frac{\pi}{2}$

$\sin x \leq x \leq \tan x$

$1 \leq \frac{x}{\sin x} \leq \frac{\sin x}{\cos x} \times \frac{1}{\sin x}$

$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$

$1 \leq \lim_{x \longrightarrow 0} \frac{x}{\sin x} \leq \lim_{x \longrightarrow 0} \frac{1}{\cos x}$

Now

$\lim_{x \longrightarrow 0} \frac{1}{\cos x} = 1$, since $\cos 0 = 1$

Hence

$1 \leq \lim_{x \longrightarrow 0} \frac{x}{\sin x} \leq 1$

BY THE SQUEEZE (SANDWICH) THEOREM

$$\lim_{x \longrightarrow 0} \frac{x}{\sin x} = 1$$

Similarly

$$\lim_{x \longrightarrow 0} \frac{\sin x}{x} = 1$$

3 Formulas used

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos^2 x = 1 - \sin^2 x$

$\sin^2 x = 1 - \cos^2 x$

....

Evaluate the limit:

$$\lim_{\theta \longrightarrow 0} \frac{\sin \sqrt{2\theta}}{\sqrt{2\theta}}$$

Solution

Putting.

$\sqrt{2\theta} = x$

$\theta \longrightarrow 0^+ \Longleftrightarrow x \longrightarrow 0^+, \quad x > 0, \theta > 0$

$$\lim_{\theta \longrightarrow 0^+} \frac{\sin\sqrt{2\theta}}{\sqrt{2\theta}} = \lim_{x \longrightarrow 0^+} \frac{\sin x}{x} = 1$$

As $\sqrt{2\theta}$ is undefined for $\theta < 0$, so

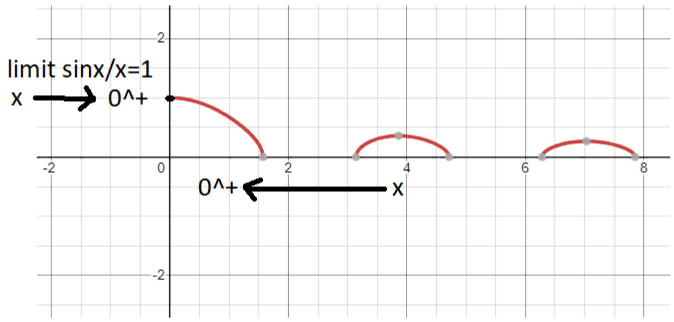
$\lim_{x \longrightarrow o^-} \frac{\sin x}{x}$ does not exist.

Since the limit does not depend on the direction from which x approaches 0 (whether from the positive side or the negative side), the limit remains the same:

$$\lim_{\theta \longrightarrow 0} \frac{\sin\sqrt{2\theta}}{\sqrt{2\theta}} = \lim_{x \longrightarrow 0} \frac{\sin x}{x} = 1$$

Intuitive Reasoning:

As x gets closer to zero, $\sin x$ simultaneously gets closer to values of x in such a way that the ratio $\frac{\sin x}{x}$ gets closer to 1. The graph below shows the behaviour of $\frac{\sin x}{x}$ as x get closer to zero .


limit sinx/x=1
x 0^+
0^+ x
2
-2
0
2
4
6
8
-2