1. Vector Differentiation

Introduction: Vector calculus or vector analysis, is concerned with differentiation and integration of vector fields. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow.

Point Function: A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function.

Scalar Point Function: If to each point *P (x, y, z)* of a region *R* in space there corresponds a unique scalar *f(P)*, then *f* is called a scalar point function.

Examples.

(i) Temperature distribution in a heated body,

(ii) Density of a body & **(iii)** Potential due to gravity.

Vector Point Function: If to each point *P (x, y, z)* of a region *R* in space there corresponds a unique vector *f(P)*, then *f* is called a vector point function.

Examples.

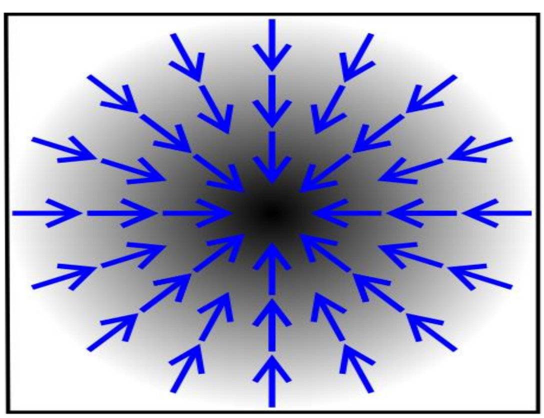
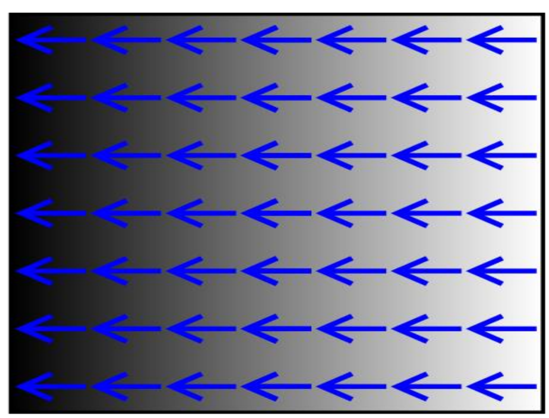
(i) Forest wind, **(ii)** The velocity of a moving fluid & **(iii)** Gravitational force.

1.1 Gradient of a scalar point function

The gradient is closely related to the derivative, but it is not itself a derivative.

The value of the gradient at a point is a tangent vector.

The gradient can be interpreted as the "direction and rate of fastest increase"

The unit vector normal to the surface $\emptyset = \frac{grad\ \emptyset}{|grad\ \emptyset|} = \frac{-2\hat{\imath}+4\hat{\jmath}+4\hat{k}}{6} = \frac{-\hat{\imath}+2\hat{\jmath}+2\hat{k}}{3}$

Example 4. If $\nabla\emptyset = (y^2 - 2xyz^3)\hat{\imath} + (3 + 2xy - x^2z^3)\hat{\jmath} + (6z^3 - 3x^2yz^2)\hat{k}$, find $\emptyset$.

Solution. Let $\vec{F} = \nabla\emptyset \Rightarrow \vec{F}.d\vec{r} = \nabla\emptyset.d\vec{r}$ [taking dot product with $d\vec{r}$]

$$\vec{F}.d\vec{r} = \left(\hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}\right).\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right) = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$$

$$\Rightarrow \vec{F}.d\vec{r} = d\emptyset \qquad \left[as\ \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz = d\emptyset\right]$$

Or $d\emptyset = \vec{F}.d\vec{r} = \nabla\emptyset.d\vec{r}$

$$d\emptyset = \left[(y^2 - 2xyz^3)\hat{\imath} + (3 + 2xy - x^2z^3)\hat{\jmath} + (6z^3 - 3x^2yz^2)\hat{k}\right].\left(dx\hat{\imath} + dy\hat{\jmath} + dz\hat{k}\right)$$

$$= (y^2 - 2xyz^3)dx + (3 + 2xy - x^2z^3)dy + (6z^3 - 3x^2yz^2)dz \quad \text{-----------------} \ (1)$$

$$= 3dy + 6z^3dz + y^2dx + 2xydy - 2xyz^3dx - x^2z^3dy - 3x^2yz^2dz$$

$$d\emptyset = 3dy + 6z^3dz + d(xy^2) - d(x^2yz^3)$$

Integrate, we get $\emptyset = 3y + \frac{3}{2}z^4 + xy^2 - x^2yz^3 + C$

Example 5. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.

Solution. Let $\emptyset_1 \equiv x^2 + y^2 + z^2 - 9 = 0$ and $\emptyset_2 \equiv x^2 + y^2 - 3 - z = 0$

then $\nabla\emptyset_1 = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$ and $\nabla\emptyset_2 = 2x\hat{\imath} + 2y\hat{\jmath} - \hat{k}$

Let $\quad \vec{n_1} = \nabla\emptyset_1]_{(2,-1,2)} = 4\hat{\imath} - 2\hat{\jmath} + 4\hat{k}$

and $\quad \vec{n_2} = \nabla\emptyset_2]_{(2,-1,2)} = 4\hat{\imath} - 2\hat{\jmath} - \hat{k}$

Let θ is the angle between the vectors $\vec{n_1}$ and $\vec{n_2}$ which are normal to the given surfaces $\emptyset_1$ and $\emptyset_2$ respectively. Then

$$\cos\theta = \frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{4.4 + (-2).(-2) + 4.(-1)}{\sqrt{16+4+16}\sqrt{16+4+1}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \boldsymbol{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

Example 6. If $u = x + y + z,\ v = x^2 + y^2 + z^2,\ w = yz + zx + xy$, prove that $grad\ u,\ grad\ v\ and\ grad\ w\ are\ coplanar\ vectors.$

Solution. Here, we have

$$grad\ u = \nabla u = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x + y + z) = \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$grad\ v = \nabla v = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2) = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$grad\ w = \nabla w = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(yz + zx + xy)$$

$$\left[(X-1)\hat{\imath}+(Y-2)\hat{\jmath}+(Z-2)\hat{k}\right].\left(4\hat{\imath}+2\hat{\jmath}+2\,\hat{k}\right)=0$$

Or $4X+2Y+2Z=12$ Or $2Z+Y+Z=6$

Equations of the normal to the given surface at the point $(1,-1,2)$ are:

$$\frac{X-x}{\frac{\partial\emptyset}{\partial x}}=\frac{Y-y}{\frac{\partial\emptyset}{\partial y}}=\frac{Z-z}{\frac{\partial\emptyset}{\partial z}} \text{ i.e. } \frac{X-1}{4}=\frac{Y-2}{2}=\frac{Z-2}{2}\Rightarrow \frac{X-1}{2}=\frac{Y-2}{1}=\frac{Z-2}{1}$$

Example 3. Find the equations of the tangent plane and normal to the surface $yz-zx+xy+5=0$ at the point $(1,-1,2)$.

Solution. Let $\emptyset\equiv yz-zx+xy+5=0$

$$\nabla\emptyset=(y-z)\hat{\imath}+(z+x)\hat{\jmath}+(y-x)\hat{k}\Rightarrow\nabla\emptyset]_{1,-1,2}=-3\hat{\imath}+3\hat{\jmath}-2\hat{k}$$

Equation of tangent at the point $(1,-1,2)$ is given by

$$\left[(X-1)\hat{\imath}+(Y+1)\hat{\jmath}+(Z-2)\hat{k}\right].\left(-3\hat{\imath}+3\hat{\jmath}-2\hat{k}\right)=0$$

Or $-3X+3Y-2Z+10=0\Rightarrow 3X-3Y+2Z=10$

Equations of the normal to the given surface at the point $(1,-1,2)$ are:

$$\frac{X-x}{\frac{\partial\emptyset}{\partial x}}=\frac{Y-y}{\frac{\partial\emptyset}{\partial y}}=\frac{Z-z}{\frac{\partial\emptyset}{\partial z}} \text{ i.e. } \frac{X-1}{-3}=\frac{Y+1}{3}=\frac{Z+2}{-2}$$

Example 4. Find the equations of the tangent plane and normal to the surface $z=x^2+y^2$ at the point $(2,-1,5)$.

Solution. Let $\emptyset\equiv z-x^2-y^2=0$

$$\nabla\emptyset=-2x\hat{\imath}-2y\hat{\jmath}+\hat{k}\Rightarrow\nabla\emptyset]_{2,-1,5}=-4\hat{\imath}+2\hat{\jmath}+\hat{k}$$

Equation of tangent at the point $(1,-1,2)$ is given by

$$\left[(X-2)\hat{\imath}+(Y+1)\hat{\jmath}+(Z-5)\hat{k}\right].\left(-4\hat{\imath}+2\hat{\jmath}+\hat{k}\right)=0$$

Or $-4X+2Y+Z+5=0\Rightarrow 4X-2Y-Z=5$

Equations of the normal to the given surface at the point $(1,-1,2)$ are:

$$\frac{X-x}{\frac{\partial\emptyset}{\partial x}}=\frac{Y-y}{\frac{\partial\emptyset}{\partial y}}=\frac{Z-z}{\frac{\partial\emptyset}{\partial z}} \text{ i.e. } \frac{X-2}{-4}=\frac{Y+1}{2}=\frac{Z-5}{1}$$

Exercise

1. If $\vec{F}(x,y,z)=xz^3\hat{\imath}-2x^2yz\hat{\jmath}+2yz^4\hat{k}$ find *divergence and curl* of $\vec{F}(x,y,z)$.

2. Find the *divergence and curl* of the vector field $\vec{V}=x^2y^2\hat{\imath}+2xy\hat{\jmath}+(y^2-xy)\hat{k}$.

3. A fluid motion is given by $\vec{V}=(y+z)\hat{\imath}+(z+x)\hat{\jmath}+(x+y)\hat{k}$

(i) Is this motion irrotational? If so, find the velocity potential.

(ii) Is the motion possible for an incompressible fluid?

2. Vector Integration

Introduction: The process of integration to compute the integrals of vector functions of a real variable is known vector integration. i.e. we compute integrals of functions of the type

$\vec{f}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k}$.

2.1 Line Integral: Any integral that is evaluated along a curve is called a line integral. The terms path integral, curve integral, and curvilinear integral are also used for line integral.

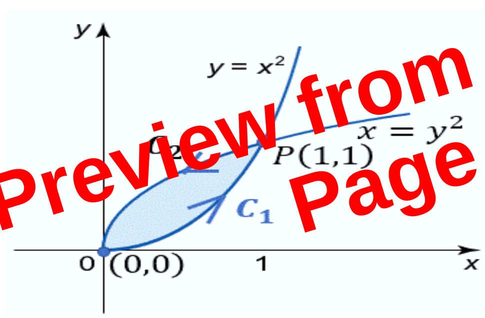
The line integral of $\vec{f}(t)$ along the curve C is denoted by $\int_C \vec{f}(t)dt$, if C is a closed curve then the integral sign $\int_C$ is replaced by $\oint_C$.

2.1.1 Work done by a force: Let $\vec{F}$ be the force acting on a moving particle along the path C, then the total work done (W) by $\vec{F}$ during displacement from $A\ to\ B$ on C is given by

$W = \int_A^B \vec{F}.d\vec{r}$ where, $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $d\vec{r} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$

Example 1. If $\vec{A} = (x-y)\hat{\imath} + (x+y)\hat{\jmath}$, evaluate $\oint_C \vec{A}.d\vec{r}$ around the curve C consisting of $y = x^2$ and $x = y^2$.

Solution. Let $I = \oint_C \vec{A}.d\vec{r}$; $C: y = x^2$ and $x = y^2$



On C_1; $y = x^2 \Rightarrow dy = 2xdx$ and $x \to 0\ to\ 1$

On C_2; $x = y^2 \Rightarrow dx = 2ydy$ and $y \to 0\ to 1$

$\vec{A}.d\vec{r} = [(x-y)\hat{\imath} + (x+y)\hat{\jmath}].(\hat{\imath}dx + \hat{\jmath}dy) = (x-y)dx + (x+y)dy$

Now, $I = \oint_C \vec{A}.d\vec{r} = \oint_{C_1} \vec{A}.d\vec{r} + \oint_{C_2} \vec{A}.d\vec{r}$

$$= \int_0^1 [(x-x^2)dx + (x+x^2)2xdx] + \int_1^0 [(y^2-y)2ydy + (y^2+y)dy]$$

$$= \int_0^1 (2x^3 + x^2 + x)dx + \int_1^0 (2y^3 - y^2 + y)dy$$

$$= \frac{2}{4} + \frac{1}{3} + \frac{1}{2} - \frac{2}{4} + \frac{1}{3} - \frac{1}{2} = \mathbf{\frac{2}{3}}$$

Example 2. Evaluate the line integral $\int_C [(x^2 + xy)dx + (x^2 + y^2)dy]$, where C is the square formed by the lines $\mathrm{y} = \pm 1\ and\ x = \pm 1$.

$$= \int_{x=0}^{3} (6-2x) \left(\frac{y^2}{2}\right)_0^{6-2x} dx$$

$$= \frac{1}{2}\int_{x=0}^{3} (6-2x)^3\, dx$$

$$= \frac{1}{2}\int_{x=0}^{3} (6-2x)^3\, dx = 4\int_0^3 (3-x)^3\, dx$$

$$= 4\left[\frac{(3-x)^4}{-4}\right]_0^3 = 81$$

$$\boxed{\iint_S \vec{A}.\hat{n} = 81}$$

Example 2: Evaluate $\iint_S \vec{A}.\hat{n}\, dS$, where $\vec{A} = z\hat{\imath} + x\hat{\jmath} - 3y^2z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Solution: We have $\vec{A} = z\hat{\imath} + x\hat{\jmath} - 3y^2z\hat{k}$

To find $\hat{n}$

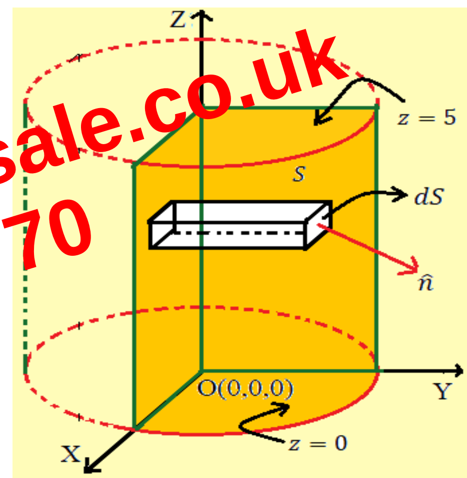
A vector normal to the surface S is given by

$$\nabla S = \nabla(x^2 + y^2 - 16) = 2x\hat{\imath} + 2y\hat{\jmath}$$

$\therefore$ $\hat{n}$ = a unit vector normal to the surface S

$$= \frac{2x\hat{\imath}+2y\hat{\jmath}}{\sqrt{(2x)^2+(2y)^2}} = \frac{x\hat{\imath}+y\hat{\jmath}}{\sqrt{x^2+y^2}} = \frac{x\hat{\imath}+y\hat{\jmath}}{4}$$

$$\boxed{\hat{n} = \frac{x\hat{\imath}+y\hat{\jmath}}{4}}$$



[$\because$ on the surface of the cylinder: $x^2 + y^2 = 16$]

$$\vec{A}.\hat{n} = (z\hat{\imath} + x\hat{\jmath} - 3y^2z\hat{k}).\left(\frac{x\hat{\imath}+y\hat{\jmath}}{4}\right) = \frac{1}{4}(xz + xy) = \frac{1}{4}x(y+z)$$

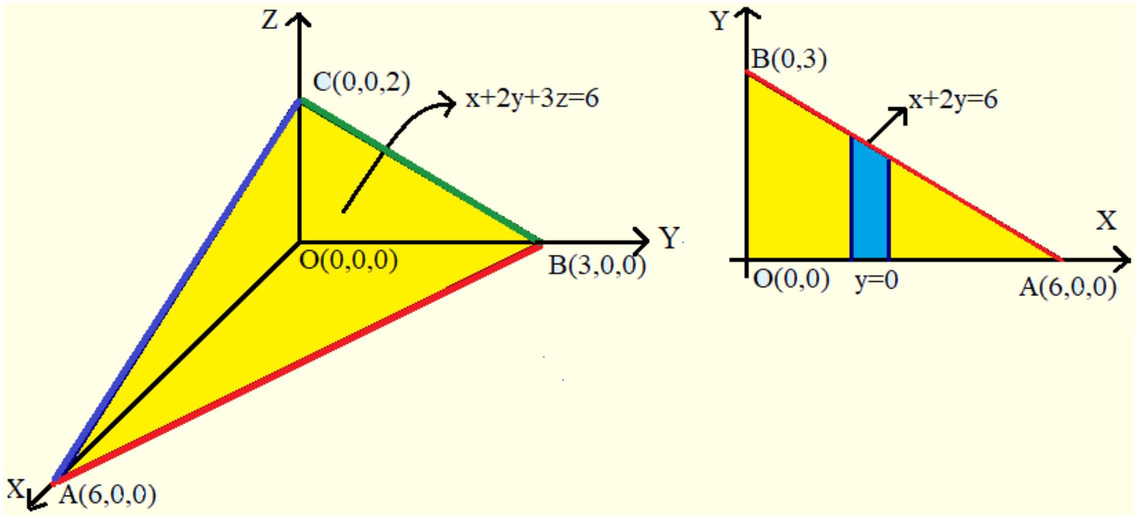
Now let R be the projection of S on yz-plane, therefore

$$\iint_S \vec{A}.\hat{n}\, dS = \iint_R \vec{A}.\hat{n}\, \frac{dy\, dz}{|\hat{n}.\hat{\imath}|} \quad \ldots\ldots (1)$$

We have

$$\hat{n}.\hat{\imath} = \left(\frac{x\hat{\imath}+y\hat{\jmath}}{4}\right).\hat{\imath} = \frac{x}{4}$$

Therefore from (1), we get



$$\therefore \iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dx\,dy\,dz$$

$$= 3\iiint_V dx\,dy\,dz = 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}}\int_{z=0}^{\frac{6-x-2y}{3}} dz\,dy\,dx = 3\int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}}\left(\frac{6-x-2y}{3}\right) dy\,dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{6-x}{2}} (6-x-2y)\,dy\,dx = \int_{x=0}^{6}\left[(6-x)y - y^2\right]_{y=0}^{\frac{6-x}{2}} dx = \int_{x=0}^{6}\left[\frac{(6-x)^2}{2} - \frac{(6-x)^2}{4}\right] dx$$

$$= \int_{x=0}^{6}\frac{(6-x)^2}{4}\,dx = \frac{1}{4}\left[\frac{(6-x)^3}{-3}\right]_0^6 = \frac{1}{12}(216) = 18$$

$$\boxed{\therefore \iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy) = 18}$$

Example 6: The vector field $\vec{F} = x^2\hat{\imath} + z\,\hat{\jmath} + yz\hat{k}$ is defined over the volume of the cuboid given by $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$ enclosing the surface S, evaluate $\iint_S \vec{F}.d\vec{S}$.

Solution: By Gauss divergence theorem, we know that

$$\boxed{\iint_S \vec{F}.\hat{n}\,dS = \iiint_V div\ \vec{F}\ dV}$$

where V is the region bounded by the closed surface S.

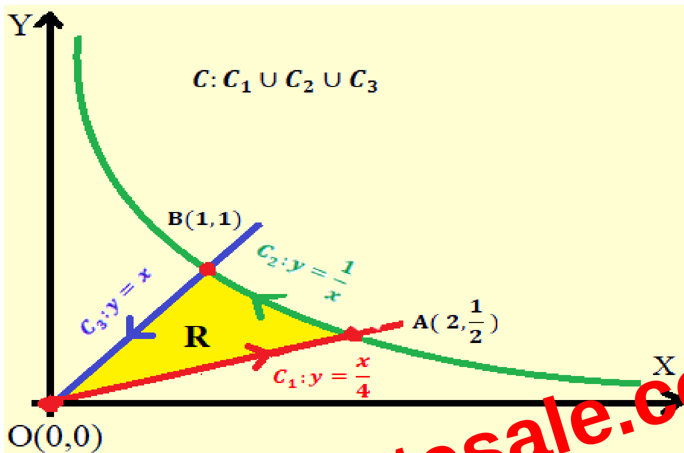
$$\therefore \iint_S \vec{F}.d\vec{S} = \iint_S \vec{F}.\hat{n}\,dS = \iiint_V div\ \vec{F}\ dV$$

Example 5: Using Green's theorem, find the area of the region in the first quadrant bounded by the curves $y = x,\ y = \frac{1}{x}, y = \frac{x}{4}$.

Solution: By Green's theorem, we know that the area bounded by a closed curve C is given by

$$\boxed{A = \frac{1}{2}\oint_C (x\,dy - y\,dx)}$$

Here the curve C consists of three curves: $C_1: y = \frac{x}{4};\ C_2: y = \frac{1}{x}\ ;\ C_3: y = x$



$$\therefore A = \frac{1}{2}\left[\int_{C_1}(x\,dy - y\,dx) + \int_{C_2}(x\,dy - y\,dx) + \int_{C_3}(x\,dy - y\,dx)\right]$$

$$= \frac{1}{2}(I_1 + I_2 + I_3) \quad \ldots\ldots\ldots\ldots(1)$$

Along $C_1: y = \frac{x}{4}$, $dy = \frac{1}{4}dx$, x varies from 0 to 2.

$$\therefore I_1 = \int_{C_1}(x\,dy - y\,dx) = \int_{x=0}^{2}\left(\frac{x}{4}dx - \frac{x}{4}dx\right) = 0$$

$$\boxed{I_1 = 0}$$

Along $C_2: y = \frac{1}{x}$, $dy = -\frac{1}{x^2}dx$, x varies from 2 to 1.

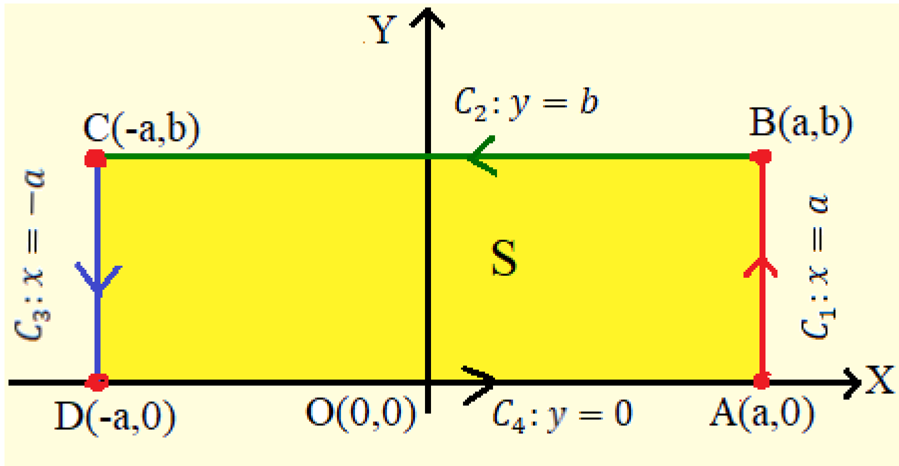
$$\therefore I_2 = \int_{C_2}(x\,dy - y\,dx) = \int_{x=2}^{1}\left[x\left(-\frac{1}{x^2}\right)dx - \frac{1}{x}dx\right] = -2\int_2^1 \frac{1}{x}dx = -2(\log x)_2^1$$

$$= -2(\log 1 - \log 2) = 2\log 2$$

$$\boxed{I_2 = 2\log 2}$$

Along $C_3: y = x, dy = dx$, x varies from 1 to 0. **A**(**2**, $\mathbf{\frac{1}{2}}$)

$$\oint_C \vec{F}.d\vec{r} = \oint_C [(x^2+y^2)\hat{\imath} - 2xy\hat{\jmath}].(\hat{\imath}\,dx + \hat{\jmath}\,dy) = \oint_C [(x^2+y^2)dx - 2xy\,dy]$$



Here the closed curve C consists of four lines AB, BC, CD and DA. Let these straight lines are denotes by C_1 , C_2 , C_3 and C_4 respectively.

$\therefore \oint_C \vec{F}.d\vec{r} = \int_{C_1}[(x^2+y^2)dx - 2xy\,dy] + \int_{C_2}[(x^2+y^2)dx - 2xy\,dy] + \int_{C_3}\Big[(x^2+y^2)dx - 2xy\,dy + \int_{C_4}[(x^2+y^2)dx - 2xy\,dy]\Big]$

or, $\oint_C \vec{F}.d\vec{r} = I_1 + I_2 + I_3 + I_4$ ------------------------------------ (1)

Now we have $I_1 = \int_{C_1}[(x^2+y^2)dx - 2xy\,dy]$

Along $C_1: x = a, dx = 0$ and y varies from 0 to b.

$\therefore I_1 = \int_{y=0}^{b} (0 - 2ay\,dy) = -2a\int_{y=0}^{b} y\,dy = -a(y^2)_0^b = -ab^2$

$$I_2 = \int_{C_2} [(x^2+y^2)dx - 2xy\,dy]$$

Along $C_2: y = b, dy = 0$ and x varies from a to $-a$.

$\therefore I_2 = \int_{x=a}^{-a} (x^2+b^2)dx = \left(\frac{x^3}{3} + b^2x\right)_a^{-a} = -\frac{2}{3}a^3 - 2ab^2$

$$I_3 = \int_{C_3} [(x^2+y^2)dx - 2xy\,dy]$$

Along $C_3: x = -a, dx = 0$ and y varies from b to 0.

$\therefore I_3 = \int_{y=b}^{0} (0 + 2ay\,dy) = 2a\int_b^0 y\,dy = a(y^2)_b^0 = -ab^2$

$$\therefore \oint_C \vec{F}.d\vec{r} = \oint_C x\,dy \qquad [\because z = 0 \implies dz = 0]$$

$$= \int_0^{2\pi} a\cos t\,.\,a\cos t\,dt = a^2\int_0^{2\pi} cos^2 t\,dt = a^2\int_0^{2\pi}\left(\frac{1+\cos 2t}{2}\right)dt$$

$$= \frac{a^2}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} = \frac{a^2}{2}(2\pi) = \pi a^2$$

$$\boxed{\oint_C \vec{F}.d\vec{r} = \pi a^2} \quad \text{--} \ (1)$$

To evaluate: $\iint_S \boldsymbol{curl\,\vec{F}.\hat{n}\,dS}$

We have $curl\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$

$$= \hat{\imath}\left[\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial z}(x)\right] - \hat{\jmath}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial z}(z)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z)\right]$$

$$= (1-0)\hat{\imath} - (0-1)\hat{\jmath} + (1-0)\hat{k} = \hat{\imath} + \hat{\jmath} + \hat{k}$$

$$\boxed{curl\,\vec{F} = \hat{\imath} + \hat{\jmath} + \hat{k}}$$

To find $\hat{n}$

If $\hat{n}$ is a unit vector along outward drawn normal at any point (x, y, z) on the surface S $i.e.$ the surface $\varphi(x, y, z) \equiv x^2 + y^2 + z^2 = a^2$, then

$$\hat{n} = \frac{grad\varphi}{|grad\,\varphi|} = \frac{\nabla\varphi}{|\nabla\varphi|}$$

$$\nabla\varphi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(x^2 + y^2 + z^2 - a^2) = 2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}$$

$$\therefore \hat{n} = \frac{2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}}{|2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}|} = \frac{2x\hat{\imath} + 2y\hat{\jmath} + 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{a^2}} \qquad [\text{ since on } S,\ x^2 + y^2 + z^2 = a^2]$$

$$\boxed{\hat{n} = \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{a}}$$

$$\therefore\ curl\,\vec{F}.\hat{n} = (\hat{\imath} + \hat{\jmath} + \hat{k}).\left(\frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{a}\right) = \frac{x+y+z}{a}$$

$$\boxed{curl\,\vec{F}.\hat{n} = \frac{x + y + z}{a}}$$

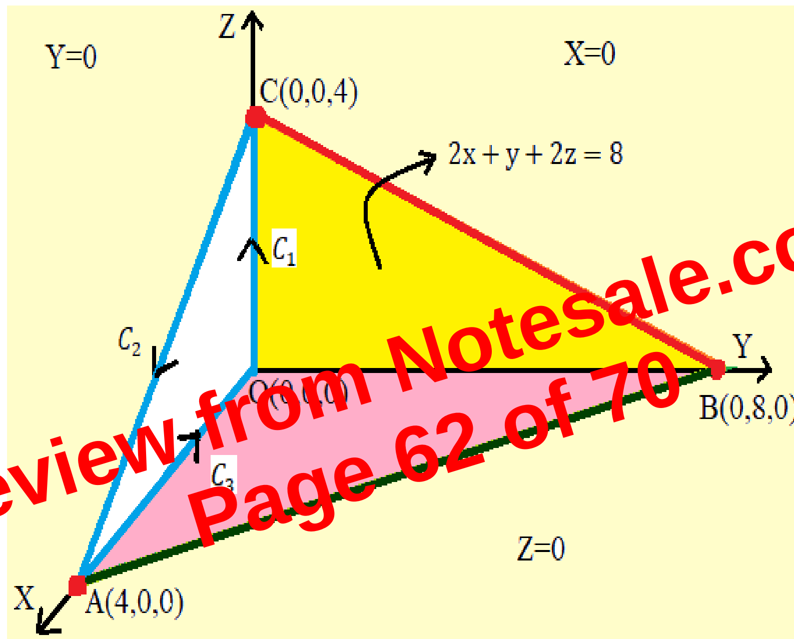
$$\oint_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.\hat{n}\,dS$$

where S is an open surface bounded by a closed surface C.

To evaluate: $\oint_C \vec{F}.d\vec{r}$

$$\oint_C \vec{F}.d\vec{r} = \oint_C (xz\hat{\imath} - y\hat{\jmath} + x^2y\hat{k}).(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k})$$

$= \oint_C (zx\,dx - y\,dy + x^2y\,dz)$



Here the closed curve C consists of three straight lines OB, BA and AO. Let these straight lines are denotes by C_1 , C_2 and C_3 respectively.

$$\therefore \oint_C \vec{F}.d\vec{r} = \oint_{C_1}(zx\,dx - y\,dy + x^2y\,dz) + \oint_{C_2}(zx\,dx - y\,dy + x^2y\,dz)$$

$$+ \oint_{C_3}(zx\,dx - y\,dy + x^2y\,dz)$$

Or, $\oint_C \vec{F}.d\vec{r} = I_1 + I_2 + I_3$ ---------------------------- (1)

We have

$$I_1 = \oint_{C_1} (zx\,dx - y\,dy + x^2y\,dz)$$

Along $C_1 : x = 0, y = 0, dx = 0, dy = 0$ and z varies from 0 to 4.

3. E-resources:

https://www.youtube.com/watch?v=fZ231k3zsAA&t=57s

https://www.youtube.com/watch?v=qOcFJKQPZfo

https://www.youtube.com/watch?v=3TkKm2mwR0Y

https://www.youtube.com/watch?v=ynzRyIL2atU

https://www.youtube.com/watch?v=Cxc7ihZWq5o

https://www.youtube.com/watch?v=vvzTEbp9lrc

https://www.youtube.com/watch?v=3edJPRkCV9k

https://www.youtube.com/watch?v=CHW6krHTtXE

https://www.youtube.com/watch?v=7FUNdFN6ZKI

https://www.youtube.com/watch?v=AqcbyjaSQ10

https://www.youtube.com/watch?v=y-gsqWf3Gms

https://www.youtube.com/watch?v=I1dfwkPY75A

https://www.youtube.com/watch?v=BVAbztfd2JM

https://www.youtube.com/watch?v=ZtQyuN7DdKE

https://www.youtube.com/watch?v=YXf3aKxgELY

https://www.youtube.com/watch?v=eKD6aDwJ2lI

https://www.youtube.com/watch?v=8SwKD5_VL5o