$$\left(\frac{\partial\theta}{\partial y}\right)_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$\left(\frac{\partial\theta}{\partial y}\right)_x = \frac{x}{x^2+y^2}.$$

Example-5: If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

a). $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$, and b) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Solution: Here we have, $u = \log(x^3 + y^3 + z^3 - 3xyz)$ … … … … … … … … (1)

Differentiating equation partially with respect to 'x' we get

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3+y^3+z^3-3xyz)}(3x^2-3yz) \quad \ldots\ldots\ldots\ldots (2)$$

Similarly differentiating equation (1) partially with respect to 'y' and 'z' we get

$$\frac{\partial u}{\partial y} = \frac{1}{(x^3+y^3+z^3-3xyz)}(3y^2-3zx) \quad \ldots\ldots\ldots\ldots (3)$$

$$\frac{\partial u}{\partial z} = \frac{1}{(x^3+y^3+z^3-3xyz)}(3z^2-3xy) \quad \ldots\ldots\ldots\ldots (4)$$

Adding equations (2), (3) & (4) we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{1}{(x^3+y^3+z^3-3xyz)}(3x^2+3y^2+3z^2-3xy-3yz-3zx)$$

$$\boldsymbol{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}} \quad \ldots\ldots\ldots\ldots (5)$$

Now,

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$\boldsymbol{\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}}.$$

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$ and $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$ respectively.

Example-7: If $u = x^3 + y^3$ where $x = acost\,, y = asint$. Find the value of $\frac{du}{dt}$ and hence verify the result.

Solution: Here we have, $u = x^3 + y^3$ ……………………………..(1)

Where, $x = acost\,, y = asint$ ……………………………… (2)

Therefore, u is a composite function in single variable t. So we shall have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now from equations (1) & (2)

$$\frac{\partial u}{\partial x} = 3x^2 \;\; \& \;\; \frac{\partial u}{\partial y} = 3y^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (4)$$

$$\frac{dx}{dt} = -asint \quad \& \quad \frac{dy}{dt} = acost \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Using equations (4) & (5) in equation (3), we get

$$\frac{du}{dt} = 3x^2(-asint) + 3y^2(acost)$$

$$\frac{du}{dt} = 3a^3(sin^2t.cost - cos^2t.sint) \ldots\ldots.... \quad (6)$$

{Putting values of x and y from equation (2)}

Again, from equations (1) and (2), we get

$$u = a^3cos^3t + a^3sin^3t$$

So, differentiating both sides with respect to t we get

$$\frac{du}{dt} = a^3[(3cos^2t).(-sint) + (3sint^2t).cost$$

$$\frac{du}{dt} = 3a^3(sin^2t.cost - cos^2t.sint) \quad \ldots\ldots\ldots\ldots\ldots \quad (7)$$

From equations (6) & (7) result is verified

Example-8: If $u = x^2 - y^2 + sinyz$, where $y = e^x, z = logx$, then find $\frac{du}{dx}$.

Solution: Here we have, $u = x^2 - y^2 + sinyz$ …………………… (1)

Where $y = e^x, z = logx$ ……………………………………………..(2)

Here u will be composite function in single variable x. So we shall have

$$\frac{du}{dx} = \frac{\partial u}{\partial x}.\frac{dx}{dx} + \frac{\partial u}{\partial y}.\frac{dy}{dx} + \frac{\partial u}{\partial z}.\frac{dz}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Now, differentiating equations (1) and (2), we get

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y + zcosyz \,\&\, \frac{\partial u}{\partial z} = ycosyz \quad \ldots\ldots\ldots\ldots (4)$$

$$\frac{dy}{dx} = e^x \& \frac{dz}{dx} = \frac{1}{x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Using equations (4) and (5) in equation (3), we get

$$\frac{du}{dx} = 2x + (-2y + zcosyz)e^x + ycosyz\left(\frac{1}{x}\right)$$

Putting values of y & z from equation (2), we get

$$\frac{du}{dx} = 2x + (-2e^x + logx.cosyz)e^x + e^x\left(\frac{1}{x}\right)cosyz$$

Example-9: If $v = f(2x - 3y, 3y - 4z, 4z - 2x)$ compute the value of $6v_x + 4v_y + 3v_z$.

Solution: Given that $v = f(2x - 3y, 3y - 4z, 4z - 2x)$

Let $v = f(r, s, t)$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

Where $r = 2x - 3y, s = 3y - 4z, t = 4z - 2x$ $\ldots\ldots\ldots (2)$

So, from equation (2), we get

$$\frac{\partial r}{\partial x} = 2, \frac{\partial r}{\partial y} = -3, \frac{\partial r}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{\partial s}{\partial x} = 0, \frac{\partial s}{\partial y} = 3, \frac{\partial s}{\partial z} = -4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\frac{\partial t}{\partial x} = -2, \frac{\partial t}{\partial y} = 0, \frac{\partial t}{\partial z} = 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now, $v_x = \frac{\partial v}{\partial x} = v_x = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}.\frac{\partial t}{\partial x}$

$v_x = \frac{\partial f}{\partial r}(2) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-2)$ Or $v_x = 2\left(\frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}\right)$ $\ldots\ldots\ldots\ldots\ldots (6)$

$$v_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial y} + \frac{\partial v}{\partial t}.\frac{\partial t}{\partial y}$$

$v_y = \frac{\partial f}{\partial r}(-3) + \frac{\partial f}{\partial s}(3) + \frac{\partial f}{\partial t}(0)$ Or $v_y = 3\left(-\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}\right)$ $\ldots\ldots\ldots\ldots (7)$

$$v_z = \frac{\partial v}{\partial z} = \frac{\partial v}{\partial r}.\frac{\partial r}{\partial z} + \frac{\partial v}{\partial s}.\frac{\partial s}{\partial z} + \frac{\partial v}{\partial t}.\frac{\partial t}{\partial z}$$

$v_z = \frac{\partial f}{\partial r}(0) + \frac{\partial f}{\partial s}(-4) + \frac{\partial f}{\partial t}(4)$ Or $v_z = 4\left(-\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t}\right)$ $\ldots\ldots\ldots\ldots (8)$

Practice Exercise:

Q-1: Find all first order derivatives of function $u = \sin^{-1}\frac{x}{y}$.

Ans: $\frac{\partial u}{\partial x} = \frac{y}{\sqrt{y^2-x^2}}$ & $\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}}$.

Q-2: If $z = f(x+ct) + \varphi(x-ct)$, show that $\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$.

Q-3: If $x = e^{r\cos\theta}.\cos(r\sin\theta)$, $y = e^{r\cos\theta}.\sin(r\sin\theta)$, prove that $\frac{\partial x}{\partial r} = \frac{1}{r}\frac{\partial y}{\partial \theta}$, $\frac{\partial y}{\partial r} = -\frac{1}{r}\frac{\partial x}{\partial \theta}$.

Q-4: If $u = u(y-z, z-x, x-y)$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Q-5: $u = f(r,s,t), r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Q-6: If $x + y = 2e^{\theta}\cos\varphi$, $x - y = 2i\, e^{\theta}\sin\varphi$, then show that $\frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial \varphi^2} = 4xy\frac{\partial^2 v}{\partial x \partial y}$.

Q-7: If $z = u^2 + v^2, u = r\cos\theta, v = r\sin\theta$. Find values of $\frac{\partial z}{\partial r}$ & $\frac{\partial z}{\partial \theta}$.

Ans: $\frac{\partial z}{\partial r} = 2r$ & $\frac{\partial z}{\partial \theta} = 0$.

Q-8: If three thermodynamic variables P, V, T are connected by a relation $f(P,V,T) = 0$. Then show that $\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$.

Q-9: If $u = f(x^2 + 2yz, y^2 + 2zx)$, then prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.

Q-10: If $f(x,y) = 0, \varphi(y,z) = 0$. Show that $\frac{\partial f}{\partial y}.\frac{\partial \varphi}{\partial z}.\frac{dz}{dx} = \frac{\partial f}{\partial x}.\frac{\partial \varphi}{\partial y}$.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n\varphi\left(\frac{y}{x}\right) \quad or\ \boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu}$$

Note: Similarly, if $u=f(x,y,z)$ is a homogeneous function in three variables x, y, z of degree n then by Euler's theorem we shall have,

$$\boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=nu}$$

Example-1: If $u=(\sqrt{x}+\sqrt{y})^5$, then find value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

Solution: Here we have $u=(\sqrt{x}+\sqrt{y})^5$ … … … … … … … … … … … … … … … … … … … (1)

From (1) we have $u=(\sqrt{x})^5\left(1+\sqrt{\frac{y}{x}}\right)^5$ or $u=x^{\frac{5}{2}}\left(1+\sqrt{\frac{y}{x}}\right)^5$

This is of form $u=x^n\varphi\left(\frac{y}{x}\right)$, so given function u is a homogeneous function in variables x, y of degree $n=\frac{5}{2}$.Therefore from Euler's theorem for homogeneous function of degree n, we shall have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \quad \boldsymbol{or} \quad \boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{5}{2}u}$$

Example-2: If $u=f\left(\frac{y}{x}\right)$, then find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$.

Solution: Here $u=f\left(\frac{y}{x}\right)$, so it is expressible as $u=x^0f\left(\frac{y}{x}\right)$

Therefore, u is a homogeneous function in variables x, y of degree $n=0$.

Hence by Euler's theorem for homogeneous functions

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,u \quad \boldsymbol{or} \quad \boldsymbol{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0}$$

Example-3: Verify Euler's theorem for homogeneous function for $u=x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)$.

Solution: Here we have $u=x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)$ … … … … … … … … … … … … … (1)

Replacing $x\ by\ \mu x\ \&\ y\ by\ \mu y$ in equation (1) we get

$$u=\mu^{\left(\frac{1}{3}-\frac{4}{3}\right)}x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right) \quad or \quad u=\mu^{(-1)}x^{\frac{1}{3}}y^{-\frac{4}{3}}\tan^{-1}\left(\frac{y}{x}\right)$$

Clearly u is a homogeneous function in variables x, y of degree -1.

Therefore, from Euler's theorem on homogeneous function, we must have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-u \quad \dots \dots \dots \quad (2)$$

Now differentiating equation (1) partially with respect to x we get

Practice Exercise:

Q-1: State Euler's theorem for homogeneous function and verify it for $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$.

Q-2: Verify Euler's theorem for $z = \frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$.

Q-3: If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8++y^8+z^8}}\right)$. show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\, tanu = 0$.

Q-4: Apply Euler's theorem for homogeneous function for $u = (x^2 - 2xy + y^2)^{\frac{3}{2}}$ to evaluate values of (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ and (ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Ans: (i) 3u and (ii) 6u.

Q-5: If $u = \cos^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, then show that $xu_x + yu_y + zu_z = -2\, cotu$.

Q-6: If $u = \csc^{-1}\sqrt{\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}}$, then evaluate $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Ans: $-\frac{1}{12}\boldsymbol{tanu}$.

Q-7: If $u = \tan^{-1}\frac{y^2}{x}$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u\, \sin^2 u$.

At $(\boldsymbol{a},\boldsymbol{a}) rt - s^2 = 27a^2 > 0$

$f(x,y)$ has extreme value at $(0,0)$

$$r = 6a$$

if $a > 0, r > 0$ so that $f(x,y)$has a minimum value at (a,a) and minimum value = $-a^3$

if $a > 0, r < 0$ so that $f(x,y)$has a maximum value at $(a,a)$and maximum value = a^3

Example 3: Examine for minimum and maximum values $sin\,x + sin\,y + sin\,(x + y)$

Solution: Here $f(x,y) = sin\,x + sin\,y + sin\,(x + y)$

$$f_x = \cos x + \cos(x + y)$$

$$f_y = \cos y + \cos(x + y)$$

$$r = f_{xx} = -\sin x - \sin(x + y)$$

$$s = f_{xy} = \cos x - \sin(x + y)$$

$$t = f_{yy} = -\sin y - \sin(x + y)$$

By equating the first order partial derivative to zero

$\cos x + \cos(x + y) = 0$ ……………………………(1)

$\cos y + \cos(x + y) = 0$ ……………………………(2)

By subtracting equation (2) from (1)

$cos\,x = cos\,y\ or\ x = y$

put the value in equ (1) $cos\,2x = -cosx = cos(\pi - x)$

$x = \frac{\pi}{3} = y$

The stationary point is $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

At $\left(\frac{\pi}{3},\frac{\pi}{3}\right) r = -\sqrt{3}, s = \frac{\sqrt{3}}{2}, t = -\sqrt{3}$

$rt - s^2 = \frac{9}{4} > 0\ and\ r < 0$

So $f(x,y)$ has a maximum value at $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$ and maximum value is$\frac{3\sqrt{3}}{2}$.

Example 4: A balloon is in the form of right circular cylinder of radius 1.5m and length 4m and is surmounted by hemispherical ends. If the radius in increased by .001m and length by .05m. Find the percentage change in the volume of balloon.

Solution: Given that $r = 1.5, h = 4, \delta r = .001$ and $\delta h = .05$

Volume of the cone $V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$

$$\delta V = \pi . 2r\delta r . h + \pi r^2 \delta h + \frac{4}{3}\pi . 3r^2 \delta r$$

$$\delta V = \pi r(2h\delta r + r\delta h + 4r\delta r)$$

$$\frac{\delta V}{V} = \frac{\pi r(2h\delta r + r\delta h + 4r\delta r)}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

Substituting the values of $r, h, \delta r, \delta h$, we get

$$\frac{\delta V}{V} = \frac{.215}{9}$$

Percentage error, $\frac{\delta V}{V}.100 = \frac{.215}{9}.100 = 2.389\%$

Example 5: What error in the common logarithm of a number will be produced by an error of 1% in the number?

Solution: Let x be any number and

$$y = log_{10} x$$

Then $\delta y = \frac{1}{x} log_{10} e(\delta x) = \left(\frac{\delta x}{x}.100\right)\left(\frac{1}{100} log_{10} e\right) = \frac{1}{100} log_{10} e$

$$\delta y = \frac{.43429}{100} = .0043429$$

Example 6: In estimating the number of bricks in a pile which is measured to be $(5m. 10m. 5m)$, the count of bricks is taken as 100 bricks per m^3. Find the error in the cost when the tape is stretched 2% beyond its standard length. The cost of bricks is 2000 rs. per thousand bricks.

Solution: Let x, y and z be the length, breadth, and height of the pile.

Volume of pile, $V = xyz = (5m. 10m. 5m) = 250$

Taking log on both sides

$$logV = logx + logy + logz$$

Differentiating $\frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$

E- Link for more understanding

1. https://www.youtube.com/watch?v=XzaeYnZdK5o&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=1
2. https://www.youtube.com/watch?v=9-tir2V3vYY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=14
3. https://www.youtube.com/watch?v=aqfSOOiO2kI&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=15
4. https://www.youtube.com/watch?v=GoyeNUaSW08&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=16
5. https://www.youtube.com/watch?v=jiEaKYI0ATY&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=17
6. https://www.youtube.com/watch?v=G0V_yp0jz5c&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=18
7. https://www.youtube.com/watch?v=G0V_yp0jz5c&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=18
8. https://www.youtube.com/watch?v=McT-UsFx1Es&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=9
9. https://www.youtube.com/watch?v=XzaeYnZdK5o&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=1
10. https://www.youtube.com/watch?v=btLWNJdHzSQ&list=PLtKWB-wrvn4nA2h8TFxzWL2zy8O9th_fy&index=12
11. https://www.youtube.com/watch?v=pgfcur31PTY
12. https://www.youtube.com/watch?v=hJ0FMHVZVSc
13. https://www.youtube.com/watch?v=6iTAY9i_v9E
14. https://www.youtube.com/watch?v=NpR91wexqHA
15. https://www.youtube.com/watch?v=gLWUrF_cOwQ
16. https://www.youtube.com/watch?v=pAb1autRHGA
17. https://www.youtube.com/watch?v=HeKB72M2Puw
18. https://www.youtube.com/watch?v=eTp5wq-cSXY
19. https://www.youtube.com/watch?v=6tQTRlbkbc8
20. https://www.youtube.com/watch?v=8ZAucbZscNA