$$= \int_0^1 \frac{1}{\sqrt{(1-y^2)}} [sin^{-1}x]_0^1 \, dy$$

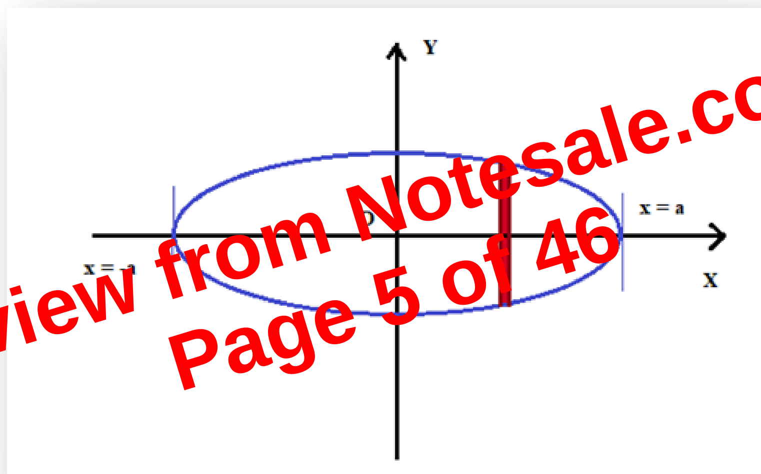
$$= \int_0^1 \frac{1}{\sqrt{(1-y^2)}} \frac{\pi}{2} dy$$

$$= \frac{\pi}{2} [sin^{-1}y]_0^1 = \frac{\pi}{2} . \frac{\pi}{2} = \frac{\pi^2}{4} \text{ (Answer)}$$

Example 2: Evaluate $\iint (x+y)^2 \, dxdy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: (Here we have area bounded by the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, depending on variables x and y so we have to construct a strip parallel to any one axis to observe variable limits of one variable.)

For the ellipse we may write $\frac{y}{b} = \pm\sqrt{1 - \frac{x^2}{a^2}}$ or $y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$



∴ The region of integration R can be expressed as

$-a \leq x \leq a, \; -\frac{b}{a}\sqrt{a^2 - x^2} \leq y \leq \frac{b}{a}\sqrt{a^2 - x^2}$,

where we have chosen variable limits of y and constant limits of x.

So first we will integrate w.r.t. y,

$$\therefore \iint (x+y)^2 \, dxdy = \iint_R (x^2 + y^2 + 2xy) dxdy$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\left(\cos^2\phi + \sin^2\phi\right) = r.$$

$$\Rightarrow \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r\cos\phi, r\sin\phi, z)\,r\,dr\,d\phi\,dz.$$

- **For full volume of the cylinder** $x^2 + y^2 = a^2 \,\&\, z = b \,to\, z = c; 0 \le r \le a, b \le z \le c, 0 \le \phi \le 2\pi.$
- **For first(positive) octant of the cylinder**

  $x^2 + y^2 = a^2 \,\&\, z = b \,to\, z = c; 0 \le r \le a, b \le z \le c, 0 \le \phi \le \pi/2.$

4.3.4.5 Solved examples

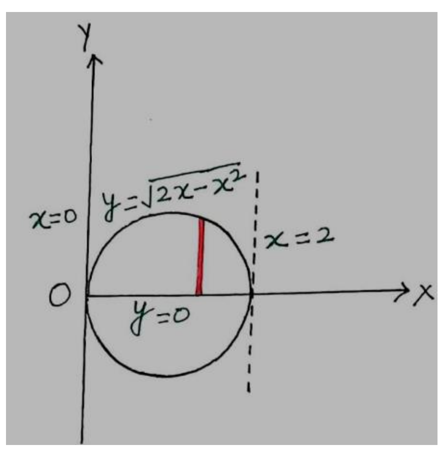
Example 1: Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing into polar co-ordinates.

Solution: In the given integral

x varies from 0 to 2

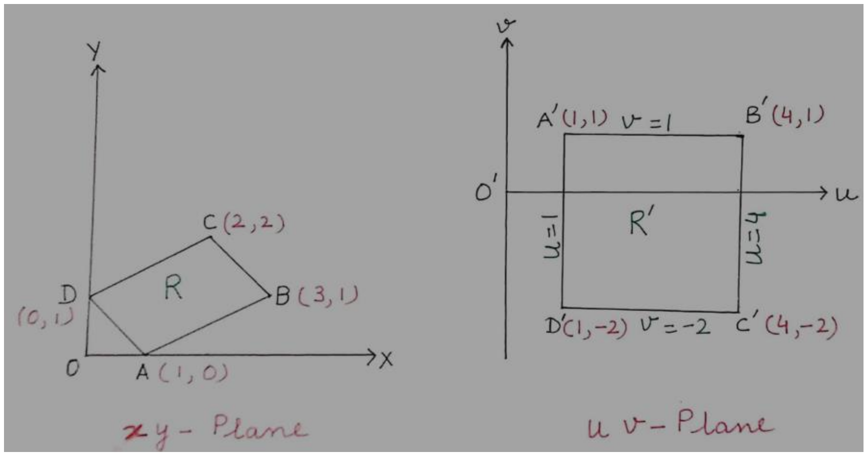
y varies from 0 to $\sqrt{2x - x^2}$

Now, $y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2 \; or \; x^2 + y^2 = 2x$.



In polar co-ordinates, we have $x = r\cos\theta; y = r\sin\theta$,

Solution: The region R in *xy*-plane i.e., parallelogram ABCD with vertices A (1,0), B (3,1), C (2,2), D (0,1) becomes region $R^{'}$ in *uv*-plane i.e., rectangle $A^{'}B^{'}C^{'}D^{'}$ with vertices $A^{'}$ (1,1), $B^{'}$ (4,1), $C^{'}$ (4,-2), $D^{'}$ (1,-2).



Solving the given equations for x and y, we get $x=\frac{1}{3}(2u+v),\ y=\frac{1}{3}(u-v)$.

Here

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{vmatrix}=-\frac{1}{3}.$$

$$\therefore \iint_R (x+y)^2\,dx\,dy=\iint_{R'} u^2\,|J|\,du\,dv=\int_{-2}^{1}\int_{1}^{4} u^2\frac{1}{3}\,du\,dv=\int_{-2}^{1} 7\,dv=21\ (\text{Answer}).$$

Example 6: Using the transformation $x-y=u,\ x+y=v,$ show that $\iint_R \sin\left(\frac{x-y}{x+y}\right)dx\,dy=0,$ where R is the region bounded by the co-ordinate axes and $x+y=1$ in first quadrant.

Solution: Here, region R is a triangle OAB in *xy-plane* having sides $x=0,\ y=0$ and $x+y=1$.

Also

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=2.$$

Using given transformation, we get

If $x=0,\ y=0$ then $u=-v,\ u=v.$

If $x+y=1$ then $v=1.$

Thus corresponding region $R^{'}$ in *uv*-plane is a triangle OPQ bounded by $u=-v,\ u=v,\ v=1.$

$$A = 2\int_{\theta=0}^{\pi/2}\int_{r=parabola}^{r=cardioid} r\,dr\,d\theta$$

$$A = 2\int_{0}^{\pi/2}\left(\frac{r^2}{2}\right)_{\frac{1}{1+\cos\theta}}^{1+\cos\theta} d\theta$$

$$A = \int_{0}^{\pi/2}\left[(1+\cos\theta)^2 - \frac{1}{(1+\cos\theta)^2}\right]d\theta$$

$$A = \int_{0}^{\pi/2}(1+\cos\theta)^2\,d\theta - \int_{0}^{\pi/2}\frac{1}{(1+\cos\theta)^2}\,d\theta$$

$$A = \int_{0}^{\pi/2}\left(1+\cos^2\theta + 2\cos\theta\right)d\theta - \frac{1}{4}\int_{0}^{\pi/2}\sec^4\frac{\theta}{2}\,d\theta$$

Let $I_1 = \int_{0}^{\pi/2}\left(1+\cos^2\theta + 2\cos\theta\right)d\theta$

and $I_2 = \frac{1}{4}\int_{0}^{\pi/2}\sec^4\frac{\theta}{2}\,d\theta$, then

$$I_1 = \int_{0}^{\pi/2}\left(1+\cos^2\theta + 2\cos\theta\right)d\theta$$

$$I_1 = \frac{\pi}{2} + \frac{1}{2}.\frac{\pi}{2} + 2 = \frac{3\pi}{4} + 2$$

$$I_2 = \frac{1}{4}\int_{0}^{\pi/2}\sec^4\frac{\theta}{2}\,d\theta$$

$$I_2 = \frac{1}{4}\int_{0}^{\pi/4}\sec^4\phi.2d\phi \qquad let\ \frac{\theta}{2} = \phi$$

$$\Rightarrow d\theta = 2d\phi$$

$$I_2 = \frac{1}{2}\int_{0}^{\pi/4}\left(1+\tan^2\phi\right)\sec^2\phi\,d\phi \qquad let\ t = \tan\phi$$

$$\Rightarrow dt = \sec^2\phi d\phi$$

$$I_2 = \frac{1}{2}\int_{0}^{1}\left(1+t^2\right)dt$$

$$I_2 = \frac{1}{2}\left(t + \frac{t^3}{3}\right)_0^1 = \frac{2}{3}$$

Hence required area= $\frac{3\pi}{4} + 2 - \frac{2}{3} = \frac{3\pi}{4} + \frac{4}{3}(Answer)$.

Therefore, required volume

$$V = \int_{y=-2}^{2} \int_{x=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} z\, dx\, dy$$

$$V = 2\int_{-2}^{2} \int_{0}^{\sqrt{4-y^2}} (4-y)dx\, dy$$

$$V = 2\int_{-2}^{2} (4-y)\sqrt{4-y^2}\, dy$$

$$V = 2\int_{-2}^{2} \left(4\sqrt{4-y^2}\right)dy - 2\int_{-2}^{2} y\sqrt{4-y^2}\, dy \qquad \{\text{using the property of odd function}\}$$

$$V = 8\int_{-2}^{2} \sqrt{4-y^2}\, dy = 16\int_{0}^{2} \sqrt{4-y^2}\, dy$$

$$V = 16\left[\frac{y\sqrt{4-y^2}}{2} + 2\sin^{-1}\frac{y}{2}\right]_0^2$$

$$V = 16\left[2\sin^{-1}1\right] = 32 \times \frac{\pi}{2} = 16\pi\,(Answer)$$

Example2: A triangular prism is formed by planes whose equations are $ay = bx$, $y = 0$ and $x = a$. Find the volume of the prism between the planes $z = 0$ and the surface $z = c + xy$.

Solution: The volume of the bounded region

$$V = \int_{x=0}^{a} \int_{y=0}^{\frac{bx}{a}} \int_{z=0}^{c+xy} dz\, dy\, dx$$

$$V = \int_{0}^{a} \int_{0}^{\frac{bx}{a}} (c+xy)\, dy\, dx$$

$$V = \int_{0}^{a} \left(cx + xy^2/2\right)_0^{\frac{bx}{a}} dx$$

$$V = \int_{0}^{a} \left(\frac{cxb}{a} + \frac{b^2}{2a^2}x^3\right)dx$$

$$V = \frac{cb}{a}\left(\frac{x^2}{2}\right)_0^a + \frac{b^2}{2a^2}\left(\frac{x^4}{4}\right)_0^a$$

$$V = \frac{abc}{2} + \frac{a^2b^2}{8}\,(Answer).$$

For paraboloid: $x = r\cos\phi, y = r\sin\phi$ so that $z = \dfrac{r^2}{a}$.

For cylinder : $r^2 = R^2$ or $r = R$.

Now, using the symmetry of the bounded region, the required volume is

$$V = 4\int_{\varphi=0}^{\pi/2}\int_{r=0}^{R}\int_{z=0}^{r^2/a} r\,dz dr\,d\varphi$$

$$V = 4\int_{0}^{\pi/2}\int_{0}^{R} r.r^2/a\,dr\,d\varphi$$

$$V = 4\int_{0}^{\pi/2}\left[\frac{r^4}{4a}\right]_0^R d\varphi = \frac{1}{a}\int_{0}^{\pi/2} R^4 d\varphi = \frac{\pi R^4}{2a}(Answer).$$

Example 5: Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution: It is clear from given equations that the base of one cylinder is on $xy-plane$ and of other is on $xz-plane$. So the volume of the common region of both cylinders is

$$V = \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{z=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$V = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}[z]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$$

$$V = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\,dy\,dx$$

$$V = 4\int_{-a}^{a}\left(a^2 = x^2\right)dx = 8\int_{0}^{a}\left(a^2 = x^2\right)dx = \frac{16a^3}{3}(Answer).$$

Example 6: Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $x^2 + y^2 = z^2$.

Solution: The bounded region lies between the cone $x^2 + y^2 = z^2$ and the sphere $x^2 + y^2 + z^2 = a^2$.

So required volume is

$V = \iiint_V dz\,dy\,dx$.

Here, $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{a^2 - x^2 - y^2}$;

Also the intersection of the given surfaces gives, Changing into Spherical polar coordinates

$$\bar{x} = \frac{\iiint_V \mu x^2 yz\,dx\,dy\,dz}{\iiint_V \mu xyz\,dx\,dy\,dz} = \frac{\iiint_V (r\sin\theta\cos\phi)^2 (r\sin\theta\sin\phi)(r\cos\theta) r^2 \sin\theta dr\,d\theta\,d\phi}{\iiint_V (r\sin\theta\cos\phi)(r\sin\theta\sin\phi)(r\cos\theta) r^2 \sin\theta dr\,d\theta\,d\phi}$$

$$\bar{x} = \frac{\iiint_V r^6 \sin^4\theta\cos\theta\sin\phi\cos^2\phi dr\,d\theta\,d\phi}{\iiint_V r^5 \sin^3\theta\cos\theta\sin\phi\cos^2\phi dr\,d\theta\,d\phi}$$

Now, for the volume in positive octant

$$\bar{x} = \frac{\iiint_V \mu x^2 yz\,dx\,dy\,dz}{\iiint_V \mu xyz\,dx\,dy\,dz} = \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^6 \sin^4\theta\cos\theta\sin\phi\cos^2\phi dr\,d\theta\,d\phi}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^5 \sin^3\theta\cos\theta\sin\phi\cos^2\phi dr\,d\theta\,d\phi}$$

$$\bar{x} = \frac{\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{r^7}{7}\right]_0^a \sin^4\theta\cos\theta\sin\phi\cos^2\phi dr\,d\theta\,d\phi}{\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{r^6}{6}\right]_0^a \sin^3\theta\cos\theta\sin\phi\cos^2\phi dr\,d\theta\,d\phi}$$

$$\bar{x} = \frac{6a}{7}\frac{\int_0^{\pi/2}\frac{3.1}{5.3.1}\sin\phi\cos^2\phi d\phi}{\int_0^{\pi/2}\frac{2}{4.2}\sin\phi\cos\phi d\phi} = \frac{24a}{35}.\frac{2}{3} = \frac{16a}{35}.$$

Using symmetry,

$\bar{x} = \frac{16a}{35}, \bar{y} = \frac{16a}{35}, \bar{z} = \frac{16a}{35}$. Therefore, the position of centre of gravity (C.G.) = $\left(\frac{16a}{35}, \frac{16a}{35}, \frac{16a}{35}\right)$ (Answer).

4.4.3.4 Practice problems

1. Find the mass of a lamina in the form of the cardioid $r = a(1+\cos\theta)$ whose density at any point varies as the square of its distance from the initial line. Ans: $\frac{21}{32}\mu\pi a^4$

2. A plate in the form of a quadrant of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{a}\right)^2 = 1$ is of small but of varying thickness, the thickness at any point being proportional to the product of the distance of that point from the major and the minor axes. Find the coordinates of the centre of gravity of the plane. Ans: $\left(\frac{8a}{15}, \frac{8b}{15}\right)$