Rules of Integration

Rule 1. Constant functions

$$\int k dx = kx + c, \quad \text{where } k \text{ is any constant.}$$

Example

Evaluate the following integrals

(i) $\int -7 dx$

(ii) $\int \frac{1}{3} dx$

(iii) $\int \sqrt{3}\, dx$

(iv) $\int -\frac{10}{7} dx$

(v) $\int 0 dx$

(vi) $\int \frac{\sqrt{3}}{2} dx$

Solution:

(i) $\int -7 dx = -7x + c$

(ii) $\int \frac{1}{3} dx = \frac{1}{3} x + c$

(iii) $\int \sqrt{3}\, dx = \sqrt{3}x + c$

(iv) $\int -\frac{10}{7} dx = -\frac{10}{7} x + c$

(v) $\int 0 dx = 0x + c = c$

(vi) $\int \frac{\sqrt{3}}{2} dx = \frac{\sqrt{3}}{2} x + c$

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Rule 2: Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \text{ provided that } n \neq -1.$$

Example

Evaluate the following integrals

(i) $\int x dx$

(ii) $\int x^2 dx$

(iii) $\int \sqrt{x}dx$

(iv) $\int \frac{1}{x^4}dx$

(v) $\int \sqrt[3]{x}dx$

(vi) $\int \sqrt[7]{x}dx$

(vii) $\int \frac{1}{\sqrt{x}}dx$

Solution:

(i) $\int xdx = \frac{x^{1+1}}{1+1} + c = \frac{x^2}{2} + c$

(ii) $\int x^2dx = \frac{x^{2+1}}{2+1} + c = \frac{x^3}{3} + c$

(iii) $\int \sqrt{x}dx = \int x^{1/2}dx = \frac{x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c$

(iv) $\int \frac{1}{x^4}dx = \int x^{-4}dx = \frac{x^{-4+1}}{-4+1} + c = \frac{-1}{3}x^{-3} + c$

(v) $\int \sqrt[3]{x}dx = \int x^{1/3}dx = \frac{x^{1/3+1}}{1/3+1} + c = \frac{x^{4/3}}{4/3} + c = \frac{3}{4}x^{4/3} + c$

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(vi) $\int \sqrt[7]{x}dx = \int x^{1/7}dx = \frac{x^{1/7+1}}{1/7+1} + c = \frac{x^{8/7}}{8/7} + c = \frac{7}{8}x^{8/7} + c$

(vii) $\int \frac{1}{\sqrt{x}}dx = \int \frac{1}{x^{1/2}}dx = \int x^{-1/2}dx = \frac{x^{-1/2+1}}{-1/2+1} + c = \frac{x^{1/2}}{1/2} + c = 2x^{1/2} + c$

Rule 3: Constant Times a Function

$$\int kf(x)dx = k\int f(x)dx$$, where k is any constant.

Example

Evaluate the following integrals

(i) $\int 4xdx$

(ii) $\int \frac{x^2}{5}dx$

(iii) $\int 3\sqrt{x}dx$

Evaluate the following.

(i) $\int \frac{2x+3}{x^2+3x+7}dx$

(ii) $\int \frac{dx}{\sqrt{x}+x}$

(iii) $\int \frac{x^2}{\sqrt[5]{3x^3+7}}dx$

(iv) $\int x^4\left(1+x^5\right)^{1/3} dx$

(v) $\int 2xe^{x^2} dx$

(vi) $\int x^2 e^{3x^3} dx$

(vii) $\int x^2\sqrt{x^3+5}\, dx$

(viii) $\int 2x\sqrt{1+x^2}\, dx$

(ix) $\int \frac{1}{\sqrt{2x+1}}dx$

(x) $\int e^x\sqrt{1+e^x}\, dx$

Solution:

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(i) $\int \frac{2x+3}{x^2+3x+7}dx$

Let $u = x^2+3x+7$, then $\frac{du}{dx} = 2x+3$ or $du = (2x+3)dx$

$$\int \frac{2x+3}{x^2+3x+7}dx = \int \frac{du}{u} = \ln u + c = \ln\left(x^2+3x+7\right)+c$$

(ii) $\int \frac{dx}{\sqrt{x}+x} = \int \frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)}$

Let $u = 1+\sqrt{x}$, then $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ or $2du = \frac{dx}{\sqrt{x}}$

$$\int \frac{du}{\sqrt{x}\left(1+\sqrt{x}\right)} = \int \frac{2du}{u} = 2\int \frac{1}{u}du = 2\ln u + c = 2\ln\left(1+\sqrt{x}\right)+c$$

(iii) $\int \frac{x^2}{\sqrt[5]{3x^3+7}}dx = \int \frac{x^2}{\left(3x^3+7\right)^{1/5}}dx$

Let $u = 3x^3+7$, then $\frac{du}{dx} = 9x^2$ or $du = 9x^2dx$ or $\frac{du}{9} = x^2dx$

(ix) $\int \frac{1}{\sqrt{2x+1}}dx = \int \frac{1}{(2x+1)^{1/2}}dx$

Let $u = 2x+1$, then $\frac{du}{dx} = 2$ or $\frac{du}{2} = dx$

$$\int \frac{1}{\sqrt{2x+1}}dx = \int \frac{1}{(2x+1)^{1/2}}dx = \int \frac{1}{u^{1/2}}\frac{du}{2} = \frac{1}{2}\int u^{-1/2}du = \frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right] + c = u^{1/2} + c$$

$$= (2x+1)^{1/2} + c$$

(x) $\int e^x\sqrt{1+e^x}\,dx = \int e^x(1+e^x)dx$

Let $u = 1+e^x$, then $\frac{du}{dx} = e^x$ or $du = e^x dx$

$$\int e^x\sqrt{1+e^x}\,dx = \int e^x(1+e^x)^{1/2}dx = \int u^{1/2}du = \frac{2}{3}u^{3/2} + c = \frac{2}{3}(1+e^x)^{3/2} + c$$

Exercise

Evaluate

(i) $\int \frac{\ln x}{x}dx$

(ii) $\int 4x^3e^{x^4}dx$

(iii) $\int (2x+1)e^{(x^2+x)}dx$

(iv) $\int x(x^2-3)^4dx$

(v) $\int x\sqrt{1-x^2}dx$

(vi) $\int \sqrt{x}\sqrt{1+x^{3/2}}dx$

(vii) $\int x^3(x^4+2)^2dx$

(viii) $\int e^x(1-e^x)^3dx$

(ix) $\int xe^{x^2+1}dx$

(x) $\int \frac{x^2}{(x^3+8)^4}dx$

(xi) $\int \frac{dx}{e^x+1}dx$

(xii) $\int \frac{1}{x(1+\ln x)}dx$

(xiii) $\int (e^x+e^{-x})^2(e^x-e^{-x})dx$

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Revenue function

The total revenue function $R(x)$ is given, then the marginal revenue function MR, is the derivative of the total revenue function i.e., $MR = \frac{dR}{dx}$. Since integration is the inverse of differentiation, therefore, the total revenue function is the integral of the marginal revenue function,

$$\text{i.e. } R(x) = \int MR dx + k$$

where k is the arbitrary constant of integration, which can be evaluated from the fact that the total revenue $R(x)$ is zero when the output is zero.

Also we know that $R(x) = px$, where p is the price $\Rightarrow p = \frac{R(x)}{x}$, which is the demand function.

Example

1. The marginal cost function of manufacturing x shoes is $6 + 10x - 6x^2$. The total cost of producing a pair of shoes is £12. Find the total and average cost function.

Solution:

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We know that

$$MC = \frac{d}{dx}(TC)$$

Thus $TC = \int MC dx + k$

$$= \int (6 + 10x - 6x^2) dx + k$$

$$\therefore C(x) = 6x + \frac{10}{2}x^2 - \frac{6}{3}x^3 + k$$

Where k is the constant of integration.

Also we are given when $x = 2$, then $C(x) = 12$

$$\therefore 12 = 6 \times 2 + 10 \times \frac{4}{2} - \frac{6}{3} \times 2^3 + k$$

$$12 = 12 + 20 - 16 + k$$

$$k = -4$$