Unit -2

Syllabus:

Laplace transform, Existence theorem, Laplace transforms of derivatives and integrals, Initial and final value theorems, Unit step function, Dirac- delta function, Laplace transform of periodic function, Inverse Laplace transform, Convolution theorem, Application to solve simple linear and simultaneous differential equations.

CO-2. The basic knowledge of Laplace Transforms and its applications in solving differential equations.

Index:

Preview from Notesale.co.uk Page 1 of 46

Example 7. Compute $L\{t^2+1\}^2$

Solution $= L\{t^4+1+2t^2\}$

$= \int_0^\infty e^{-st}.t^4\,dt + \int_0^\infty e^{-st}.1\,dt +$
$2\int_0^\infty e^{-st}.t^2\,dt$

$=\frac{4}{s^5}+\frac{1}{s}+\frac{4}{s^3}$

EXERCISE

1. L{ 3t - 5} Ans. $f(s) = \frac{3}{s^2} - 5s$
2. L{ $2t^3 - 6t + 8$ } Ans. $f(s) = \frac{12}{s^4} - \frac{6}{s^2} + \frac{8}{s}$
3. L { 6 sin 2t – 5 cos 2t } Ans. $f(s) = \frac{12-5s}{s^2+4}$
4. L{ $(t^2 + 1)^2$ } Ans. $f(s) = s^4 + 4s^2 + \frac{24}{s^5}$
5. L{ 2 $e^{3t} - e^{-3t}$ } Ans $f(s) = \frac{s+9}{s^2-9}$
6. L{ cosh at – cos at } Ans $f(s) = \left(\frac{2a^2s}{s^4-a^4}\right)$

1.5 Some basic formula of Laplace Transform:

S No.	F(t)	Laplace Transform
01.	L{f(t)} = f(s)	$L^{-1}\{f(s)\} = f(t)$
02.	1	$\frac{1}{s}$
03.	t	$\frac{1}{s^2}$
04.	t^2	$\frac{2!}{s^3}$
05.	t^n n = 0,1,2,3,...	$\frac{n!}{s^{n+1}}$
06.	e^{at}	$\frac{1}{s-a}$
07.	e^{-at}	$\frac{1}{s+a}$
08	$\frac{t^{(n-1)}\,e^{at}}{(n-1)!}$	$\frac{1}{(s-a)^n}$ n = 1,2,3,....
09.	$\frac{t^{(k-1)}\,e^{at}}{\Gamma(k)}$	$\frac{1}{(s-a)^k}$ k = 1,2,3,.....
10.	sin at	$\frac{a}{s^2+a^2}$

Preview from Notesale.co.uk
Page 6 of 46

$$\lim_{t \to 0} F(t) = \lim_{s \to \infty} s.\frac{(s+1)}{(s+2)}$$

Initial value theorem is not applicable in this case.

Note: This theorem is applicable strictly if F(s) is proper fraction i.e. the numerator polynomial is of lower order than the Denominator Polynomial.

Example 3:- If $L\{F(t)\} = \frac{2s+51}{47s^2+67s}$ find

$\lim_{t \to \infty} F(t)$

Solution: By Final Value theorem

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} s.L\{F(t)\}$$

$$= \lim_{s \to 0} s.\left(\frac{2s+51}{47s^2+67s}\right)$$

$$= \lim_{s \to 0} s.\frac{s}{s}\left(\frac{2s+51}{47s+67}\right)$$

$$= \frac{51}{67}$$

1.17 Laplace Transform of integrals:

Preview from Notesale.co.uk
Page 18 of 46

Theorem: If $L\{F(t)\} = f(s)$ then

$$L\left\{\int_0^t F(t)dt\right\} = \frac{1}{s}f(s)$$

Proof: Let $G(t) = \int_0^t F(t)dt$, then

$G'(t) = F(t)$ and $G(0) = 0$

Taking Laplace Transform both side, we get

$$L\{G'(t)\} = sL\{G'(t)\} - G(0) = sL\{G(t)\}$$

$$L\{F(t)\} = L\left\{\int_0^t F(t)dt = \frac{1}{s}f(s)\right.$$

EXAMPLES

Example 1: Find the Laplace Transform of $\int_0^t \frac{\sin t}{t} dt$

Solution: $L\left\{\int_0^t F(t)dt\right\} = \frac{L\{F(t)\}}{s}$ [Laplace Transform of Integrals]

Here $F(t) = \frac{\sin t}{t}$

$$L\left\{\int_0^t \frac{\sin}{t} dt\right\} = \frac{L\left(\frac{\sin t}{t}\right)}{s} = \frac{Cot^{-1}s}{s}$$

Example 2: Find Laplace Transform of $\int_0^t e^t \frac{\sin t}{t} dt$

Solution: We know

$L\left\{\int_0^t F(t)dt\right. = \frac{f(s)}{s}$ (Laplace transform of integrals)

where $f(s) = L\{F(t)\} = L\left\{\frac{e^t \sin t}{t}\right\}$

$$= \left(L\left\{\frac{\sin t}{t}\right\}\right)_{s\to s-1}$$

$$= \left(Cot^{-1} s\right)_{s\to s-1}$$

$$= Cot^{-1}(s-1)$$

Hence $L\int_0^t \frac{e^t \sin t}{t} dt = \frac{\cot^{-1}(s-1)}{s}$

Preview from Notesale.co.uk
Page 19 of 46

EXERCISE

1. Find initial value of the transformed function.

   $F(s) = \frac{0.9(s+1)}{2.1s^2+5s+16}$ Ans $\frac{0.9}{2.1}$

2. Find final value of the function.

   $F(s) = \frac{3}{s(s-2)}$ Ans. we can't apply **Final Value Theorem**

3. Find the final values of the given $F(s) = \frac{9s}{s(5s+9)}$ Ans.1
4. Find initial value function of $F(s) = \frac{s}{s^2+2s^4}$ Ans.0

$= e^{-4s} L\,(t^2 + 16 + 8t)$

$= e^{-4s}(\frac{2}{s^3} + \frac{16}{s} + \frac{8}{s^2})$

1.19. Laplace Transform of Dirac Delta or unit Impulse function:-

So, the **Dirac Delta function** is a **function** that is zero everywhere except one point and at that point it can be thought of as either undefined or as having an “infinite” value.

The unit impulse function is consider as limiting form of the function

$$\delta_\epsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & a \le t \le a+\epsilon \\ 0, & otherwise \end{cases} \qquad \text{where } \epsilon > 0$$

Now Laplace Transform of unit Impulse function

$$L\{\delta_\epsilon(t-a)\{ = \int_a^{a+\epsilon} e^{-st}.\frac{1}{\epsilon}dt$$

$$= \frac{1}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\epsilon}$$

$$= \frac{e^{-as}}{\epsilon s}(1 - e^{-st})$$

As $\epsilon \to 0$ we get

Preview from Notesale.co.uk
Page 21 of 46

$$L\{\delta(t-a)\} = \lim_{\epsilon \to 0} \frac{e^{-as}}{s}\frac{(1-e^{-s\epsilon})}{\epsilon} \quad \text{form } \frac{0}{0} \text{ Apply L Hospital rule}$$

$$= \lim_{\epsilon \to 0}{}' \frac{e^{-as}}{s}\frac{0 + se^{-s\epsilon}}{1}$$

$$= L\{\delta(t-a)\} = e^{-as}$$

Put a = 0 therefore $L\{\delta(t)\} = 1$

i.e Laplace Transform of unit Impulse function is 1.

Remark: Unit impulse function $\delta(t-a)$ is defined as follows

$$\delta(t-a) = \begin{cases} 0\,, & for\ t \neq a \\ \infty\,, & for\ t = a \end{cases}$$

$$= \frac{1}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} + \left(\frac{e^{-st}}{s}\right)_{a/2}^{a}\right]$$

$$= \frac{1}{s\,(1-e^{-as})}\left[1-e^{-\frac{as}{2}}\right]^2$$

$$= \frac{1}{s}\left[\frac{e^{\frac{as}{4}}-e^{-\frac{as}{4}}}{e^{\frac{as}{4}}+e^{\frac{-as}{4}}}\right]$$

$$= \frac{1}{s}\tanh\left(\frac{as}{4}\right)$$

EXERCISE

1. Find the Laplace Transform of $t^3 u(t-3)$

   Ans $\frac{e^{-3s}}{s3}(2+6s+9s^2)$

2. Find the Laplace Transform of unit step function of $f(t)=\begin{Bmatrix} 8 & t<2 \\ 6 & t>2 \end{Bmatrix}$

   Ans. $\frac{8}{s}-\frac{2e^{-2s}}{s}$

3. Find Laplace Transform of unit step function $f(t)=\begin{Bmatrix} 2t & for\ 0\le t\le 5 \\ 10 & for\ t>5 \end{Bmatrix}$

   Ans. $\frac{2}{s^2}(1-e^{-5s})$

4. Find Laplace transform of the Periodic function

   $$f(t)=\frac{kt}{4} \quad for\ \ 0<t<T : f(t+T)=f(t)$$

   Ans. $\frac{K}{s^2T}-\frac{Ke^{-st}}{s(1-e^{-st})}$

5. Find Laplace transform of the Periodic function $f(t)=\begin{cases} t & 0<t<1 \\ 0 & 1<t<2 \end{cases}$ where $F(t)$ has period 2.

   Ans. $\frac{1-e^{-s}(1+s)}{s^2(1-e^{-2s})}$

Preview from Notesale.co.uk
Page 25 of 46

Examples 3: Using Laplace Transformation, solve the differential equation

$$\frac{d^2x}{dt^2}+9x=\cos 2t \qquad \text{If} \qquad x\,(0)=1$$

$$x\left(\frac{\pi}{2}\right)=-1$$

Solution: Solution:- The given equation is $x'' + 9x = \cos 2\,t$

Taking the Laplace transform of both sides, we get

L (x'') + 9 L (x) = L (cos 2t)

$$[s^2\bar{x}-sx(0)-x'(0)]+9\bar{x}=\frac{s}{s^2+4}$$

$(s^2+9)\bar{x}-s-A=\frac{s}{s^2+4}$ Where x' (0) = A Say

$$\bar{x} \quad =\frac{s}{(s^2+4)(s^2+9)}+\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

$$\bar{x}=\frac{1}{s}\left(\frac{s}{s^2+4}-\frac{s}{s^2+9}\right)+\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

Preview from Notesale.co.uk
Page 41 of 46

Taking the inverse Laplace transform of both sides, we get

$$x=\frac{1}{5}L^{-1}\left(\frac{s}{s^2+4}\right)-\frac{1}{5}L^{-1}\left(\frac{s}{s^2+9}\right)+L^{-1}\left(\frac{s}{s^2+9}\right)+AL^{-1}\left(\frac{1}{s^2+9}\right)$$

$$x= \quad \frac{1}{5}Cos2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{A}{3}Sin3t$$

But $x\left(\frac{\pi}{2}\right)=-1$

$$-1= \quad \frac{1}{5}(-1)-\frac{1}{5}(0)+0+\frac{A}{3}(-1)$$

$$A=\frac{12}{5}$$

So, $x=\frac{1}{5}[Cos2t-Cos3t]+Cos3t+\frac{12}{5}Sin3t$