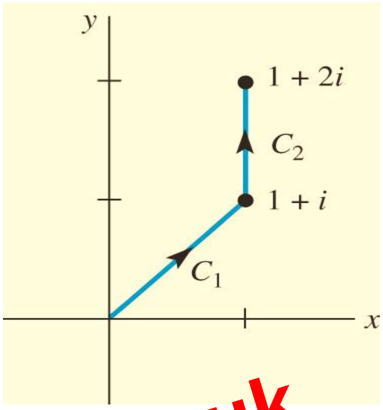
Properties of Contour Integrals

- $\int_C kf(z)\ dz = k\int_C f(z)\ dz$, k is a constant
- $\int_C [f(z) \pm g(z)]\ dz = \int_C f(z)\ dz \pm \int_C g(z)\ dz$
- $\int_C f(z)\ dz = \int_{C_1} f(z)\ dz + \int_{C_2} f(z)\ dz$, where $C \equiv C_1 \cup C_2$
- $\int_{-C} f(z)\ dz = -\int_C f(z)\ dz$, where $-C$ denotes the

  curve having the opposite orientation of C.

Example. Evaluate $\int_C (x^2 + iy^2)dz$, where C is the contour in given figure:

Solution. We have $\int_C (x^2 + iy^2)dz = \int_{C_1}(x^2 + iy^2)dz + \int_{C_2}(x^2 + iy^2)dz$



on C_1: $y = x \Rightarrow dy = dx$ and $x \to 0$ *to* 1

Now, $\int_{C_1}(x^2 + iy^2)dz = \int_0^1 (x^2 + ix^2)(1+i)dx$

$$= (1+i)^2 \int_0^1 x^2 dx = \frac{2}{3}\boldsymbol{i}$$

on C_2: $x = 1 \Rightarrow dx = 0$ and $y \to 1$ *to* 2

$\int_{C_2}(x^2 + iy^2)dz = \int_1^2 (1 + iy^2)idy$

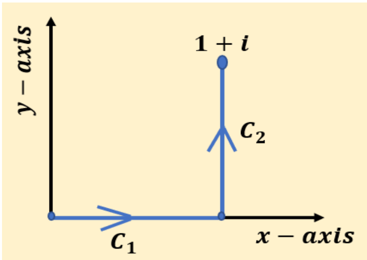
$$= -\int_1^2 y^2 dy + i\int_1^2 dy = -\frac{7}{3} + \boldsymbol{i}$$

Finally, $\int_C (x^2 + iy^2)dz = \frac{2}{3}i + \left(-\frac{7}{3} + i\right) = -\frac{7}{3} + \frac{5}{3}\boldsymbol{i}.$

Example. Evaluate $\int_0^{1+i}(x - y - ix^2)\ dz$, along real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ *to* $z = 1 + i$.

Solution. Let $I = \int_0^{1+i}(x - y - ix^2)\ dz$

$$= \int_{C_1}(x - y - ix^2)dz + \int_{C_2}(x - y - ix^2)dz$$



on C_1: $y = 0 \Rightarrow dy = 0$ and $x \to 0$ *to* 1

Now, $\int_{C_1}(x - y - ix^2)dz = \int_0^1 (x - ix^2)\ dx = \frac{1}{2} - \frac{1}{3}i$

on C_2: $x = 1 \Rightarrow dx = 0$ and $y \to 0$ *to* 1

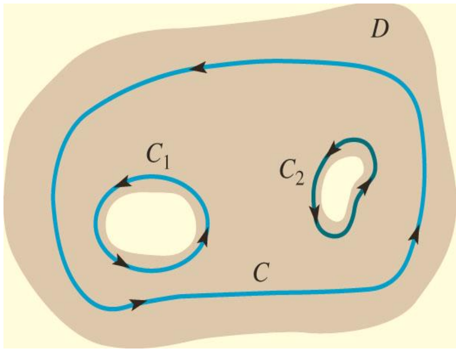
$\int_{C_2}(x - y - ix^2)dz = \int_0^1 (1 - i - y)idy = 1 + \frac{1}{2}i$

$\therefore I = \frac{1}{2} - \frac{1}{3}i + 1 + \frac{1}{2}i = \frac{3}{2} + \frac{1}{6}i.$

Example. Evaluate $\int_C (z - z^2)\ dz$, where C is the upper half of the circle. $|z - 2| = 3$. What is the value of integral if C is the lower half of the circle?

Similarly, for triply connected region, We can show that

$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$



In general, suppose $C, C_1, C_2, \ldots., C_n$ are simple closed curves with a positive orientation such that $C_1, C_2, \ldots., C_n$ are interior to C but the regions interior to each C_k, $k = 1,2,3, \ldots \ldots, n$ have no points in common.

If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k = 1,2,3, \ldots \ldots, n$

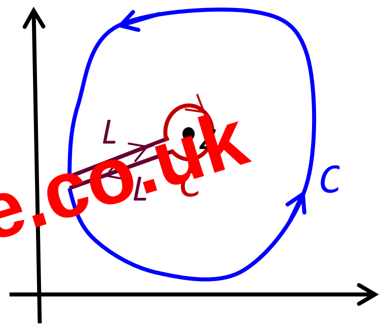
$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$

Cauchy's integral formula

Statement: If $f(z)$ is analytic within and on a closed curve C and z_0 is any point within C, then

$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$ OR $\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i. f(z_0)$

Proof. By Cauchy's theorem, we have



$\oint_C \frac{f(z)}{z-z_0}dz + \oint_{L_1} \frac{f(z)}{z-z_0}dz + \oint_{C_0} \frac{f(z)}{z-z_0}dz + \oint_{L_2} \frac{f(z)}{z-z_0}dz = 0$

$\oint_C \frac{f(z)}{z-z_0}dz = -\oint_{C_0} \frac{f(z)}{z-z_0}dz$

Now,

The equation of the circle C_0 is $|z - z_0| = r$

Or $z - z_0 = re^{i\theta} \Rightarrow dz = rie^{i\theta}d\theta$

$$\therefore \oint_C \frac{f(z)}{z-z_0}dz = \int_{2\pi}^{0} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} rie^{i\theta}d\theta$$

$= i\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta$

$= i\int_0^{2\pi} f(z_0)\, d\theta$ [as $r \to 0$]

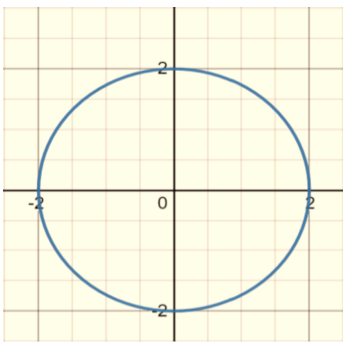
$= 2\pi i. f(z_0)$

OR $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$

Example. Evaluate $\oint_C \frac{z^2-4z+4}{z+i}dz$ where C is the circle $|z| = 2$.

Solution. Let $f(z) = z^2 - 4z + 4$

which is analytic and $z_0 = -i$ is within C. Thus



$\oint_C \frac{z^2-4z+4}{z+i}dz = 2\pi i. f(-i)$

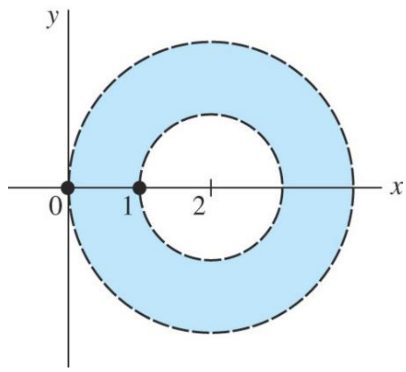
$= 2\pi i. [(-i)^2 - 4(-i) + 4]$

$= 2\pi i. [-1 + 4i + 4]$

$= 2\pi i(3 + 4i) = 2\pi(-4 + 3i)$

Example. Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid $1 < |z-2| < 2$

Solution. Here, $f(z)$ can be written as:



$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z)$$

where, $f_1(z) = -\frac{1}{z} = -\frac{1}{2+z-2} = -\frac{1}{2}\left[\frac{1}{1+\left(\frac{z-2}{2}\right)}\right]$

$$= -\frac{1}{2}\left[1 - \frac{z-2}{2} + \left(\frac{z-2}{2}\right)^2 - \left(\frac{z-2}{2}\right)^3 + \cdots \ldots\ldots\right]$$

$$= -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots \ldots$$

This series converges for $\left|\frac{z-2}{2}\right| < 1$ or $|z-2| < 2$

and

$$f_2(z) = \frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2}\left[\frac{1}{1+\frac{1}{z-2}}\right]$$

$$f_2(z) = \frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2}\left[\frac{1}{1+\frac{1}{z-2}}\right]$$

$$= \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \left(\frac{1}{z-2}\right)^2 - \left(\frac{1}{z-2}\right)^3 + \cdots \ldots\right]$$

$$= \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \cdots \ldots$$

This series converges for $\left|\frac{1}{z-2}\right| < 1$ or $1 < |z-2|$

Example. Expand $f(z) = \frac{8z+1}{z(1-z)}$ in a Laurent series valid for $0 < |z| < 1$

Solution. We can write

$$f(z) = \frac{8z+1}{z(1-z)} = \left[\frac{8z+1}{z}\right]\left[\frac{1}{1-z}\right] = \left(8 + \frac{1}{z}\right)\left[1 + z + z^2 + \cdots \ldots\right]$$

$$= \frac{1}{z} + 9 + 9z + 9z^2 + \cdots \ldots$$

Example. *Expand* $f(z) = \frac{z^3}{(z-1)^2}$ *about* $z = 1$.

Solution. $f(z) = \frac{z^3}{(z-1)^2} = \frac{[(z-1)+1]^3}{(z-1)^2} = \frac{(z-1)^3 + 3(z-1)^2 + 3(z-1) + 1}{(z-1)^2}$

$$= \frac{1}{(z-1)^2} + \frac{3}{z-1} + 3 + (z-1)$$

Example. *Expand* $f(z) = \frac{1}{z^2-1}$ *about* $z = i$.

Solution. $f(z) = \frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{2i+z-i}\right)$

$$= \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{2i} \cdot \frac{1}{1+\frac{z-i}{2i}}\right) = \frac{1}{2i}\frac{1}{z-i} - \frac{1}{(2i)^2}\sum_{n=0}^{\infty}\left(-\frac{1}{2i}\right)^n (z-i)^n$$

$$= -\frac{i}{2}\frac{1}{z-i} + \frac{1}{4} + \frac{i}{8}(z-i) \ldots \ldots \cdots$$

Zero of an Analytic function

The value of z for which the analytic function $f(z)$ becomes zero is said to be the zero of $f(z)$.

An analytic function $f(z)$ is said to have a zero of order m if $f(z)$ is expressible as

$$f(z) = (z-a)^m\,\phi(z), \text{ where } \phi(z) \text{ is analytic and } \phi(a) \neq 0$$

Singularity:

A point z_0 is said to be a singular point or singularity of $f(z)$ if $f(z)$ fails to be analytic at z_0 but is analytic at some point in the neighbourhood of z_0 .

Isolated and Non-Isolated Singularity

If $z = a$ is a singularity of $f(z)$ and if there is no singularity within a small circle surrounding the point $z = a$, then $z = a$ is said to be an isolated singularity of the function $f(z)$, otherwise it is called non-isolated.

Example. Consider the function $f(z) = \frac{z+1}{z(z-2)}$

It is analytic everywhere except at $z = 0$ and $z = 2$. Thus $z = 0$ and $z = 2$ are the only singularities of this function. There are no other singularities of $f(z)$ in the neighbourhood of $z = 0$ and $z = 2$. Hence $z = 0$ and $z = 2$ are the isolated singularities of this function.

Again, consider the function

$$f(z) = \frac{1}{\tan\left(\frac{\pi}{z}\right)} = \cot\left(\frac{\pi}{z}\right)$$

It is not analytic at the points where $\tan\left(\frac{\pi}{z}\right) = 0 = \tan n\pi$ $i.e.$ at the points where $\frac{\pi}{z} = n\pi$

$$\Rightarrow z = \frac{1}{n}\,(1,\ 2,\ 3,\ \ldots\ldots\ldots\ldots)$$

Thus $z = 1, \frac{1}{2}, \frac{1}{3},\ \ldots\ldots\ldots\ldots,\ z = 0$ are the singularities of the function all of which are isolated except $z = 0$ because in the neighbourhood of $z = 0$, there are infinite number of other singularities $z = \frac{1}{n}$ where n is large. Therefore, $z = 0$ is an non-isolated singularity of the given function.

Types of Singularity

If z_0 is an isolated singularity, then in some deleted neighbourhood of z_0 the function is analytic and hence its Laurent's expansion exists as

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n} \quad \text{where } 0 < |z-a| <$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots\ldots\ldots + \frac{b_n}{(z-z_0)^n} + \cdots\ldots$$

The first series on the right side of Laurent expansion is the regular part (Taylor series) while series with negative powers of $(z-z_0)$ is called Principal Part.

(i) Removable Singularity: If in some neighbourhood of an isolated singularity z_0, the Laurent series expansion of the function has no Principal Part (P.P.) then z_0 is called a removable singularity. Moreover, the nomenclature is justified as it can be removed by appropriately defining the function at $z = z_0$.

Note, that the function

$$f(z) = \frac{\sin z}{z},\ z \neq 0\ ; \text{ is undefined at } z = 0.$$

Note that it has a removable singularity at $z = 0$ due to the absence of the Principal Part in the Laurent expansion.

$(z-z_1)(z-z_2)=0 \Rightarrow z=z_1, z_2$ (both are simple)

it is given that $a>|b| \Rightarrow |z_2|>1$

$\because z_1.z_2=-1 \therefore |z_1.z_2|=1$

$\Rightarrow |z_1|.|z_2|=1 \Rightarrow |z_1|<1 \quad [as\ |z_2|>1]$

Hence only z_1 lies inside the unit circle C

Now, $[Res.\ of\ f(z)]_{z=z_1}=\lim\limits_{z\to z_1}(z-z_1)f(z)$

$$=\lim_{z\to z_1}\frac{1}{(z-z_2)}=\frac{1}{z_1-z_2}=\frac{b}{2i\sqrt{a^2-b^2}}$$

$$\therefore \oint_C f(z)dz=2\pi i\left(\frac{b}{2i\sqrt{a^2-b^2}}\right)=\frac{b\pi}{\sqrt{a^2-b^2}}$$

From (1), $I=\frac{2}{b}\left[\frac{b\pi}{\sqrt{a^2-b^2}}\right]=\frac{2\pi}{\sqrt{a^2-b^2}}$

Example. Evaluate $\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}d\theta$

Solution. Let $\mathrm{I}=\int_0^{2\pi}\frac{\cos 3\theta}{5-4\cos\theta}\mathrm{d}\theta$

$$I=Real\ part\ of\ \int_0^{2\pi}\frac{e^{3i\theta}}{5-4\cos\theta}\mathrm{d}\theta \quad = \mathrm{R.P.}\left[\int_0^{2\pi}\frac{e^{3i\theta}}{5-2(e^{i\theta}+e^{-i\theta})}\mathrm{d}\theta\right]$$

put $e^{i\theta}=z$, $d\theta=\frac{dz}{iz}$

$\therefore \mathrm{I}=\mathrm{R.P.}\left[\oint_C\frac{z^3}{5-2(z+z^{-1})}\left(\frac{dz}{iz}\right)\right]$, where $C:|z|=1$

$$=\mathrm{R.P.}\left[-\frac{1}{i}\oint_C\frac{z^3}{2z^2-5z-2}dz\right]=\mathrm{R.P.}\left[-\frac{1}{2i}\oint_C\frac{z^3}{z^2-\frac{5}{2}z-1}dz\right]$$

Let $f(z)=\frac{z^3}{(z-z_1)(z-z_2)}$, where $z_1=\frac{1}{2}$ and $z_2=2$

$\therefore I=\mathrm{R.P.}\left[-\frac{1}{2i}\oint_C f(z)dz\right]$ ---------------- (1)

Poles of $f(z)$ are given by $(z-z_1)(z-z_2)=0 \Rightarrow z=z_1, z_2$

(both are simple) it is very clear that only z_1 lies inside the unit circle C.

Now, $[Res.\ of\ f(z)]_{z=z_1}=\lim\limits_{z\to z_1}(z-z_1)f(z)=\lim\limits_{z\to z_1}\frac{z^3}{(z-z_2)}=\frac{{z_1}^3}{z_1-z_2}=-\frac{1}{12}$

$$\therefore \oint_C f(z)dz=2\pi i\left(-\frac{1}{12}\right)=-\frac{i\pi}{6}$$

From (1), $I=\mathrm{R.P.}\left[-\frac{1}{2i}\left(-\frac{i\pi}{6}\right)\right]=\frac{\boldsymbol{\pi}}{\mathbf{12}}$.

Example. Evaluate $\int_0^{2\pi}\frac{sin^2\theta-2\cos\theta}{2+\cos\theta}d\theta$

Sol. Let $\mathrm{I}=\int_0^{2\pi}\frac{sin^2\theta-2\cos\theta}{2+\cos\theta}d\theta$

$$=\frac{1}{2}\int_0^{2\pi}\frac{2sin^2\theta-4\cos\theta}{2+\cos\theta}d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\frac{1-\cos 2\theta-4\cos\theta}{2+\cos\theta}d\theta$$