Question 8: $\int (1-x)\sqrt{x}\,dx$

$$\int (1-x)\sqrt{x}\,dx$$
$$= \int \left(\sqrt{x} - x^{\frac{3}{2}} \right) dx$$
$$= \int x^{\frac{1}{2}} dx - \int x^{\frac{3}{2}} dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + C$$
$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{2}{5} x^{\frac{5}{2}} + C$$

Question 9: $\int \sqrt{x}\left(3x^2 + 2x + 3\right) dx$

$$\int \sqrt{x}\left(3x^2 + 2x + 3\right) dx$$
$$= \int \left(3x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 3x^{\frac{1}{2}} \right) dx$$
$$= 3\int x^{\frac{5}{2}} dx + 2\int x^{\frac{3}{2}} dx + 3\int x^{\frac{1}{2}} dx$$
$$= 3\left(\frac{x^{\frac{7}{2}}}{\frac{7}{2}} \right) + 2\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}} \right) + 3\frac{\left(x^{\frac{3}{2}} \right)}{\frac{3}{2}} + C$$
$$= \frac{6}{7} x^{\frac{7}{2}} + \frac{4}{5} x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + C$$

Question 10: $\int \left(2x - 3\cos x + e^x\right) dx$

$$\int \left(2x - 3\cos x + e^x\right) dx$$
$$= 2\int x\,dx - 3\int \cos x\,dx + \int e^x dx$$
$$= \frac{2x^2}{2} - 3(\sin x) + e^x + C$$
$$= x^2 - 3\sin x + e^x + C$$

Question 11: $\int \left(2x^2 - 3\sin x + 5\sqrt{x}\right) dx$

$$\int \left(2x^2 - 3\sin x + 5\sqrt{x}\right) dx$$
$$= 2\int x^2 dx - 3\int \sin x\,dx + 5\int x^{\frac{1}{2}} dx$$
$$= \frac{2x^3}{3} - 3(-\cos x) + 5\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$
$$= \frac{2}{3} x^3 + 3\cos x + \frac{10}{3} x^{\frac{3}{2}} + C$$

Question 12: $\int \sec x\left(\sec x + \tan x\right) dx$

$$\int \sec x\left(\sec x + \tan x\right) dx$$
$$= \int \left(\sec^2 x + \sec x \tan x\right) dx$$
$$= \int \sec^2 x\,dx + \int \sec x \tan x\,dx$$
$$= \tan x + \sec x + C$$

Question 13: $\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$

$$\int \frac{\sec^2 x}{\operatorname{cosec}^2 x} dx$$
$$= \int \frac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}} dx$$
$$= \int \frac{\sin^2 x}{\cos^2 x} dx$$
$$= \int \tan^2 x\,dx$$
$$= \int \left(\sec^2 x - 1\right) dx$$
$$= \int \sec^2 x\,dx - \int 1\,dx$$
$$= \tan x - x + C$$

Question 14: $\int \frac{2 - 3\sin x}{\cos^2 x} dx$

$$\int \frac{2 - 3\sin x}{\cos^2 x} dx$$
$$= \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x} \right) dx$$
$$= 2\int \sec^2 x\,dx - 3\int \tan x \sec x\,dx$$
$$= 2\tan x - 3\sec x + C$$

Question 15:Find the anti derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)$

Solution:

$$\left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$$
$$= \int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{2}} dx$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= \frac{2}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

3 mark questions:

If $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$ such that $f(2)=0$, find $f(x)$

Solution: It is given that $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$

∴Anti derivative of $4x^3-\frac{3}{x^4}=f(x)$

$$\therefore f(x)=\int 4x^3-\frac{3}{x^4}\,dx$$

$$f(x)=4\int x^3dx-3\int(x^{-4})dx$$

$$f(x)=4\left(\frac{x^4}{4}\right)-3\left(\frac{x^{-3}}{-3}\right)+C$$

$$\therefore f(x)=x^4+\frac{1}{x^3}+C$$

Also,

$f(2)=0$

$$\therefore f(2)=(2)^4+\frac{1}{(2)^3}+C=0$$

$$\Rightarrow 16+\frac{1}{8}+C=0$$

$$\Rightarrow C=-\left(16+\frac{1}{8}\right)$$

$$\Rightarrow C=\frac{-129}{8}$$

$$\therefore f(x)=x^4+\frac{1}{x^3}-\frac{129}{8}$$

INTEGRATION BY SUBSTITUTION:

ONE MARK QUESTIONS:

1`. Evaluate $\int \tan^2(2x).dx$

$$\int \tan^2(2x).dx=\int(\sec^2 2x-1)dx$$

Solution: $=\frac{1}{2}\tan 2x-x+c$

2. Evaluate $\int \operatorname{cosec}^2\left(\frac{x}{2}\right)dx$.

= $-2\cot\frac{x}{2}+c$

TWO MARK QUESTIONS:

Integrate the following w.r.t x

1. $\frac{2x}{1+x^2}$

Hint: $1+x^2=t$ Ans: $\log(1+x^2)+c$

2. $\frac{(\log x)^2}{x}$

Hint: $\log|x|=t$ $\therefore \frac{1}{x}dx=dt$

$$\Rightarrow \int\frac{(\log|x|)^2}{x}dx=\int t^2dt$$

$$=\frac{t^3}{3}+C$$

$$=\frac{(\log|x|)^3}{3}+C$$

Question 3: $\frac{1}{x+x\log x}$

$$\frac{1}{x+x\log x}=\frac{1}{x(1+\log x)}$$

Let $1+\log x=t$

$$\therefore \frac{1}{x}dx=dt$$

$$\Rightarrow \int\frac{1}{x(1+\log x)}dx=\int\frac{1}{t}dt$$

$$=\log|t|+C$$

$$=\log|1+\log x|+C$$

Question 4: $\sin x\cdot\sin(\cos x)$

$\sin x\cdot\sin(\cos x)$

Let $\cos x=t$

$\therefore -\sin x\,dx=dt$

$$\Rightarrow \int\sin x\cdot\sin(\cos x)dx=-\int\sin t\,dt$$

$$=-[-\cos t]+C$$

$$=\cos t+C$$

$$=\cos(\cos x)+C$$

Question 3: $\cos 2x \cos 4x \cos 6x$

It is known that,

$$\cos A\cos B = \frac{1}{2}\{\cos(A+B)+\cos(A-B)\}$$

$$\therefore \int \cos 2x(\cos 4x\cos 6x)dx = \int \cos 2x\left[\frac{1}{2}\{\cos(4x+6x)+\cos(4x-6x)\}\right.$$

$$= \frac{1}{2}\int\{\cos 2x\cos 10x + \cos 2x\cos(-2x)\}\,dx$$

$$= \frac{1}{2}\int\{\cos 2x\cos 10x + \cos^2 2x\}\,dx$$

$$= \frac{1}{2}\int\left[\left\{\frac{1}{2}\cos(2x+10x)+\cos(2x-10x)\right\}+\left(\right.\right.$$

$$= \frac{1}{4}\int(\cos 12x + \cos 8x + 1 + \cos 4x)\,dx$$

$$= \frac{1}{4}\left[\frac{\sin 12x}{12}+\frac{\sin 8x}{8}+x+\frac{\sin 4x}{4}\right]+C$$

Question 4: $\sin^3(2x+1)$

Let $I = \int \sin^3(2x+1)$

$$\Rightarrow \int \sin^3(2x+1)dx = \int \sin^2(2x+1)\cdot\sin(2x+1)dx$$

$$= \int(1-\cos^2(2x+1))\sin(2x+1)dx$$

Let $\cos(2x+1) = t$

$$\Rightarrow -2\sin(2x+1)dx = dt$$

$$\Rightarrow \sin(2x+1)dx = \frac{-dt}{2}$$

$$\Rightarrow I = \frac{-1}{2}\int(1-t^2)dt$$

$$= \frac{-1}{2}\left\{t - \frac{t^3}{3}\right\}$$

$$= \frac{-1}{2}\left\{\cos(2x+1) - \frac{\cos^3(2x+1)}{3}\right\}$$

$$= \frac{-\cos(2x+1)}{2} + \frac{\cos^3(2x+1)}{6} + C$$

Question 5: $\sin^3 x \cos^3 x$

Let $I = \int \sin^3 x\cos^3 x\cdot dx$

$$= \int \cos^3 x\cdot\sin^2 x\cdot\sin x\cdot dx$$

$$= \int \cos^3 x(1-\cos^2 x)\sin x\cdot dx$$

Let $\cos x = t$

$$\Rightarrow -\sin x\cdot dx = dt$$

$$\Rightarrow I = -\int t^3(1-t^2)dt$$

$$= -\int(t^3 - t^5)dt$$

$$= -\left\{\frac{t^4}{4}-\frac{t^6}{6}\right\}+C$$

$$= -\left\{\frac{\cos^4 x}{4}-\frac{\cos^6 x}{6}\right\}+C$$

$$= \frac{\cos^6 x}{6}-\frac{\cos^4 x}{4}+C$$

Question 6: $\sin x \sin 2x \sin 3x$

It is known that $\sin A\sin B = \frac{1}{2}\{\cos(A-B)-\cos(A+B)\}$

$$\therefore \int \sin x\sin 2x\sin 3x\,dx = \int\left[\sin x\cdot\frac{1}{2}\{\cos(2x-3x)-\cos(2x+3x)\}\right]dx$$

$$= \frac{1}{2}\int(\sin x\cos(-x) - \sin x\cos 5x)\,dx$$

$$= \frac{1}{2}\int(\sin x\cos x - \sin x\cos 5x)\,dx$$

$$= \frac{1}{2}\int\frac{\sin 2x}{2}\,dx - \frac{1}{2}\int \sin x\cos 5x\,dx$$

$$= \frac{1}{4}\left[\frac{-\cos 2x}{2}\right] - \frac{1}{2}\int\left\{\frac{1}{2}\sin(x+5x)+\sin(x-5x)\right\}dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\int(\sin 6x + \sin(-4x))\,dx$$

$$= \frac{-\cos 2x}{8} - \frac{1}{4}\left[\frac{-\cos 6x}{3}+\frac{\cos 4x}{4}\right]+C$$

$$= \frac{-\cos 2x}{8} - \frac{1}{8}\left[\frac{-\cos 6x}{3}+\frac{\cos 4x}{2}\right]+C$$

$$= \frac{1}{8}\left[\frac{\cos 6x}{3}-\frac{\cos 4x}{2}-\cos 2x\right]+C$$

Question 7: $\sin 4x \sin 8x$

It is known that $\sin A\sin B = \frac{1}{2}\cos(A-B)-\cos(A+B)$

$$\therefore \int \sin 4x\sin 8x\,dx = \int\left\{\frac{1}{2}\cos(4x-8x)-\cos(4x+8x)\right\}dx$$

$$= \frac{1}{2}\int(\cos(-4x)-\cos 12x)\,dx$$

$$= \frac{1}{2}\int(\cos 4x - \cos 12x)\,dx$$

$$= \frac{1}{2}\left[\frac{\sin 4x}{4}-\frac{\sin 12x}{12}\right]$$

Question 8: $\frac{1-\cos x}{1+\cos x}$

$$\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} \qquad \left[2\sin^2\frac{x}{2} = 1-\cos x \text{ and } 2\cos^2\frac{x}{2} = 1+\cos x\right]$$

$$= \tan^2\frac{x}{2}$$

$$= \left(\sec^2\frac{x}{2}-1\right)$$

$$\therefore \int\frac{1-\cos x}{1+\cos x}dx = \int\left(\sec^2\frac{x}{2}-1\right)dx$$

$$= \left[\frac{\tan\frac{x}{2}}{\frac{1}{2}} - x\right]+C$$

$$= 2\tan\frac{x}{2} - x + C$$

Question 11: $\frac{1}{x\left(x^n+1\right)}$

[Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$]

$\frac{1}{x\left(x^n+1\right)}$ Multiplying numerator and denominator by x^{n-1}, we obtain

$$\frac{1}{x\left(x^n+1\right)}=\frac{x^{n-1}}{x^{n-1}x\left(x^n+1\right)}=\frac{x^{n-1}}{x^n\left(x^n+1\right)}$$

Let $x^n=t \Rightarrow x^{n-1}dx=dt$

$$\therefore \int\frac{1}{x\left(x^n+1\right)}dx=\int\frac{x^{n-1}}{x^n\left(x^n+1\right)}dx=\frac{1}{n}\int\frac{1}{t\left(t+1\right)}dt$$

Let $\frac{1}{t\left(t+1\right)}=\frac{A}{t}+\frac{B}{\left(t+1\right)}$

$$1=A\left(1+t\right)+Bt \qquad \ldots(1)$$

Substituting $t = 0, -1$ in equation (1), we obtain

$A = 1$ and $B = -1$

$$\therefore \frac{1}{t\left(t+1\right)}=\frac{1}{t}-\frac{1}{\left(1+t\right)}$$

$$\Rightarrow \int\frac{1}{x\left(x^n+1\right)}dx=\frac{1}{n}\int\left\{\frac{1}{t}-\frac{1}{\left(t+1\right)}\right\}dx$$

$$=\frac{1}{n}\left[\log\left|t\right|-\log\left|t+1\right|\right]+C$$

$$=-\frac{1}{n}\left[\log\left|x^n\right|-\log\left|x^n+1\right|\right]+C$$

$$=\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$$

Question 12: $\frac{\cos x}{\left(1-\sin x\right)\left(2-\sin x\right)}$ [Hint: Put $\sin x = t$]

$\frac{\cos x}{\left(1-\sin x\right)\left(2-\sin x\right)}$

Let $\sin x=t \Rightarrow \cos x\,dx=dt$

$$\therefore \int\frac{\cos x}{\left(1-\sin x\right)\left(2-\sin x\right)}dx=\int\frac{dt}{\left(1-t\right)\left(2-t\right)}$$

Let $\frac{1}{\left(1-t\right)\left(2-t\right)}=\frac{A}{\left(1-t\right)}+\frac{B}{\left(2-t\right)}$

$$1=A\left(2-t\right)+B\left(1-t\right) \qquad \ldots(1)$$

Substituting $t = 2$ and then $t = 1$ in equation (1), we obtain $A = 1$ and $B = -1$

$$\therefore \frac{1}{\left(1-t\right)\left(2-t\right)}=\frac{1}{\left(1-t\right)}-\frac{1}{\left(2-t\right)}$$

$$\Rightarrow \int\frac{\cos x}{\left(1-\sin x\right)\left(2-\sin x\right)}dx=\int\left\{\frac{1}{1-t}-\frac{1}{\left(2-t\right)}\right\}dt$$

$$=-\log\left|1-t\right|+\log\left|2-t\right|+C$$

$$=\log\left|\frac{2-t}{1-t}\right|+C$$

$$=\log\left|\frac{2-\sin x}{1-\sin x}\right|+C$$

Question 13: $\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$

$\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}$ Let $x^2 = t \Rightarrow 2x\,dx = dt$

$$\therefore \int\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx=\int\frac{dt}{\left(t+1\right)\left(t+3\right)} \qquad \ldots(1)$$

Let $\frac{1}{\left(t+1\right)\left(t+3\right)}=\frac{A}{\left(t+1\right)}+\frac{B}{\left(t+3\right)}$

$$1=A\left(t+3\right)+B\left(t+1\right) \qquad \ldots(1)$$

Substituting $t = -3$ and $t = -1$ in equation (1),

we obtain $A=\frac{1}{2}$ and $B=-\frac{1}{2}$

$$\therefore \frac{1}{\left(t+1\right)\left(t+3\right)}=\frac{1}{2\left(t+1\right)}-\frac{1}{2\left(t+3\right)}$$

$$\Rightarrow \int\frac{2x}{\left(x^2+1\right)\left(x^2+3\right)}dx=\int\left\{\frac{1}{2\left(t+1\right)}-\frac{1}{2\left(t+3\right)}\right\}dt$$

$$=\frac{1}{2}\log\left|\left(t+1\right)\right|-\frac{1}{2}\log\left|t+3\right|+C$$

$$=\frac{1}{2}\log\left|\frac{t+1}{t+3}\right|+C$$

$$=\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$$

Question 14: $\frac{1}{x\left(x^4-1\right)}$

$\frac{1}{x\left(x^4-1\right)}$

Multiplying numerator and denominator by x^3, we obtain

$$\frac{1}{x\left(x^4-1\right)}=\frac{x^3}{x^4\left(x^4-1\right)}$$

$$\therefore \int\frac{1}{x\left(x^4-1\right)}dx=\int\frac{x^3}{x^4\left(x^4-1\right)}dx$$

Let $x^4 = t \Rightarrow 4x^3dx = dt$

$$\therefore \int\frac{1}{x\left(x^4-1\right)}dx=\frac{1}{4}\int\frac{dt}{t\left(t-1\right)}$$

Let $\frac{1}{t\left(t-1\right)}=\frac{A}{t}+\frac{B}{\left(t-1\right)}$

$$1=A\left(t-1\right)+Bt \qquad \ldots(1)$$

Substituting $t = 0$ and 1 in (1), we obtain

$A = -1$ and $B = 1$

$$\Rightarrow \frac{1}{t\left(t+1\right)}=\frac{-1}{t}+\frac{1}{t-1}$$

SOLUTIONS: ASSIGNMENTS: INDEFINITE INTEGRALS

(i) Integration by substitution

LEVEL I — 1. $\tan(\log_e x) + C$ — 2. $\frac{1}{m}e^{m\tan^{-1}x} + C$ — 3. $e^{\sin^{-1}x} + C$

LEVEL II — 1. $2\log_e\left|1+\sqrt{x}\right| + C$ — 2. $\frac{1}{3}\sec^{-1}x^3 + C$ — 3. $\log_e\left|1-e^x\right| + C$

LEVEL III — 1. $2\sqrt{\tan x} + C$ — 2. $-\tan^{-1}(\cos x) + C$ — 3. $\frac{\tan^2 x}{2} + \log_e\left|\tan x\right| + C$

(ii)) Application of trigonometric function in integrals

LEVEL I — 1. $-\frac{3}{4}\cos x + \frac{1}{12}\cos 3x + C$ — 2. $\frac{1}{2}\left[x + \frac{\sin 6x}{6}\right] + C$

3. $\frac{x}{4} + \frac{1}{4}\sin 6x + \frac{1}{16}\sin 4x + \frac{1}{8}\sin 2x + C$

LEVEL II — 1. $\frac{1}{4}\sec^4 x + C$ OR $\frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} + C$ — 2. $\frac{2}{3}\sin 3x + 2\sin x + C$

LEVEL III — 1. $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$ — 2. $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

(iii) Integration using Standard results

LEVEL I — 1. $\frac{1}{2}\log_e\left|x + \frac{1}{2}\sqrt{4x^2-9}\right| + C$ — 2. $\frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C$ — 3. $\frac{1}{9}\tan^{-1}\left(\frac{3x+2}{3}\right) + C$

LEVEL II — 1. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$ — 2. $\tan^{-1}(\sin x + 2) + C$ — 3. $\sin^{-1}\left(\frac{2x-1}{5}\right) + C$

LEVEL III — 1. $\sin^{-1}\left(\frac{2x^2-1}{5}\right) + C$ — 2. $x + \log\left|x^2 - x + 1\right| + \frac{2}{\sqrt{3}}\log\left|\frac{2x-1}{\sqrt{3}}\right| + C$

3. $\sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C$

4. $\sin^{-1}x + \sqrt{1-x^2} + C$ [Hint: Put x=cos2θ]

5. $6\sqrt{x^2-9x+20} + 34\log\left|\left(\frac{2x-9}{2}\right) + \sqrt{x^2-9x+20}\right| + C$

(iv) Integration using Partial Fraction

LEVEL I — 1. $\frac{1}{3}\log(x+1) + \frac{5}{3}\log(x-2) + C$ — 2. $\frac{1}{2}\log(x-1) - 2\log(x-2) + \frac{3}{2}\log(x-3) + C$

3. $\frac{11}{4}\log\left(\frac{x+1}{x+3}\right) + \frac{5}{2(x+1)} + C$

LEVEL II — 1. $x - 11\log(x-1) + 16\log(x-2) + C$ — 2. $\frac{1}{4}\log x - \frac{1}{2x} + \frac{3}{4}\log(x+2) + C$

3. $\frac{3}{8}\log(x-1) - \frac{1}{2(x-1)} + \frac{5}{8}\log(x+3) + C$

Detail of the concepts to be mastered by every student of class second PUC with exercises and examples of NCERT Text Book.

Indefinite Integrals	(i) Integration by substitution	*	Text book , Vol. II Examples 5&6 Page 300, 302,301,303
	(ii)) Application of trigonometric function in integrals	**	Text book , Vol. II Ex 7 Page 306, Exercise 7.3 Q13&Q24
	(iii) Integration of some particular function $\int \frac{dx}{x^2 \pm a^2}$, $\int \frac{dx}{\sqrt{x^2 \pm a^2}}$, $\int \frac{1}{\sqrt{a^2 - x^2}} dx$, $\int \frac{dx}{ax^2 + bx + c}$, $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, $\int \frac{(px+q)dx}{ax^2 + bx + c}$, $\int \frac{(px+q)dx}{\sqrt{ax^2 + bx + c}}$	***	Text book , Vol. II Exp 8, 9, 10 Page 311,312,313, Exercise 7.4 Q 3,4,8,9,13&23
	(iv) Integration using Partial Fraction	***	Text book , Vol. II Exp 11&12 Page 318 Exp 13 319,Exp 14 & 15 Page320
	(v) Integration by Parts	**	Text book , Vol. II Exp 18,19&20 Page 325 Exs 7.6 QNO ,10,11, 17,18,20
	(vi)Some Special Integrals $\int \sqrt{a^2 \pm x^2}\, dx$, $\int \sqrt{x^2 - a^2}\, dx$	***	Text book , Vol. II Exp 23 &24 Page 329
	(vii) Miscellaneous Questions	**	Text book , Vol. II Solved Ex. 40, 41
	viii)Some special integrals		Text book Supplimentary material Page 614,615

SYMBOLS USED :

*** : Important Questions, ** :Very Important Questions, *** : Very-Very Important Questions**