$$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2} \text{ \{second order\}}$$

1.3Degree of a Partial Differential Equation

The degree of a partial differential equation is a degree of highest order derivative occurs in it. When it has been made free from radical sign and fraction power.

For example,

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy \quad \text{\{first degree\}}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial y}\right) \quad \text{\{first degree\}}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2} \text{ \{second degree\}}$$

$$y\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} = z\left(\frac{\partial z}{\partial y}\right) \text{ \{second degree\}}$$

1.4Linear and Non-linear Partial differential Equation

A partial differential equation is said to be linear if the dependent variable and its partial derivative occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a non-linear partial differential equation.

For example

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z + xy \quad \text{\{linear\}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz \text{\{linear\}}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right) \quad \text{\{non-linear\}}$$

$$z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x \quad \text{\{non-linear\}}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2} \text{\{non-linear\}}$$

Notation:- In partial derivative we use the following notations:

$$p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$$

We usually assume x and y as independent variable and z to be as dependent variable.

Example 1: Find the order and degree of the following PDE:

(1) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2\left(\frac{\partial z}{\partial x}\right),$ Order 3, Degree 1

(2) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial x}\right)^2 = 1,$ Order 2, Degree 1

(3) $x\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z\frac{\partial z}{\partial y},$ Order 1, Degree 2

(4) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{-1/2}$

$$\Rightarrow \left(\frac{\partial^2 z}{\partial x^2}\right)^2 = \left(1 + \frac{\partial z}{\partial y}\right)^{-1}$$

(i) Grouping method

(ii) Method of multipliers

(iii) Combination of (i) & (ii)

4. Suppose $\mathrm{u} = a$ and $\mathrm{v} = b$ are two Solution of eq. (1) which obtained by (2), where u and v are function of x, y, z and a and b are constant.
5. Complete Solution of (1) is $f(\mathrm{u}, \mathrm{v}) = 0$ or $\mathrm{u} = \phi(\mathrm{v})$ or $\mathrm{v} = \psi(\mathrm{u})$or $f(a, b)$.

Example 1: Solve the following partial differential equations

(i)$yzp - xzq = xy$**(ii)**$p \tan x + q \tan y = \tan z$

Solution (i). The given differential equation is

$yzp - xzq = xy$(1)

Comparing eq. (1)with $Pp + Qq = R$, we get

$P = yz, Q = -xz$and$R = xy$

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$\Rightarrow \frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$ (2)

Taking first and second fractions of eq. (2)

$$\frac{dx}{yz} = \frac{dy}{-xz} \Rightarrow \frac{dx}{y} = \frac{dy}{-x}$$

$\Rightarrow xdx + ydy = 0$ (3)

Integrating (3), we get

$\Rightarrow x^2 + y^2 = c_1$(4)

Next, taking second and third fractions

$$\frac{dy}{-xz} = \frac{dz}{xy} \Rightarrow \frac{dy}{-z} = \frac{dz}{y}$$

Integrating, we get

$$\Rightarrow y^2 + z^2 = c_2$$

Therefore, the general Solution is

$$\phi(x^2 + y^2, y^2 + z^2) = 0$$

(ii) Given differential equation

$p \tan x + q \tan y = \tan z$(1)

The Lagrange's auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$(2)

Taking first two fraction of (2)

From first fraction, we have $dp = 0 \Rightarrow p = a$

Putting $p = a$ in eq. (1), we get $q = \frac{y(z-ax)}{(a+y)}$.

Putting these values of p and q in the equation $dz = pdx + qdy$, we get

$$dz = adx + \frac{y(z-ax)}{(a+y)}dy$$

$$\Rightarrow \frac{dz - adx}{z - ax} = \frac{y}{(a+y)}dy$$

$$\Rightarrow \frac{dz - adx}{z - ax} = \left(1 - \frac{a}{a+y}\right)dy$$

On integrating

$$(z - ax)(y + a)^a = be^y$$

Example 4: Find a complete integral of $p^2x + q^2y = z$

Solution. Given equation is $f(x, y, z, p, q) = p^2x + q^2y - z = 0$ (1)

Charpit's auxiliary equation

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} \qquad (2)$$

$$\Rightarrow \frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}, \qquad \text{(from eq. (1) and (2))} \qquad (3)$$

Now, each fraction of eq. (2)

$$= \frac{2pxdp + p^2dx}{2px(-p+p^2) + p^2(-2px)} = \frac{2qydq + q^2dy}{2qy(-q+q^2) + q^2(-2qy)}$$

$$\Rightarrow \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$$

$$\Rightarrow \log(p^2x) = \log(q^2y) + \log a$$

$\Rightarrow p^2x = q^2ya$ (4)

From (1) and (4)

$$q^2ya + q^2y = z$$

$\Rightarrow q = \{z/(1+a)y\}^{1/2}$ (5)

Using (4) and (5), we have

$$p = \left\{\frac{za}{(1+a)x}\right\}^{1/2}$$

Putting the value of p and q in $dz = pdx + qdy$,

We get $dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy,$

$$\Rightarrow (1+a)^{1/2}z^{-1/2}dz = \sqrt{a}x^{-1/2}dx + y^{-1/2}dy$$

On integrating,

$\Longrightarrow P^2 + Q^2 = x^2 + y^2$, where $P = \frac{\partial Z}{\partial x}, Q = \frac{\partial Z}{\partial y}$

$$\Longrightarrow P^2 - x^2 = -Q^2 + y^2$$

Which is in standard form III, i.e. $f_1(x, P) = f_2(y, Q)$.

Let $P^2 - x^2 = a$ and $y^2 - Q^2 = a$

$$\Longrightarrow P = \sqrt{(a + x^2)}, Q = \sqrt{(y^2 - a)}$$

$\therefore$ The complete integral is $dZ = Pdx + Qdy$,

$$\Longrightarrow dZ = \sqrt{(a + x^2)}dx + \sqrt{(y^2 - a)}dy$$

On integrating, we have

$$Z = \frac{x}{2}\sqrt{(a + x^2)} + \frac{a}{2}\log\left\{x + \sqrt{(a + x^2)}\right\} + \frac{y}{2}\sqrt{(y^2 - a)} - \frac{a}{2}\log\left\{y + \sqrt{y^2 - a}\right\} + b$$

$$\Longrightarrow z^2 = x\sqrt{(a + x^2)} + a\ \log\left\{x + \sqrt{(a + x^2)}\right\} + y\sqrt{(y^2 - a)} - a\log\{y + \sqrt{(y^2 - a)}\} + b.$$

Standard form IV (Clairaut'sForm): $z = px + qy + f(p, q)$.

Working Rule

Step I: Given equation is of the form $z = px + qy + f(p, q)$. (1)

Step 2: Put $p = a$ and $q = b$, we get complete solution or complete integral

$z = ax + by + f(a, b)$ (2)

Step 3: Differentiating (2) partially w.r.t. a and b, we get

$0 = x + \frac{\partial f}{\partial a}$ (3)

$0 = y + \frac{\partial f}{\partial b}$ (4)

The singular solution is obtained by eliminating a and b from (2), (3) and (4).

Example 1: Find the complete solution of $z = px + qy + c\sqrt{1 + p^2 + q^2}$.

Solution: Given $z = px + qy + c\sqrt{1 + p^2 + q^2}$, (1)

Which is of standard form IV i.e. Clairaut's form i.e. $z = px + qy + f(p, q)$.

Putting $p = a$ and $q = b$ in (1)

The complete integral is

$z = ax + by + c\sqrt{1 + a^2 + b^2}$.

Example 2: Find the complete and singular solution of $z = px + qy + \log pq$.

Solution: Given $z = px + qy + \log pq$ (1)

which is of Clairaut's form i.e. $z = px + qy + f(p, q)$.

Putting $p = a$ and $q = b$ in (1), we get the compete integral is

$z = ax + by + \log ab$.(2)