Pearson Edexcel
Level 3 GCE
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Mathematics

Advanced Subsidiary

Paper 1: Pure Mathematics

1. **A curve has equation**

$$y = 2x^3 - 4x + 5$$

Find the equation of the tangent to the curve at the point $P(2, 13)$.

Write your answer in the form $\mathbf{y = mx + c}$**,**

where m and c are integers to be found

Solution

$$y = 2x^3 - 4x + 5$$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - 4x + 5)$$

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3) - \frac{d}{dx}(4x) + \frac{d}{dx}(5)$$

$$\frac{dy}{dx} = 2\frac{d}{dx}(x^3) - 4\frac{dx}{dx} + \frac{d}{dx}(5)$$

$$\frac{dy}{dx} = 2(3x^2) - 4 + 0 = 6x^2 - 4$$

Slope of the line at the point $P(2, 13) = P(x, y)$

$$m = \frac{dy}{dx} = 6(2)^2 - 4 = 20$$

Where m denotes the slope at the point $P(2, 13)$

Putting $P(2, 13) = P(x, y)$ and $m = 20$ in :

$$y = mx + c$$

$$13 = 20 \times 2 + c$$

$$13 = 40 + c$$

$$40 + c = 13$$

$$c = 13 - 40 = -27$$

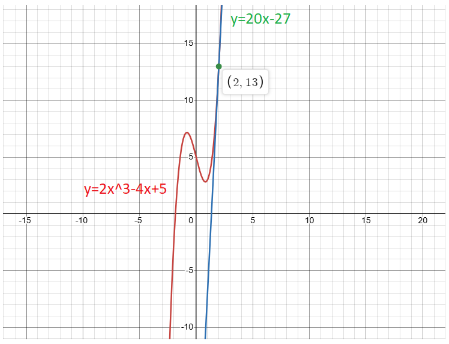
Putting m=20, c=-27 , equation of tangent to the curve at the point P(2,13)

$$y = mx + c = 20x - 27$$

Or

$$y = 20x - 27$$

Graph of solution



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Solution

(a) Calculate the bearing on which the boat is moving.

To calculate the bearing, we first determine the displacement vector. The displacement vector $\mathbf{d}$ is given by:

$$\mathbf{d} = \text{final position} - \text{initial position}.$$

Substituting the positions:

$$\mathbf{d} = (-3\mathbf{i} - 5\mathbf{j}) - (4\mathbf{i} - 2\mathbf{j}),$$

$$\mathbf{d} = -3\mathbf{i} - 5\mathbf{j} - 4\mathbf{i} + 2\mathbf{j},$$

$$\mathbf{d} = -7\mathbf{i} - 3\mathbf{j}.$$

The direction of the displacement vector determines the bearing. We calculate the angle θ it makes with the positive $\mathbf{i}$-axis (due east). The angle θ is given by:

$$\tan\theta = \frac{\text{opposite component}}{\text{adjacent component}} = \frac{-3}{-7} = \frac{3}{7}.$$

Using a calculator:

$$\theta = \tan^{-1}\left(\frac{3}{7}\right) \approx 23.2^\circ.$$

This angle is measured counterclockwise from the negative $\mathbf{i}$-axis because both components of the vector are negative. To find the bearing (measured clockwise from north), we compute:

$$\text{Bearing} = 180^\circ + \theta = 180^\circ + 23.2^\circ = 203.2^\circ.$$

Final Answer: The bearing on which the boat is moving is:

$$\boxed{203.2^\circ}.$$

(b) Calculate the speed of the boat.

To calculate the speed, we first determine the magnitude of the displacement vector:

$$\|\mathbf{d}\| = \sqrt{(-7)^2 + (-3)^2} = \sqrt{49 + 9} = \sqrt{58}.$$

Thus:

$$\|\mathbf{d}\| \approx 7.62\,\text{km}.$$

The time elapsed between $10 : 00$ and $12 : 45$ is

$2\,\text{hours} + 45\,\text{minutes}$, which is:

$$2 + \frac{45}{60} = 2.75\,\text{hours}.$$

The speed of the boat is given by:

$$\text{Speed} = \frac{\text{distance}}{\text{time}} = \frac{\|\mathbf{d}\|}{2.75}.$$

Substituting the values:

$$\text{Speed} = \frac{7.62}{2.75} \approx 2.77\,\text{km/h}.$$

Final Answer: The speed of the boat is:

$$\boxed{2.77\,\text{km/h}}.$$

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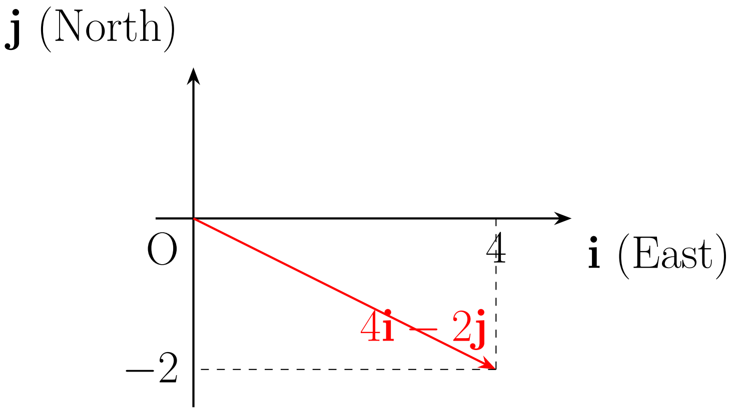
Vector Diagram

The vector $4\mathbf{i} - 2\mathbf{j}$ can be represented as follows:



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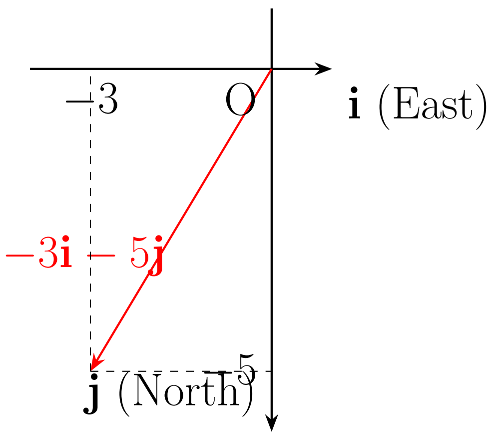
Vector Diagram

The vector $-3\mathbf{i} - 5\mathbf{j}$ can be represented as follows:



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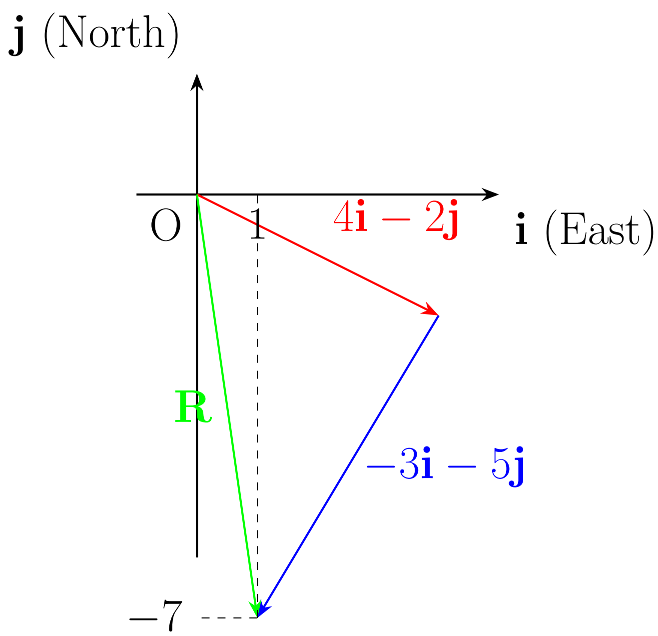
Vector Addition Diagram

The vectors $4\mathbf{i} - 2\mathbf{j}$ and $-3\mathbf{i} - 5\mathbf{j}$ are added graphically as follows:



Resultant Vector: Adding the two vectors gives:

$$\mathbf{R} = (4 - 3)\mathbf{i} + (-2 - 5)\mathbf{j} = \mathbf{i} - 7\mathbf{j}.$$

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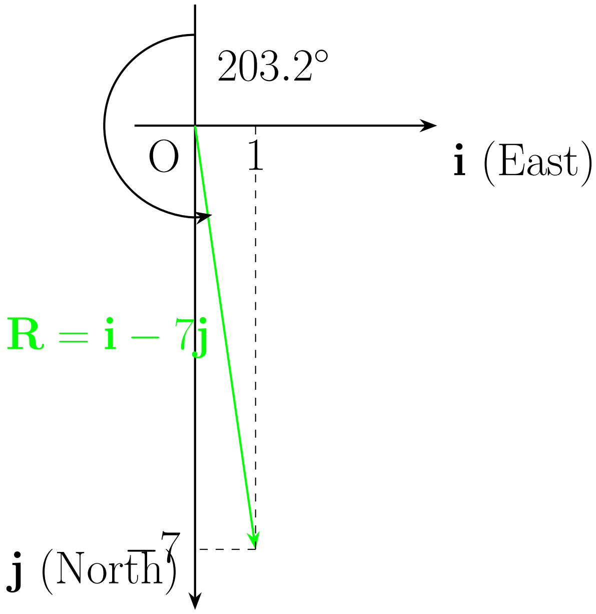
Final Bearing Diagram

The resultant vector $\mathbf{R} = \mathbf{i} - 7\mathbf{j}$ is represented graphically below, with the bearing indicated.



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3. (i) Solve the equation

$$x\sqrt{2} - \sqrt{18} = x$$

writing the answer as a surd in simplest form.

(ii) Solve the equation

$$4^{3x-2} = \frac{1}{2\sqrt{2}}$$

Solution(i)

$$x\sqrt{2} - \sqrt{18} = x$$

$$x\sqrt{2} - \sqrt{9 \times 2} = x$$

$$x\sqrt{2} - \sqrt{9} \times \sqrt{2} = x$$

$$x\sqrt{2} - 3\sqrt{2} = x$$

$$x\sqrt{2} - x = 3\sqrt{2}$$

$$x(\sqrt{2} - 1) = 3\sqrt{2}$$

$$x(\sqrt{2} - 1)(\sqrt{2} + 1) = 3\sqrt{2}(\sqrt{2} + 1)$$

$$x[(\sqrt{2})^2 - 1^2] = 3\sqrt{2}(\sqrt{2} + 1)$$

$$x[2 - 1] = 3\sqrt{2}(\sqrt{2} + 1)$$

$$x = 3\sqrt{2}(\sqrt{2}+1)$$

Solution(ii)

$$4^{3x-2} = \frac{1}{2\sqrt{2}}$$

$$4^{3x} \times 4^{-2} = \frac{1}{2\sqrt{2}}$$

$$4^{3x} \times \frac{1}{4^2} = \frac{1}{2\sqrt{2}}$$

$$4^{3x} \times \frac{1}{16} = \frac{1}{2\sqrt{2}}$$

$$4^{3x} = \frac{16}{2\sqrt{2}}$$

$$4^{3x} = \frac{8}{\sqrt{2}}$$

$$(2^2)^{3x} = \frac{2^3}{2^{\frac{1}{2}}}$$

$$2^{6x} = 2^{3-\frac{1}{2}}$$

$$2^{6x} = 2^{\frac{5}{2}}$$

$\Longrightarrow$

$6x = \frac{5}{2}$

$x = \frac{5}{12}$

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4. In 1997 the average CO_2 emissions of new cars in the UK was 190 g/km. In 2005 the average CO_2 emissions of new cars in the UK had fallen to 169g/km. Given Ag/km is the average CO_2 emissions of new cars in the UK n years after 1997 and using a linear model,

(a) form an equation linking A with n.

In 2016 the average CO_2 emissions of new cars in the UK was 120 g/km.

(b) Comment on the suitability of your model in light of this information.

Solution

Solution

(a) Form an equation linking A with n

We are given two data points:

- In 1997 ($n = 0$), $A = 190\,\text{g/km}$.
- In 2005 ($n = 8$), $A = 169\,\text{g/km}$.

Assuming a linear model, the equation for A in terms of n can be expressed as:

$$A = mn + c$$

where m is the slope of the line, and c is the y-intercept.

Step 1: Calculate the slope m The slope m is given by the rate of change of A with respect to n:

$$m = \frac{\Delta A}{\Delta n} = \frac{169 - 190}{8 - 0} = \frac{-21}{8} = -2.625$$

Step 2: Determine the y-intercept c Using the point $(n = 0, A = 190)$:

$$A = mn + c \implies 190 = -2.625(0) + c \implies c = 190$$

Thus, the equation linking A with n is:

$$A = -2.625n + 190$$

(b) Comment on the suitability of the model in light of 2016 data

In 2016 ($n = 19$), the average A is 120 g/km. Using the model:

$$A = -2.625n + 190$$

Substituting $n = 19$:

$$A = -2.625(19) + 190 = -49.875 + 190 = 140.125\,\text{g/km}$$

However, the actual emissions in 2016 were 120 g/km, which is significantly lower than the predicted 140.125 g/km.

Suitability of the model The linear model is reasonable over short periods, such as from 1997 to 2005, as the data aligns closely. However:

1. Over longer periods, the model may become less accurate due to factors like technological advancements, policy changes, and increased environmental awareness.

2. The actual decrease in CO_2 emissions between 1997 and 2016 is not constant, suggesting a nonlinear trend.

3. A better fit for the data may involve a nonlinear model, such as an exponential or piecewise function, to account for faster reductions in emissions over time.

In summary, the linear model provides a rough approximation but is not suitable for accurately modeling long-term trends.

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5.

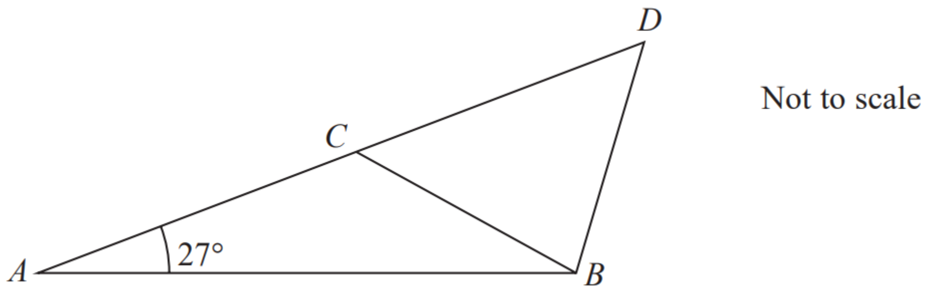


Figure 1

Figure 1 shows the design for a structure used to support a roof.

The structure consists of four steel beams, AB, BD, BC and AD.

Given AB $= 12m$, $BC = BD = 7m$ **and** $\angle B\hat{A}C = 27°$

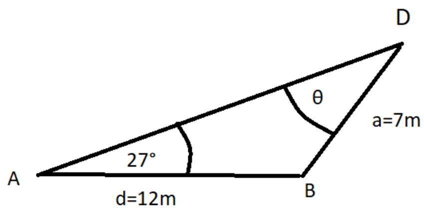
(a) find, to one decimal place, the size of $\angle A\hat{C}B$

The steel beams can only be bought in whole meter lengths.

(b) Find the minimum length of steel that needs to be bought to make the complete structure.

Solution

In the $\triangle ABD$



$\angle ADB = \theta$

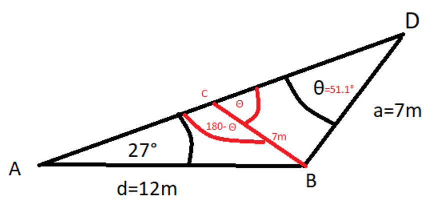
$$\frac{7}{\sin 27} = \frac{12}{\sin \theta}$$

$$7 \sin \theta = 12(\sin 27)$$

$$\sin \theta = \frac{12(\sin 27)}{7}$$

$$\theta = \sin^{-1}[\frac{12(\sin 27)}{7}]$$

$$\theta = 51.1^{\circ}$$



Since BD=BC=7m

$$\angle BCD = \theta = \angle BDC = 51.1^{\circ}$$

Since

$\angle A\hat{C}B + \angle B\hat{C}D = 180°$

$\angle A\hat{C}B + 51.1° = 180°$

$\angle A\hat{C}B = 180° - 51.1°$

$\angle A\hat{C}B = 128.9°$

Also

$\angle B\hat{A}D + \angle B\hat{D}A + \angle A\hat{B}D = 180°$

$27° + 51.1° + \angle A\hat{B}D = 180°$

$78.1° + \angle A\hat{B}D = 180°$

$\angle A\hat{B}D = 180° - 78.1°$

$\angle A\hat{B}D = 101.9°$

Now

$$\frac{b}{\sin\angle ABD} = \frac{a}{\sin\angle BAD}$$
$$\frac{b}{\sin 101.9°} = \frac{7}{\sin 27°}$$
$$b = \frac{7}{\sin 27°} \times \sin 101.9°$$

$$b = 15.08m = AD$$

Minimum length of the steel required=AB+BD+AD+BC

=12+7+15.08+7= 41.08 m

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6. (a) Find the first 4 terms, in ascending powers of x, in the binomial expansion of

$$(1 + kx)^1 0$$

where k is a non-zero constant. Write each coefficient as simply as possible

Given that in the expansion of $(1 + kx)^1 0$ **the coefficient** x^3 **3 is 3 times the coefficient of x**

(b) find the possible values of k.

Solution

Part (a): Finding the first 4 terms in ascending powers of x in the binomial expansion of $(1+kx)^{10}$

The binomial expansion of $(1+kx)^{10}$ is given by:

$$(1+kx)^{10} = \sum_{r=0}^{10} \binom{10}{r} (1)^{10-r} (kx)^r$$

The first 4 terms in ascending powers of x are:

$$T_0 = \binom{10}{0} (1)^1 (kx)^0 = 1$$

$$T_1 = \binom{10}{1} (1)^9 (kx)^1 = 10kx$$

$$T_2 = \binom{10}{2} (1)^8 (kx)^2 = 45k^2x^2$$

$$T_3 = \binom{10}{3} (1)^7 (kx)^3 = 120k^3x^3$$

Thus, the first 4 terms of the expansion are:

$$1 + 10kx + 45k^2x^2 + 120k^3x^3$$

Part (b): Given that the coefficient of x^3 is 3 times the coefficient of x, find the possible values of k.

From Part (a):

- Coefficient of $x^3 = 120k^3$
- Coefficient of $x = 10k$

It is given that:

$$120k^3 = 3(10k)$$

Simplify:

$$120k^3 = 30k$$

Divide through by k (since $k \neq 0$):

$$120k^2 = 30$$

Solve for k^2:

$$k^2 = \frac{30}{120} = \frac{1}{4}$$

Take the square root of both sides:

$$k = \pm\frac{1}{2}$$

Final Answer:

(a) The first 4 terms in the expansion are:

$$1 + 10kx + 45k^2x^2 + 120k^3x^3$$

(b) The possible values of k are:

$$k = \pm\frac{1}{2}$$

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7. Given that k is a positive constant and

$\int_1^k (\frac{5}{2\sqrt{x}} + 3) \quad dx = 4$

(a) show that $3k + 5\sqrt{k} - 12 = 0$

(b) Hence, using algebra, find any values of k such that

$$\int_1^k (\frac{5}{2\sqrt{x}} + 3) \quad dx = 4$$

Solution

$$\int_1^k (\frac{5}{2\sqrt{x}} + 3) \quad dx = 4$$

$$\int_1^k \frac{5}{2\sqrt{x}} \quad dx + \int_1^k 3 \quad dx = 4$$

$$\frac{5}{2}\int_1^k \frac{1}{\sqrt{x}} \quad dx + 3\int_1^k \quad dx = 4$$

$$\frac{5}{2}\int_1^k x^{-\frac{1}{2}} \quad dx + 3\int_1^k \quad dx = 4$$

$$\frac{5}{2}[\frac{1}{-\frac{1}{2}+1}]|x^{-\frac{1}{2}+1}|_1^k + 3|x|_1^k = 4$$

$$\frac{5}{2}[\frac{1}{\frac{1}{2}}]|x^{\frac{1}{2}}|_1^k + 3|k-1| = 4$$

$$\frac{5}{2}[\frac{2}{1}]|k^{\frac{1}{2}} - 1^{\frac{1}{2}}| + 3k - 3 = 4$$

$$5|k^{\frac{1}{2}} - 1| + 3k - 3 = 4$$

$$5k^{\frac{1}{2}} - 5 + 3k - 3 = 4$$

$$5k^{\frac{1}{2}} + 3k - 8 - 4 = 0$$

$$5k^{\frac{1}{2}} + 3k - 12 = 0$$

$$3k + 5k^{\frac{1}{2}} - 12 = 0$$

(b)

$$3k + 5k^{\frac{1}{2}} - 12 = 0$$

Putting $k^{\frac{1}{2}} = y$

$(k^{\frac{1}{2}})^2 = y^2$

$k = y^2$

$\Longrightarrow$

$$3y^2 + 5y - 12 = 0$$

$$3y^2 + (9 - 4)y - 12 = 0$$

$$3y^2 + 9y - 4y - 12 = 0$$

$$(3y^2 + 9y) - (4y + 12) = 0$$

$$3y(y + 3) - 4(y + 3) = 0$$

$$(3y - 4)(y + 3) = 0$$

$$3y - 4 = 0 \qquad Or \qquad y + 3 = 0$$

$$3y = 4 \qquad Or \qquad y = -3$$

$$y = \frac{4}{3} \qquad Or \qquad y = -3$$

Putting in $k = y^2$

$\implies$

$k = (\frac{4}{3})^2 = \frac{16}{9}$

Or

$k = (-3)^2 = 9$

Checking the values

Putting $k = \frac{16}{9}$ in :

$$3k + 5\sqrt{k} - 12 = 0$$

$$3(\frac{16}{9}) + 5(\frac{16}{9})^{\frac{1}{2}} - 12 = 0$$

$$(\frac{16}{3}) + 5(\frac{4^2}{3^2})^{\frac{1}{2}} - 12 = 0$$

$$(\frac{16}{3}) + 5(\frac{4}{3}) - 12 = 0$$

$$(\frac{16}{3}) + (\frac{20}{3}) - 12 = 0$$

$$(\frac{16+20}{3}) - 12 = 0$$

$$12 - 12 = 0$$

Hence $k = \frac{16}{9}$:

Similarly putting $k = (-3)^2 = 9$

$$3k + 5\sqrt{k} - 12 = 0$$

$\Longrightarrow$

$$3(9) + 5\sqrt{9} - 12 = 0$$

$$27 + 15 - 12 = 0$$

Not possible.

Hence

$k \neq 9$

Now putting k=$\frac{16}{9}$ in

$$\int_1^{\frac{16}{9}} (\frac{5}{2\sqrt{x}} + 3) \quad dx$$

$$= 5(\frac{16}{9})^{\frac{1}{2}} - 5 + 3(\frac{16}{9}) - 3$$
$$= 5(\frac{4}{3}) - 5 + (\frac{16}{3}) - 3$$

$$= (\frac{20}{3}) - 5 + (\frac{16}{3}) - 3$$

$$= (\frac{20+16}{3}) - 8$$

$$= (12 - 8) = 4$$

Hence Value of k=$\frac{16}{9}$

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8. The temperature, $\theta°C$, of a cup of tea t minutes after it was placed on a table in a room, is modeled by the equation

$$\theta = 18 + 65e^{-\frac{t}{8}} \qquad t \geq 0$$

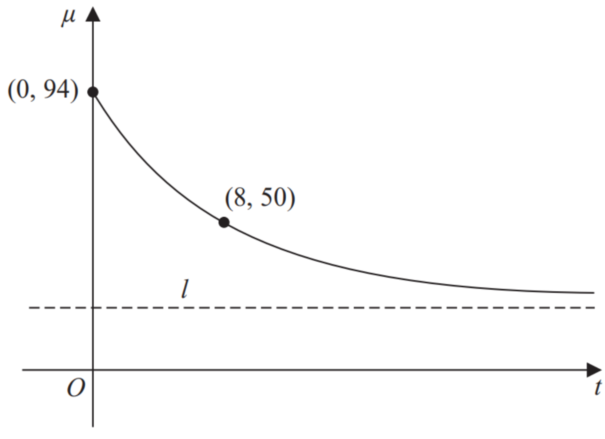
Find, according to the model,

(a) the temperature of the cup of tea when it was placed on the table.

(b) the value of t, to one decimal place,

when the temperature of the cup of tea was $35°C$.

(c) Explain why, according to this model, the temperature of the cup of tea could not fall to $15°C$.



The temperature, $\mu°C$, of a second cup of tea t minutes after it was placed on a table in a different room, is modeled by the equation

$$\mu = A + Be^{-\frac{t}{8}} \qquad t \geq 0$$

where A and B are constants. Figure 2

shows a sketch of μ against t with two data points that lie on the curve .

The line l, also shown on Figure 2, is the asymptote to the curve

Using the equation of this model and the information given in Figure 2

(d) find an equation for the asymptote l.

Solution

(a)

$$\theta = 18 + 65e^{-\frac{t}{8}} \qquad t \geq 0$$

Putting t=0

$$\theta = 18 + 65e^{-\frac{0}{8}}$$

$$\theta = 18 + 65e^{0} = 18 + 65(1) = 18 + 65 = 83^{\circ}$$

(b)

Again

$$\theta = 18 + 65e^{-\frac{t}{8}} \qquad t \geq 0$$

Putting $\theta = 35°C$.

$\implies$

$$35 = 18 + 65e^{-\frac{t}{8}} \qquad t \geq 0$$

$$18 + 65e^{-\frac{t}{8}} = 35 \qquad t \geq 0$$

$$65e^{-\frac{t}{8}} = 35 - 18 \qquad t \geq 0$$

$$65e^{-\frac{t}{8}} = 17 \qquad t \geq 0$$

$$e^{-\frac{t}{8}} = \frac{17}{65} \qquad t \geq 0$$

$$\frac{1}{e^{\frac{t}{8}}} = \frac{17}{65} \qquad t \geq 0$$

$$\frac{65}{17} = e^{\frac{t}{8}} \qquad t \geq 0$$

$$e^{\frac{t}{8}} = \frac{65}{17} \qquad t \geq 0$$

$$\ln e^{\frac{t}{8}} = \ln \frac{65}{17} \qquad t \geq 0$$

$$\frac{t}{18} \ln e = \ln \frac{65}{17} \qquad t \geq 0$$

$$\frac{t}{18} = \ln \frac{65}{17} \qquad t \geq 0$$

$$t = 18 \ln \frac{65}{17} = 24^\circ 8' 28.07" \qquad t \geq 0$$

(c)

Put $\theta = 15^\circ C.$ in the equation

$$\theta = 18 + 65e^{-\frac{t}{8}} \qquad t \geq 0$$

We have

$$15 = 18 + 65e^{-\frac{t}{8}}$$

$$18 + 65e^{-\frac{15}{8}} = 15$$

$$65e^{-\frac{t}{8}} = 15 - 18$$

$$65e^{-\frac{15}{8}} = -3$$

$$e^{-\frac{t}{8}} = -\frac{3}{65}$$

$$\frac{1}{e^{\frac{t}{8}}} = -\frac{3}{65}$$

Since LHS is positive for all values of t, no solution is possible . Hence $\theta = 15°C$ is not possible.

(d)

We are given the equation:

$$\mu(t) = A + Be^{-t/8}, \quad t \geq 0,$$

and need to find the equation of the horizontal asymptotes as well as determine A and B using the given conditions:

1. $\mu(94) = 0$, 2. $\mu(50) = 8$.

Step 1: Horizontal Asymptote Analysis

The exponential term $e^{-t/8}$ approaches 0 as $t \to \infty$. Thus:

$$\lim_{t \to \infty} \mu(t) = A + B \cdot 0 = A.$$

The horizontal asymptote of $\mu(t)$ is therefore given by:

$$\mu(t) = A, \quad t \to \infty.$$

Step 2: Solve for A and B

From the given conditions:

1. At $t = 94$:

$$\mu(94) = A + Be^{-94/8} = 0.$$

2. At $t = 50$:

$$\mu(50) = A + Be^{-50/8} = 8.$$

Rewrite the equations

1. $A+Be^{-94/8} = 0$, or $Be^{-94/8} = -A$. 2. $A+Be^{-50/8} = 8$.

Solve for A and B

Let $x = e^{-94/8}$ and $y = e^{-50/8}$. Substituting x and y into the equations:

1. $A + Bx = 0 \implies A = -Bx$. 2. Substituting $A = -Bx$ into the second equation:

$$-Bx + By = 8.$$

Factor out B:

$$B(y - x) = 8 \quad \Longrightarrow \quad B = \frac{8}{y - x}.$$

Substitute $B = \frac{8}{y-x}$ into $A = -Bx$:

$$A = -\frac{8}{y - x} \cdot x = -\frac{8x}{y - x}.$$

Simplify x and y:

$$x = e^{-94/8} = e^{-47/4}, \quad y = e^{-50/8} = e^{-25/4}.$$

Final expressions for A and B:

$$B = \frac{8}{e^{-25/4} - e^{-47/4}}, \quad A = -\frac{8e^{-47/4}}{e^{-25/4} - e^{-47/4}}.$$

Step 3: Horizontal Asymptote

From Step 1, the horizontal asymptote is:

$$\mu(t) = A \quad \text{as } t \to \infty.$$

Thus, the horizontal asymptote equation is:

$$\mu = -\frac{8e^{-47/4}}{e^{-25/4} - e^{-47/4}}.$$

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9.

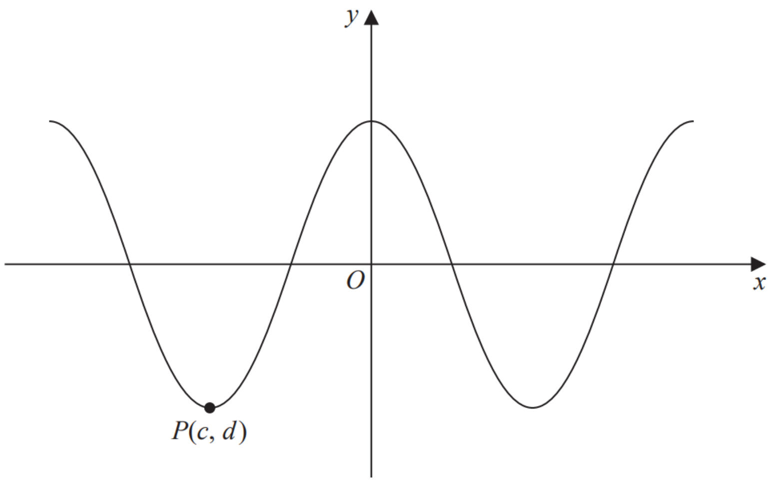


Figure 3 shows part of the curve with equation $y = 3\cos x^\circ$

The point P(c, d) is a minimum point on the curve with c being the smallest negative value of x at which a minimum occurs.

(a) State the value of c and the value of d.

(b) State the coordinates of the point to which P is mapped by the transformation which transforms the curve with equation $y = 3\cos x^\circ$ **to the curve with equation**

(i) y = $\mathbf{3}\cos\frac{x^\circ}{4}$

(ii)y=$\mathbf{3}\cos x - 36^\circ$

(c) Solve, for $450\circ \leq \theta < 720^\circ$

$$3\cos\theta = 8\tan\theta$$

giving your solution to one decimal place.

Solution

(a) Value of c and d

The curve is $y = 3\cos x^\circ$. For $\cos x^\circ$, the minimum value is -1. Hence:

$$y = 3(-1) = -3$$

This occurs at $x = -180^\circ + 360n$, where $n \in \mathbb{Z}$. The smallest negative x is -180°.

Thus:

$$c = -180^\circ, \quad d = -3$$

(b) Coordinates of the transformed point P

(i) Transformation to $y = 3\cos\left(\frac{x}{4}\right)^\circ$

The transformation involves scaling x by a factor of 4. The x-coordinate of $P(-180, -3)$ be-

comes:

$$c' = 4(-180) = -720^\circ$$

The y-coordinate remains the same. Hence, P is mapped to:

$$(-720, -3)$$

(ii) Transformation to $y = 3\cos(x - 36)^\circ$

The transformation involves a horizontal shift of 36° to the right. The x-coordinate of $P(-180, -3)$ becomes:

$$c' = -180 + 36 = -144^\circ$$

The y-coordinate remains the same. Hence, P is mapped to:

$$(-144, -3)$$

(c) Solve $3\cos\theta = 8\tan\theta$ for $450^\circ \leq \theta < 720^\circ$

1. Rewrite the equation:

$$3\cos\theta = 8\tan\theta \implies 3\cos\theta = 8\frac{\sin\theta}{\cos\theta}$$

2. Simplify:

$$3\cos^2\theta = 8\sin\theta \implies 3(1-\sin^2\theta) = 8\sin\theta$$

3. Expand and rearrange:

$$3 - 3\sin^2\theta = 8\sin\theta \implies 3\sin^2\theta + 8\sin\theta - 3 = 0$$

4. Solve the quadratic equation: Let $u = \sin\theta$.

The equation becomes:

$$3u^2 + 8u - 3 = 0$$

Solve using the quadratic formula:

$$u = \frac{-8 \pm \sqrt{8^2 - 4(3)(-3)}}{2(3)} = \frac{-8 \pm \sqrt{64 + 36}}{6} = \frac{-8 \pm 10}{6}$$

$$u = \frac{-8 + 10}{6} = \frac{2}{6} = \frac{1}{3}, \quad u = \frac{-8 - 10}{6} = \frac{-18}{6} = -3$$

Since $\sin\theta$ **must lie in** $[-1, 1]$**, only** $\sin\theta = \frac{1}{3}$ **is valid.**

5. **Find** θ**: Using** $\sin\theta = \frac{1}{3}$**, find the angles in the range** $450° \leq \theta < 720°$**. The principal angle is:**

$$\theta = \arcsin\left(\frac{1}{3}\right) \approx 19.5°$$

In the range $450° \leq \theta < 720°$**, the solutions**

are in the 3rd and 4th quadrants:

$$\theta = 540^\circ + 19.5^\circ = 559.5^\circ, \quad \theta = 720^\circ - 19.5^\circ = 700.5^\circ$$

Final Answers

- **(a)** $c = -180^\circ$**,** $d = -3$
- **(b)**
  - **(i)** $(-720, -3)$
  - **(ii)** $(-144, -3)$
- **(c)** $\theta = 559.5^\circ, 700.5^\circ$

....

10.

$$g(x) = 2x^3 + x^2 - 41x - 70$$

(a) Use the factor theorem to show that g(x) is divisible by $x - 5$**.**

(b) Hence, showing all your working, write g(x) as a product of three linear factors

The finite region R is bounded by the curve with equation y = g(x) and the x-axis, and lies below the x-axis.

(c) Find, using algebraic integration, the exact value of the area of R.

Solution

(a) Use the factor theorem to show that $g(x)$ is divisible by $(x-5)$:

The given function is:

$$g(x) = 2x^3 + x^2 - 41x - 70.$$

Using the factor theorem, we evaluate $g(5)$:

$$g(5) = 2(5)^3 + (5)^2 - 41(5) - 70.$$

$g(5) = 2(125) + 25 - 205 - 70 = 250 + 25 - 205 - 70 = 0.$

Since $g(5) = 0$, by the factor theorem, $(x - 5)$ is a factor of $g(x)$.

(b) Write $g(x)$ as a product of three linear factors:

We divide $g(x)$ by $(x - 5)$ using polynomial long division.

Divide $g(x) = 2x^3 + x^2 - 41x - 70$ by $(x - 5)$:

1. **Divide the leading term:** $\dfrac{2x^3}{x} = 2x^2$.

2. **Multiply:** $(x-5)(2x^2) = 2x^3 - 10x^2$.

3. **Subtract:** $(2x^3 + x^2 - 41x - 70) - (2x^3 - 10x^2) = 11x^2 - 41x -$

4. **Divide the next term:** $\dfrac{11x^2}{x} = 11x$.

5. **Multiply:** $(x-5)(11x) = 11x^2 - 55x$.

6. **Subtract:** $(11x^2 - 41x - 70) - (11x^2 - 55x) = 14x - 70$.

7. **Divide:** $\dfrac{14x}{x} = 14$.

8. **Multiply:** $(x-5)(14) = 14x - 70$.

9. **Subtract:** $(14x - 70) - (14x - 70) = 0$.

The quotient is $2x^2 + 11x + 14$, so:

$$g(x) = (x-5)(2x^2 + 11x + 14).$$

Next, we factorize $2x^2 + 11x + 14$:

$$2x^2+11x+14 = 2x^2+7x+4x+14 = x(2x+7)+2(2x+7) = (2x+7)(x+2).$$

Thus, the fully factorized form of $g(x)$ is:

$$g(x) = (x-5)(2x+7)(x+2).$$

(c) Find the exact value of the area of the region R:

The region R is bounded by the curve $y = g(x)$ and the x-axis, and lies below the x-axis. The x-intercepts of $g(x)$ are:

$$x - 5 = 0 \quad \Rightarrow \quad x = 5,$$

$$2x + 7 = 0 \quad \Rightarrow \quad x = -\frac{7}{2},$$

$$x + 2 = 0 \quad \Rightarrow \quad x = -2.$$

The integral for the area is:

$$\textbf{Area} = \int_{-\frac{7}{2}}^{5} -g(x)\,dx = \int_{-\frac{7}{2}}^{5} -(2x^3+x^2-41x-70)\,dx.$$

Step 1: Integrate $-g(x)$:

$$\int g(x)\,dx = \frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{41}{2}x^2 - 70x + C.$$

$$\int -g(x)\,dx = -\left(\frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{41}{2}x^2 - 70x\right) + C.$$

Step 2: Evaluate the definite integral:

$$\textbf{Area} = \left[-\frac{1}{2}x^4 - \frac{1}{3}x^3 + \frac{41}{2}x^2 + 70x\right]_{-\frac{7}{2}}^{5}.$$

At $x = 5$:

$$-\frac{1}{2}(5)^4-\frac{1}{3}(5)^3+\frac{41}{2}(5)^2+70(5) = -\frac{1}{2}(625)-\frac{1}{3}(125)+\frac{41}{2}(25)+350.$$

At $x = -\frac{7}{2}$, substitute $x = -\frac{7}{2}$ into the same expression. Finally, subtract the value at $x = -\frac{7}{2}$ from the value at $x = 5$ to find the area.

Find the exact value of the area of the region R:

The area is given by:

$$\textbf{Area} = \int_{-\frac{7}{2}}^{5} -g(x)\,dx = \int_{-\frac{7}{2}}^{5} -(2x^3+x^2-41x-70)\,dx.$$

The integral of $-g(x)$ is:

$$\int -g(x)\,dx = -\left(\frac{1}{2}x^4 + \frac{1}{3}x^3 - \frac{41}{2}x^2 - 70x\right).$$

We evaluate the definite integral by substituting the limits $x = 5$ and $x = -\frac{7}{2}$.

Step 1: Evaluate at $x = 5$:

$$\textbf{At } x = 5:$$

$$-\left(\frac{1}{2}(5)^4 + \frac{1}{3}(5)^3 - \frac{41}{2}(5)^2 - 70(5)\right).$$

$$= -\left(\frac{1}{2}(625) + \frac{1}{3}(125) - \frac{41}{2}(25) - 350\right).$$

Simplify term by term:

$$\frac{1}{2}(625) = 312.5, \quad \frac{1}{3}(125) = 41.67, \quad \frac{41}{2}(25) = 512.5, \quad 70(5) = 350.$$

$$= -\left(312.5 + 41.67 - 512.5 - 350\right).$$

$$= -\left(-508.33\right) = 508.33.$$

Step 2: Evaluate at $x = -\frac{7}{2}$:

$$\textbf{At } x = -\frac{7}{2}:$$

$$-\left(\frac{1}{2}\left(-\frac{7}{2}\right)^4 + \frac{1}{3}\left(-\frac{7}{2}\right)^3 - \frac{41}{2}\left(-\frac{7}{2}\right)^2 - 70\left(-\frac{7}{2}\right)\right).$$

Simplify term by term:

$$\frac{1}{2}\left(-\frac{7}{2}\right)^4 = \frac{1}{2}\cdot\frac{2401}{16} = \frac{2401}{32} = 75.03,$$

$$\frac{1}{3}\left(-\frac{7}{2}\right)^3 = \frac{1}{3}\cdot\left(-\frac{343}{8}\right) = -\frac{343}{24} = -14.29,$$

$$-\frac{41}{2}\left(-\frac{7}{2}\right)^2 = -\frac{41}{2}\cdot\frac{49}{4} = -\frac{2009}{8} = -251.13,$$

$$-70\left(-\frac{7}{2}\right) = 245.$$

Adding these:

$$-\left(75.03 - 14.29 - 251.13 + 245\right) = -\left(54.61\right) = -54.61.$$

Step 3: Compute the definite integral:

$$\textbf{Area} = \textbf{Value at } x = 5 - \textbf{Value at } x = -\frac{7}{2}.$$

$$\textbf{Area} = 508.33 - (-54.61) = 508.33 + 54.61 = 562.94.$$

Final Answer:

$$\boxed{562.94 \textbf{ square units}}$$

....

11. (i) A circle C_1 has equation

$$x^2 + y^2 + 18x - 2y + 30 = 0$$

The line l is the tangent to C_1 at the point P(–5, 7)

Find an equation of l in the form ax + by + c = 0, where a, b and c are integers to be found.

(ii) A different circle C_2 has equation

$$x^2 + y^2 - 8x + 12y + k = 0$$

where k is a constant.

Given that C_2 lies entirely in the 4th quadrant, find the range of possible values for k.

Solution

(i)

$$x^2 + y^2 + 18x - 2y + 30 = 0$$

Differentiating with respect to x.

$$\frac{d}{dx}(x^2+y^2+18x-2y+30)=\frac{d}{dx}0$$

$$\frac{d}{dx}x^2+\frac{d}{dx}(y^2)+\frac{d}{dx}(18x)-\frac{d}{dx}(2y)+\frac{d}{dx}(30)=0$$

$$2x+\frac{d}{dy}(y^2)\frac{dy}{dx}+18\frac{dx}{dx}-2\frac{dy}{dx}+0=0$$

$$2x+2y\frac{dy}{dx}+18-2\frac{dy}{dx}=0$$

$$2y\frac{dy}{dx}-2\frac{dy}{dx}=-2x-18$$

$$2[y-1]\frac{dy}{dx}=-2(x+9)$$

$$[y-1]\frac{dy}{dx}=-(x+9)$$

$$\frac{dy}{dx}=-\frac{x+9}{y-1}$$

In particular ,slope at the point $P(x_1,y_1)$

$$\frac{dy}{dx}=-\frac{x_1+9}{y_1-1}$$

Putting $P(–5, 7) = P(x_1, y_1)$

$$\frac{dy}{dx} = -\frac{-5+9}{7-1} = -\frac{4}{6} = -\frac{2}{3} = m$$

Where m is the slope of the line l the tangent to C_1 **at the point P(–5, 7) .**

Equation of the line l tangent to the point $P(x_1, y_1)$

$$y - y_1 = m(x - x_1)$$

Putting $P(x_1, y_1) = P(-5, 7)$ **and** $= -\frac{2}{3}$

$$y - 7 = -\frac{2}{3}[x - (-5)]$$

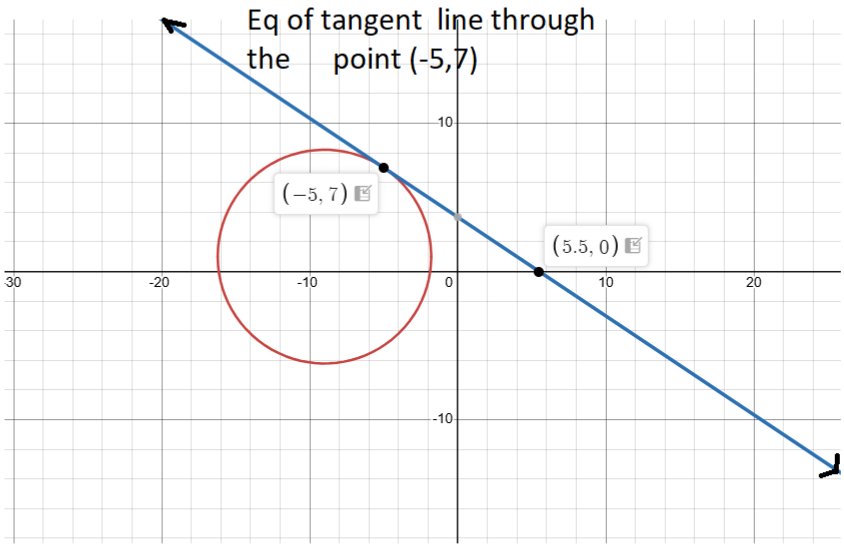
$$y - 7 = -\frac{2}{3}[x + 5]$$

$$3(y - 7) = -2[x + 5]$$

$$3y - 21 = -2x - 10$$

$$2x + 3y - 21 + 10 = 0$$

$$2x + 3y - 11 = 0$$



(ii)

To ensure the circle represented by the equation lies entirely in the fourth quadrant, we need to determine the values of k such that the circle does not cross or touch the axes.

Step 1: Rewrite the equation in standard form

The given equation is:

$$x^2 + y^2 - 8x + 12y + k = 0.$$

Group x-terms and y-terms, and complete the square:

- **For x:** $x^2 - 8x$ **Complete the square:**

$$x^2 - 8x = (x - 4)^2 - 16.$$

- **For y:** $y^2 + 12y$ **Complete the square:**

$$y^2 + 12y = (y + 6)^2 - 36.$$

Substituting back into the equation, we get:

$$(x - 4)^2 - 16 + (y + 6)^2 - 36 + k = 0.$$

Simplify:

$$(x-4)^2+(y+6)^2=52-k.$$

Step 2: Standard form of the circle

The equation is now in standard form:

$$(x-4)^2+(y+6)^2=r^2,$$

where the center of the circle is $(4,-6)$ and the radius is $r=\sqrt{52-k}$.

Step 3: Conditions for the circle to lie entirely in the fourth quadrant

The circle lies entirely in the fourth quadrant if:

1. **The center is in the fourth quadrant: $x>0$**

and $y < 0$ (which is true since the center is $(4, -6)$).

2. **The radius must be small enough so the circle does not touch or cross the x-axis or the y-axis.**

(a) Condition for the circle not touching the x-axis

The distance from the center $(4, -6)$ to the x-axis is 6. For the circle to not touch the x-axis:

$$r < 6 \quad \Rightarrow \quad \sqrt{52 - k} < 6.$$

Squaring both sides:

$$52 - k < 36 \quad \Rightarrow \quad k > 16.$$

(b) Condition for the circle not touching the y-axis

The distance from the center $(4, -6)$ to the y-axis is 4. For the circle to not touch the y-axis:

$$r < 4 \quad \Rightarrow \quad \sqrt{52 - k} < 4.$$

Squaring both sides:

$$52 - k < 16 \quad \Rightarrow \quad k > 36.$$

Step 4: Combine conditions

For the circle to lie entirely in the fourth quadrant:

$$k > 36.$$

Final Answer

$$\boxed{k > 36}$$

Note for k=36, circle touches y-axis and for $k < 36$, it will not be entirely in the 4th quadrant as shown in figure.

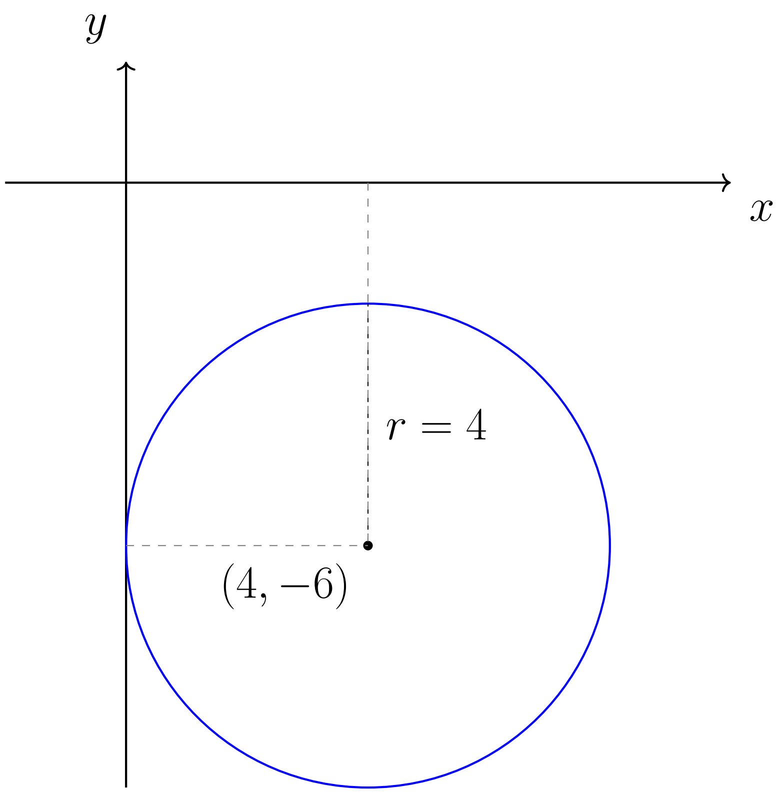
The given equation is:

$$x^2 + y^2 - 8x + 12y + 36 = 0.$$

Rewriting in standard form:

$$\begin{aligned} x^2 - 8x + y^2 + 12y &= -36, \\ (x-4)^2 - 16 + (y+6)^2 - 36 &= -36, \\ (x-4)^2 + (y+6)^2 &= 16. \end{aligned}$$

The circle has center $(4, -6)$ and radius $r = 4$.



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12. An advertising agency is monitoring the number of views of an online advert.

The equation

$$\log_{10} V = 0.072t + 2.379 \qquad 1 \leq t \leq 30, t \in N$$

is used to model the total number of views of the advert, V, in the first t days after the advert went live.

(a) Show that $V = ab^t$ **where a and b are constants to be found.**

Give the value of a to the nearest whole number and give the value of b to 3 significant figures.

(b) Interpret, with reference to the model, the value of ab.

Using this model, calculate

(c) the total number of views of the advert in the first 20 days after the advert went live. Give your answer to 2 significant figure

Solution

$$\log_{10} V = 0.072t + 2.379 \qquad 1 \leq t \leq 30, t \in N$$

In exponential form

$$10^{0.072t+2.379} = V \qquad 1 \leq t \leq 30, t \in N$$

$$10^{0.072t} 10^{2.379} = V \qquad 1 \leq t \leq 30, t \in N$$

Putting $10^{0.072} = b \Longrightarrow 10^{0.072t} = b^t$**,** $10^{2.379} = a$

Hence

$$ab^t = V \qquad 1 \leq t \leq 30, t \in N$$

$10^{2.379} = 239 = a$

$10^{0.072} = 1.180 = b$

(b)

$ab = 239 \times 1.180 = 282.1$

(c)Putting t=20

$$239(1.180)^{20} = V \qquad 1 \leq t \leq 30, t \in N$$

$$V = 6,546.94 \qquad 1 \leq t \leq 30, t \in N$$

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13. (a) Prove that for all positive values of a and b

$$\frac{4a}{b} + \frac{b}{a} \geq 4$$

(b) Prove, by counter example, that this is not true for all values of a and b. Solution

Solution

Part (a): Prove that $\frac{4a}{b} + \frac{b}{a} \geq 4$ for all positive a and b.

We start by applying the Arithmetic Mean–Geometric Mean (AM-GM) inequality:

Step 1: Use AM-GM inequality. **The AM-GM inequality states that for any two positive numbers x and y,**

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Equality holds if and only if $x = y$.

Let $x = \frac{4a}{b}$ and $y = \frac{b}{a}$. Then:

$$\frac{\frac{4a}{b} + \frac{b}{a}}{2} \geq \sqrt{\frac{4a}{b} \cdot \frac{b}{a}}.$$

Step 2: Simplify the geometric mean.

$$\sqrt{\frac{4a}{b} \cdot \frac{b}{a}} = \sqrt{4} = 2.$$

Thus,

$$\frac{\frac{4a}{b} + \frac{b}{a}}{2} \geq 2.$$

Step 3: Multiply through by 2.

$$\frac{4a}{b} + \frac{b}{a} \geq 4.$$

Equality holds if and only if $\frac{4a}{b} = \frac{b}{a}$, which simplifies to $b = 2a$. Hence, the inequality is true for all positive a and b, with equality when $b = 2a$.

Part (b): Provide a counterexample to show this is not true for all values of a and b.

The inequality $\frac{4a}{b} + \frac{b}{a} \geq 4$ assumes a and b are positive. However, the inequality may not hold if a or b are not positive.

Counterexample: **Let $a = 1$ and $b = -2$ (i.e., b is negative):**

$$\frac{4a}{b} + \frac{b}{a} = \frac{4(1)}{-2} + \frac{-2}{1} = -2 - 2 = -4,$$

which is clearly less than 4.

This proves the inequality does not hold for non-positive values of a or b.

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14. A curve has equation y = g(x).

Given that

- $g(x)$ **is a cubic expression in which the coefficient of x^3 is equal to the coefficient of x.**
- **The curve with equation $y = g(x)$ passes through the origin.**

- **The curve with equation $y = g(x)$ has a stationary point at $(2, 9)$.**

(a) find g(x),

(b) prove that the stationary point at (2, 9) is a maximum

Solution

Part (a) - Find $g(x)$

We are given:

- $g(x)$ **is a cubic expression:** $g(x) = ax^3 + bx^2 + cx + d$**,**
- **The coefficient of x^3 equals the coefficient of x:** $a = c$**,**
- **The curve passes through the origin:** $g(0) = 0$**, which implies** $d = 0$**,**

- **The curve has a stationary point at $(2, 9)$, so:**
  - $g(2) = 9$**,**
  - $g'(2) = 0$**.**

Thus, $g(x)$ takes the form:

$$g(x) = ax^3 + bx^2 + ax.$$

Step 1: Find the coefficients using $g(2) = 9$

$$g(2) = a(2)^3 + b(2)^2 + a(2) = 8a + 4b + 2a = 10a + 4b.$$

Since $g(2) = 9$, we have:

$$10a + 4b = 9. \tag{1}$$

Step 2: Use $g'(2) = 0$ to find another equation **The derivative of $g(x)$ is:**

$$g'(x) = 3ax^2 + 2bx + a.$$

Substituting $x = 2$:

$$g'(2) = 3a(2)^2 + 2b(2) + a = 12a + 4b + a = 13a + 4b.$$

Since $g'(2) = 0$, we have:

$$13a + 4b = 0. \tag{2}$$

Step 3: Solve the system of equations **From equations (1) and (2):**

$$10a + 4b = 9,$$

$$13a + 4b = 0.$$

Subtract equation (2) from equation (1):

$$(10a + 4b) - (13a + 4b) = 9 - 0,$$

$$-3a = 9 \implies a = -3.$$

Substitute $a = -3$ into equation (2):

$$13(-3)+4b = 0 \implies -39+4b = 0 \implies 4b = 39 \implies b = \frac{39}{4}.$$

Thus:

$$g(x) = -3x^3 + \frac{39}{4}x^2 - 3x.$$

Part (b) - Prove the stationary point at $(2, 9)$ is a maximum

Step 1: Find the second derivative **The first derivative is:**

$$g'(x) = 3(-3)x^2 + 2\left(\frac{39}{4}\right)x - 3 = -9x^2 + \frac{39}{2}x - 3.$$

The second derivative is:

$$g''(x) = -18x + \frac{39}{2}.$$

Step 2: Evaluate $g''(x)$ at $x = 2$

$$g''(2) = -18(2) + \frac{39}{2} = -36 + \frac{39}{2} = \frac{-72 + 39}{2} = \frac{-33}{2}.$$

Since $g''(2) < 0$, the stationary point at $(2, 9)$ is a maximum.

Final Answer

(a) $g(x) = -3x^3 + \frac{39}{4}x^2 - 3x$**,**

(b) The stationary point at $(2, 9)$ is a maximum because $g''(2) < 0$.

Solution

Part (a) - Find $g(x)$

We are given:

- $g(x)$ **is a cubic expression:** $g(x) = ax^3 + bx^2 + cx + d$**,**
- **The coefficient of** x^3 **equals the coefficient of** x**:** $a = c$**,**
- **The curve passes through the origin:** $g(0) = 0$**, which implies** $d = 0$**,**
- **The curve has a stationary point at** $(2, 9)$**, so:**
  - $g(2) = 9$**,**
  - $g'(2) = 0$**.**

Thus, $g(x)$ takes the form:

$$g(x) = ax^3 + bx^2 + ax.$$

Step 1: Find the coefficients using $g(2) = 9$

$$g(2) = a(2)^3 + b(2)^2 + a(2) = 8a + 4b + 2a = 10a + 4b.$$

Since $g(2) = 9$, we have:

$$10a + 4b = 9. \tag{1}$$

Step 2: Use $g'(2) = 0$ to find another equation **The derivative of $g(x)$ is:**

$$g'(x) = 3ax^2 + 2bx + a.$$

Substituting $x = 2$:

$$g'(2) = 3a(2)^2 + 2b(2) + a = 12a + 4b + a = 13a + 4b.$$

Since $g'(2) = 0$, we have:

$$13a + 4b = 0. \tag{2}$$

Step 3: Solve the system of equations **From equations (1) and (2):**

$$10a + 4b = 9,$$

$$13a + 4b = 0.$$

Subtracting equation (2) from equation (1):

$$(10a + 4b) - (13a + 4b) = 9 - 0,$$

$$-3a = 9 \implies a = -3.$$

Substitute $a = -3$ into equation (2):

$$13(-3)+4b = 0 \implies -39+4b = 0 \implies 4b = 39 \implies b = \frac{39}{4}.$$

Thus:

$$g(x) = -3x^3 + \frac{39}{4}x^2 - 3x.$$

Part (b) - Prove the stationary point at $(2, 9)$ is a maximum

Step 1: Find the second derivative **The first derivative is:**

$$g'(x) = 3(-3)x^2 + 2\left(\frac{39}{4}\right)x - 3 = -9x^2 + \frac{39}{2}x - 3.$$

The second derivative is:

$$g''(x) = -18x + \frac{39}{2}.$$

Step 2: Evaluate $g''(x)$ at $x = 2$

$$g''(2) = -18(2) + \frac{39}{2} = -36 + \frac{39}{2} = \frac{-72 + 39}{2} = \frac{-33}{2}.$$

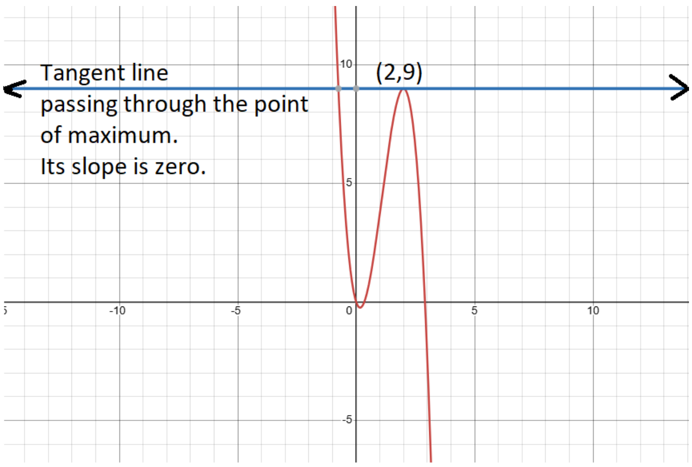
Since $g''(2) < 0$, the stationary point at $(2, 9)$

is a maximum.

Final Answer

- **(a)** $g(x) = -3x^3 + \frac{39}{4}x^2 - 3x$**,**
- **(b) The stationary point at** $(2, 9)$ **is a maximum because** $g''(2) < 0$**.**

Solution is shown below graphically .



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Derivation of the Law of Cosines

Derivation of the Law of Cosines

Consider a triangle $\triangle ABC$ with sides of lengths a, b, and c, where c is the side opposite angle C.

We will derive the formula $\cos(C) = \frac{b^2+c^2-a^2}{2bc}$.

Step 1: Place the Triangle in the Coordinate Plane

Place vertex A at the origin, so $A = (0, 0)$, and place vertex B along the positive x-axis, so $B = (b, 0)$. Let the coordinates of vertex C be (x_C, y_C).

Step 2: Distance Formula for Side c

The distance from point A to point C is given by the distance formula:

$$c^2 = x_C^2 + y_C^2$$

Step 3: Distance Formula for Side a

The distance from point B to point C is given by:

$$a^2 = (x_C - b)^2 + y_C^2$$

Expanding this:

$$a^2 = x_C^2 - 2bx_C + b^2 + y_C^2$$

Substitute $x_C^2 + y_C^2 = c^2$ from Step 2:

$$a^2 = c^2 - 2bx_C + b^2$$

Rearranging for x_C:

$$2bx_C = b^2 + c^2 - a^2$$

$$x_C = \frac{b^2 + c^2 - a^2}{2b}$$

Step 4: Relation to Cosine of Angle C

Using the definition of cosine in the coordinate plane:

$$\cos(C) = \frac{x_C}{b}$$

Substituting the expression for x_C:

$$\cos(C) = \frac{\frac{b^2+c^2-a^2}{2b}}{b} = \frac{b^2+c^2-a^2}{2bc}$$

Thus, the Law of Cosines is derived as:

$$\cos(C) = \frac{b^2+c^2-a^2}{2bc}$$

Step 5: Graphical Representation of Triangle $\triangle ABC$