Gauss and Stokes Theorem

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1 Gauss Theorem

Imagine you have a blob somewhere in space (A hollow shell, a surface), and there are some vectors ($\mathbf{F}$) inside the blob. A question you might ask is that how many vectors are flowing out of a point inside the blob? We can answer that by the divergence

$$\boldsymbol{\nabla} \cdot \mathbf{F} \tag{1}$$

To find out how much divergence is there in total inside the blob, we can simply sum up all the divergence across the whole volume of the blob via an integral, where dV is an infinitesimal volume of the blob.

$$\iiint_V \boldsymbol{\nabla} \cdot \mathbf{F} \; dV \tag{2}$$

This integral adds together all the vectors flowing out of all the tiny volumes of dV. As a shorthand, we will write the three integral signs as one. We finally have

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} \; dV \tag{3}$$

Another way to find at the total divergence is to notice that as a result from these little divergences from every tiny volume in space, there will be a total divergence passing through the surface of the blob. We can find this divergence at every bit of the surface by dotting the field with the normal vector of the blob. The reason for dotting is because the "effective" component of the vector must be normal to the blob, so dotting $\mathbf{F}$ with the normal vector of the blob "filters" out everything but the "effective" component.

$$\mathbf{F} \cdot \hat{\mathbf{n}} \tag{4}$$

Finally we can add up all the bits of divergence on the surface via a surface integral, where dS is an infinitesimally small slice of the surface.

$$\int_S \mathbf{F} \cdot \hat{\mathbf{n}} \; dS \tag{5}$$

This is equal to the previous method in (3), so we can say

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} \; dV = \int_S \mathbf{F} \cdot \hat{\mathbf{n}} \; dS \tag{6}$$

Which tells us that in a given region (enclosed by a mathematical blob), tiny divergences at every small volume in space within that region will result in a net divergence at the surface of the region.

2 Stokes Theorem

Imagine you have a plate somewhere in space, and there are some vectors ($\mathbf{F}$) on the plate. A question you might ask is how much the vectors are swirling around a given point on the surface on the plate? We can answer that by the curl

$$\nabla \times \mathbf{F} \tag{7}$$

To find how much swirling is going on in total, we can just need to add up the tiny bits of curl over the entire surface of the plate. This gives

$$\int_S \nabla \times \mathbf{F} \; dS \tag{8}$$

Where as always, dS is a tiny slice of the surface. As a result of these little curls, there will be a total curl around the whole plate. To find out how much curl there is about the borders of the plate, we can take a tiny bit of the border ($d\mathbf{l}$) and dot it with the vectors, giving

$$\mathbf{F} \cdot d\mathbf{l} \tag{9}$$

The dot product measures how much the vectors deviate from the border, which is perfect for measuring how much is rotating away from the loop. Now we can just sum over the entire closed loop by a line integral to get the total curl

$$\oint \mathbf{F} \cdot d\mathbf{l} \tag{10}$$

Relating the above to (8), we will have

$$\int_S \nabla \times \mathbf{F} \; dS = \oint \mathbf{F} \cdot d\mathbf{l} \tag{11}$$

Which simply states that a bunch of small curls on a given surface (a kind of mathematical plate that needn't be flat) will make the whole plate rotate with a net curl found by the line integral on the right.

3 Applications

3.1 Application of Gauss Theorem - Coloumb from Maxwell

An application of Gauss theorem can be found in electrodynamics. From Gauss' Law (the first Maxwell equation) in natural units, we know that

$$\nabla \cdot \mathbf{E} = \rho \tag{12}$$

A point charge at the origin is said to have 0 charge density in any point in space and an infinite charge density at the origin. We also know the charge density integrates to a charge of Q in total. This can be expressed by

$$\rho = Q\delta^3(x) \tag{13}$$

Where the shorthand $\delta^3(x) = \delta_x(x)\delta_y(y)\delta_z(z)$ is used. Now we can modify (12) to say that for a point charge,

$$\nabla \cdot \mathbf{E} = Q\delta^3(x) \tag{14}$$

We seek to find the electric field evaluated at a distance r away from the particle. Integrating both sides, we can say

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{E} \; dV = \int_V Q\delta^3(x) \; dV \tag{15}$$

We can apply the Gauss theorem to the left hand side, which states that

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{E} \; dV = \int_S \mathbf{E} \cdot \hat{\mathbf{n}} \; dS \tag{16}$$

From this, we recognise that point-source electric fields have spherical symmetry, which means that the "effective" part of the electric field will always be the same if you are at the same distance away, regardless of your angular position (looking at it from spherical coordinates, E depends only on r and not ϕ or θ). Mathematically, we can say $\mathbf{E} \cdot \hat{\mathbf{n}} = E$ (since we know that the delta function produces a field that points radially outwards, so it will always be normal to the sphere and thus parallel to $\hat{\mathbf{n}}$). This means that we can "pull" the E out, giving

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{E} \; dV = E \int_S dS \tag{17}$$

From the spherical symmetry, we know that the surface integral just finds the surface of a sphere distance r away from the charge. This gives

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{E} \; dV = E 4\pi r^2 \tag{18}$$

Now let's deal with the right hand side of (15). we have

$$\int_V Q\delta^3(x) \; dV = Q \tag{19}$$

This is from a property of the delta function that says that when multiplied by a function under an integral, it just "pulls out" a value out of the function. Here we have a constant function Q, so naturally, it just pulls out the constant itself. Now we can equate the left and right hand sides, which give

$$4\pi r^2 E = Q \tag{20}$$

After a simple rearrangement, we can find the magnitude of the electric field at any distance r.

$$E = \frac{Q}{4\pi r^2} \tag{21}$$

Which is precisely Coloumb's law in scalar form and natural units. We can convert it into SI units and vector form by simple algebra, which results in

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{4\pi r^2} \hat{\mathbf{r}} \tag{22}$$

Which is the classic Coloumb's law that we all know and love.

3.2 Application of Stokes Theorem - Orsted from Maxwell

An application of Stokes theorem can also be found in electrodynamics. This time we seek to find the magnetic field around a straight current carrying wire at distance r where the electric field does not change with time. We assume the wire is infinitely thin and runs along the x-direction infinitely with current j. This means the current density $\mathbf{j}$ can be found by

$$\mathbf{j} = < j\delta_y(y)\delta_z(z), 0, 0 > \tag{23}$$

The fourth Maxwell equation states that

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\mathbf{j} \tag{24}$$

We can take an infinitesimally thin slice of the wire and construct a mathematical "plate" of radius r around it. Now, we can take the surface integral of both sides over the plate.

$$\int_S \nabla \times \mathbf{B}\ dS = \frac{1}{c^2}\int_S \mathbf{j}\ dS \tag{25}$$

Dealing with the left hand side, we can use the Stokes theorem, which gives us

$$\int_S \nabla \times \mathbf{B}\ dS = \oint \mathbf{B} \cdot d\mathbf{l} \tag{26}$$

Again, we can pull out the magnetic field because it is the same if evaluated at a distance r and does not depend on the angle of displacement. Therefore, (26) becomes

$$\int_S \nabla \times \mathbf{B}\ dS = B \oint d\mathbf{l} \tag{27}$$

Since we were considering a perfectly circular plate, the line integral finds the circumference of the plate, which is famously given by $2\pi r$. Using this fact, we now have

$$\int_S \nabla \times \mathbf{B}\ dS = 2\pi r B \tag{28}$$

Now we can deal with the left hand side of (25). We will omit the factor of $\frac{1}{c^2}$ and only deal with the integral. A slice of the surface is given by $dS = dxdydz$.

$$\int_S j\delta_y(y)\delta_z(z)\ dxdydz = j \tag{29}$$

Which we can again get from the properties of a delta function. Now we can combine the two sides of the equation to get

$$2\pi r B = \frac{j}{c^2} \tag{30}$$

After some rearranging, we are left with an expression for B

$$B = \frac{j}{2\pi c^2 r} \tag{31}$$

Which is exactly Orsted's law in natural units and in scalar form.