Problem 10

Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$.

You want the probability that X falls within c standard deviations of the mean to be at least 98%. What is the minimum value of c such that:

$$\Pr(\mu - c\sigma \leq X \leq \mu + c\sigma) \geq 0.98$$

Solution. Standardize using the transformation $Z = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0,1)$. Then,

$$\Pr(\mu - c\sigma \leq X \leq \mu + c\sigma) = \Pr(|Z| \leq c) = 2\Phi(c) - 1,$$

where Φ is the standard normal cumulative distribution function.

Impose the requirement:

$$2\Phi(c) - 1 \geq 0.98 \quad \Rightarrow \quad \Phi(c) \geq 0.99.$$

Taking the inverse CDF:

$$c = \Phi^{-1}(0.99) \approx 2.326.$$

Hence, the minimum value ensuring at least 98% coverage is $\boxed{c \approx 2.33}$.

Problem 11

Let X be a discrete variable representing the number of trials until a machine part fails, with range $\{1, 2, 3, \ldots\}$.

(a) Define the discrete failure rate function $r(k) = \Pr(X = k \mid X \geq k)$.

Solution. The failure rate function is defined in terms of the PMF and the survival function:

$$r(k) = \frac{\Pr(X = k)}{\Pr(X \geq k)} = \frac{p_X(k)}{S_X(k)}, \quad k = 1, 2, \ldots,$$

where $p_X(k) = \Pr(X = k)$ is the PMF and $S_X(k) = \Pr(X \geq k)$ is the survival function.

(b) Suppose X is geometric with success probability $p = 0.3$. Sketch the PMF and failure rate. Explain their differences.

Solution. For the geometric distribution,

$$p_X(k) = (1-p)^{k-1}p = 0.7^{k-1} \cdot 0.3, \quad k = 1, 2, \ldots.$$

The survival function becomes:

$$S_X(k) = \sum_{j=k}^{\infty} 0.7^{j-1} \cdot 0.3 = 0.7^{k-1}.$$

Problem 13

Let $X \sim \text{Poisson}(\theta)$ be a random variable representing the number of defective widgets found during routine inspection at a factory per hour.

(a) Prove that

$$\mathbb{E}[X(X-3)] = \theta^2 - 2\theta$$

by applying the definition of expected value and manipulating the Poisson PMF.

Solution. Start from the definition:

$$\mathbb{E}[X(X-3)] = \sum_{k=0}^{\infty} k(k-3)e^{-\theta}\frac{\theta^k}{k!}.$$

Split the expression into two terms:

$$\sum_{k=0}^{\infty} k^2 e^{-\theta}\frac{\theta^k}{k!} - 3\sum_{k=0}^{\infty} k e^{-\theta}\frac{\theta^k}{k!}.$$

Rewrite $k^2 = k(k-1) + k$ and separate:

$$\sum_{k=0}^{\infty} k(k-1)e^{-\theta}\frac{\theta^k}{k!} + \sum_{k=0}^{\infty} k e^{-\theta}\frac{\theta^k}{k!}.$$

Shift indices: $k \mapsto k+2$ and $k \mapsto k+1$ respectively. The series simplify to:

$$e^{-\theta}\theta^2 \sum_{k=0}^{\infty} \frac{\theta^k}{k!} + e^{-\theta}\theta \sum_{k=0}^{\infty} \frac{\theta^k}{k!} = e^{-\theta}\theta^2 e^{\theta} + e^{-\theta}\theta e^{\theta} = \theta^2 + \theta.$$

Also, $\sum_{k=0}^{\infty} k e^{-\theta}\frac{\theta^k}{k!} = \theta$. Putting it all together:

$$\mathbb{E}[X(X-3)] = (\theta^2 + \theta) - 3\theta = \theta^2 - 2\theta.$$

(b) Use the result from part (a) and the identity $X^2 = X(X-3) + 3X$ to derive $\mathbb{E}[X^2]$, and hence find $\text{Var}(X)$.

Solution. Substitute into the identity:

$$\mathbb{E}[X^2] = \mathbb{E}[X(X-3)] + 3\mathbb{E}[X] = (\theta^2 - 2\theta) + 3\theta = \theta^2 + \theta.$$

Now compute the variance:

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = (\theta^2 + \theta) - \theta^2 = \theta.$$

Thus, the variance of a Poisson-distributed variable is $\boxed{\text{Var}(X) = \theta}$.