# The Binomial Expansion

The binomial Theorem allows us to expand many brackets without multiplying each bracket out one by one. It states:

$$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b^1+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{r}a^{n-r}b^r+\ldots+b^n,\quad \binom{n}{r}=\frac{n!}{r!(n-r)!}$$

To expand $(x+2)^4$, we could expand $(x+2)(x+2)(x+2)(x+2)$, which would be a very long winded process. Or we could just substitute for $a$, $b$ and $n$ into the expression for the binomial expansion. Example: Expand $(3+2x)^5$ $a=3$, $b=2x$, $n=5$ then

$$(3+2x)^5=3^5+\binom{5}{1}3^{5-1}*2x+\binom{5}{2}3^{5-2}(2x)^2+\binom{5}{3}3^{5-3}(2x)^3+\binom{5}{4}3^{5-4}(2x)^4+(2x)^5$$

which simplifies to

$$243+5*81*2x+15*27*4x^2+15*9*8x^3+5*3*16x^4+32x^5$$

and further to

$$243+810x+1720x^2+240x^3+32x^5$$

We may be asked to solve questions involving the coefficients. For example, the coefficient of $x^2$ in the binomial expansion of $(2+x)^n$ is equal to 3 times the coefficient of $x^3$ .Find n.

Using the binomial expansion, the coefficient of $x^2$ is $\binom{n}{2}2^{n-2}$

Using the binomial expansion, the coefficient of $x^3$ is $\binom{n}{3}2^{n-3}$

Hence we can write down the equation $\binom{n}{2}2^{n-2}=3*\binom{n}{3}2^{n-3}$

Now we have to perform some trickery:

$$\frac{2^{n-2}}{2^{n-1}}=\frac{3*\binom{n}{2}}{\binom{n}{3}}\to 2=\frac{3*3}{n-2}\to 2n-4=0\to n=\frac{13}{2}.$$

Example: **Find the equation of the normal to the curve** $x^3+4xy-16y^2=0$ **at the point** $(2,1)$.

We differentiate implicitly to get $3x^2+4y+4x\dfrac{dy}{dx}-32y\dfrac{dy}{dx}=0.$ We have to make $\dfrac{dy}{dx}$ the subject of this equation.

$$3x^2+4y+(4x-32y)\frac{dy}{dx}=0\rightarrow(4x-32y)\frac{dy}{dx}=-(3x^2+4y)\rightarrow\frac{dy}{dx}=-\frac{3x^2+4y}{4x-32y}.$$

The gradient at the point $(2,1)$ is $m=\dfrac{3*2^2+4*1}{4*2-32*1}=\dfrac{16}{-24}=-\dfrac{2}{3}.$

$$y-y_1=m(x-x_1)\rightarrow y-1=-\frac{2}{3}(x-2)\rightarrow y=-\frac{2}{3}x+\frac{7}{3.}$$

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# The Factor Theorem

We can use the factor theorem to find if a particular value of $x$ is a root of a polynomial equation or to find out if a particular linear factor divides a polynomial perfectly, with no remainder.

**The Factor Theorem**

If $(ax-b)$ is a factor of the polynomial $p(x)$ then $\frac{p(x)}{(ax-b)}$ has no remainder or equivalently, $p(\frac{b}{a})=0$.

We show uses of the factor theorems in the following examples.

**Example: Show that $(x+2)$ is a factor of $p(x)=2x^3-12x+x-6$.**

Solve $x+2=0$ to get $x=-2$.

$p(-2)=2*(-2)^3-12*-2\pm2-6=0$ therefore $(x+2)$ is a factor of $p(x)$.

**Example: Find the polynomial with with integer coefficients whose roots are** $\frac{1}{2}$, $-2$, $6$.

If these are the roots, the factors are $(x-\frac{1}{2})$ or $(2x-1)$, $(x--2)$ or $(x+2)$ and $(x-6)$, hence we expand and simplify $(2x-1)(x+2)(x-6)$ to get $2x^3-9x^2-20x+12$.#



Example: When $p(x)=2x^3+ax^2+bx+1$ is divided by $(x-2)$ or $(x+1)$ the remainder is zero. Find a and $b$, hence find the polynomial $p(x)$.

Remainder 0 when $p(x)$ is divided by $(x-2)\rightarrow p(2)=0\rightarrow16+4a+2b+1=0\rightarrow4a+2b=-15$.

Remainder 0 when $p(x)$ is divided by $(x+2)\rightarrow p(-1)=0\rightarrow-2+a-b+1=0\rightarrow a-b=1$.

Now we solve the simultaneous equations

$4a+2b=-15$ (1)

$a-b=1$ (2)

$(1)+2*(2)6a=-13\rightarrow a=-\frac{13}{6}$

$(1)-4*(2)6b=-19\rightarrow b=-\frac{19}{6}$. Hence, $p(x)=2x^3-\frac{13}{6}x^2-\frac{19}{6}x+1$

**Example: Differentiate** $\frac{e^x}{x^2-1}$

$$f=e^x\,,\ g=x^2-1\rightarrow\frac{df}{dx}=e^x\,,\ \frac{dg}{dx}=2\mathrm{x}$$

$$\frac{dh}{dx}=\frac{g*\frac{df}{dx}-f*\frac{dg}{dx}}{g^2}=\frac{(x^2-1)e^x-e^x*2\mathrm{x}}{(x^2-1)^2}=\frac{e^x(x^2-2\mathrm{x}-1)}{(x^2-1)^2}$$

# Solving Quadratic Exponential Equations by Substitution

Some exponential equations can be factorised in linear factors. The simplest can be factorised into quadratic equations. We then put each factor equal to zero and solve it.

Example: Solve $e^{2x}-9e^{x}+20=0.$ (1)

Factorise to get $(e^{x}-4)(e^{x}-5)=0$

$e^{x}-4=0\rightarrow e^{x}=4\rightarrow x=\ln 4$ or $e^{x}-5=0\rightarrow e^{x}=5\rightarrow x=\ln 5.$

The above equation has two solutions. In general, as for quadratic equations, an exponential which can be expressed as two factors can have one, two or no solutions. It is convenient to make clear the connection by expressing the original equation as a quadratic using the substitution $p=e^{x}.$ Then $p^{2}=e^{2x}$ and equation (1) above becomes $p^{2}-9p+20=0.$ This equation factorises to give $(p-4)(p-5)=0$ so $p=4, 5.$ Since the original equation was expressed in terms of $x,$ we still have to find $x,$ but we can use the substitution $p=e^{x}$ with the values of $p$ that we have found, to find $x.$

$p=e^{x}=4\rightarrow x=\ln 4$ or $p=e^{x}=5\rightarrow x=\ln 5.$

Example: Solve $3e^{4x}-8e^{2x}-3=0$

Substitute $e^{2x}=p$ to get $3p^{2}-8p-3=0$ and factorise this expression to give $(3p+1)(p-3)=0$



$3p+1=0\rightarrow p=-\frac{1}{3}\rightarrow e^{2x}=-\frac{1}{3}\rightarrow x=\frac{1}{2}\ln(-\frac{1}{3}).$ This has no solution since there is no real log of a negative number.

$p-3=0\rightarrow e^{2x}=3\rightarrow x=\frac{1}{2}\ln 3.$

Example: Solve $3e^{4x}+10e^{2x}+3=0$

Substitute $e^{2x}=p$ to get $3p^{2}+10p+3=0$ and factorise this expression to give $(3p+1)(p+3)=0$

$3p+1=0\rightarrow p=-\frac{1}{3}\rightarrow e^{2x}=-\frac{1}{3}\rightarrow x=\frac{1}{2}\ln(-\frac{1}{3}).$ This has no solution since there is no real log of a negative number. This has no solution since there is no real log of a negative number hence the equation has no solutions.

$p+3=0\rightarrow e^{2x}=-3\rightarrow x=\frac{1}{2}\ln(-3).$

Example: Solve $4\sin x = 7\cos x$

$$\frac{sinx}{cosx} = \frac{7}{4} \rightarrow tanx = \frac{7}{4} \rightarrow x = \arctan\left(\frac{7}{4}\right) = 60.26\, degrees$$

Now we we the property of the tan curve that it repeats every 180 degrees. The solutions are

60.26, 180+60.26, 360+60.26, 540+60.26, 720+60.26, 900+60.26.....degrees