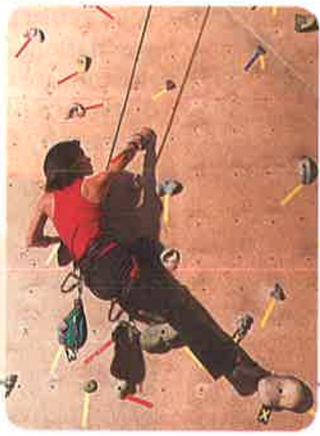
8.2 Graph Simple Rational Functions



Before You graphed polynomial functions.
Now You will graph rational functions.
Why? So you can find average monthly costs, as in Ex. 38.

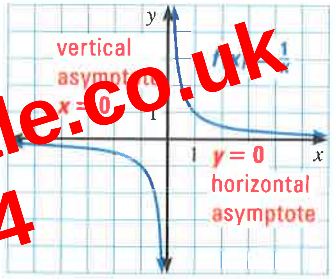
Key Vocabulary
- **rational function**
- **domain,** *p. 72*
- **range,** *p. 72*
- **asymptote,** *p. 478*

A **rational function** has the form $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. The inverse variation function $f(x) = \frac{a}{x}$ is a rational function. The graph of this function when $a = 1$ is shown below.

KEY CONCEPT *For Your Notebook*

Parent Function for Simple Rational Functions

The graph of the parent function $f(x) = \frac{1}{x}$ is a *hyperbola*, which consists of two symmetrical parts called *branches*. The domain and range are all nonzero real numbers.



Any function of the form $g(x) = \frac{a}{x}$ $(a \neq 0)$ has the same asymptotes, domain, and range as the function $f(x) = \frac{1}{x}$.

EXAMPLE 1 Graph a rational function of the form $y = \frac{a}{x}$

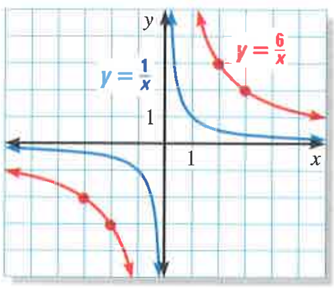
Graph the function $y = \frac{6}{x}$. Compare the graph with the graph of $y = \frac{1}{x}$.

Solution

STEP 1 **Draw** the asymptotes $x = 0$ and $y = 0$.

STEP 2 **Plot** points to the left and to the right of the vertical asymptote, such as $(-3, -2)$, $(-2, -3)$, $(2, 3)$, and $(3, 2)$.

STEP 3 **Draw** the branches of the hyperbola so that they pass through the plotted points and approach the asymptotes.



The graph of $y = \frac{6}{x}$ lies farther from the axes than the graph of $y = \frac{1}{x}$. Both graphs lie in the first and third quadrants and have the same asymptotes, domain, and range.

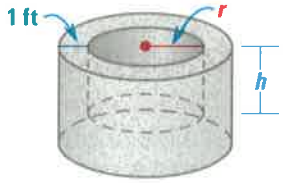
INTERPRET TRANSFORMATIONS
The graph of $y = \frac{6}{x}$ is a vertical stretch of the graph of $y = \frac{1}{x}$ by a factor of 6.

36. **CHALLENGE** You need to build a cylindrical water tank using 100 cubic feet of concrete. The sides and the base of the tank must be 1 foot thick.

a. Write an equation that gives the tank's inner height h in terms of its inner radius r.

b. Write an equation that gives the volume V of water that the tank can hold as a function of r.

c. Graph the equation from part (b). What values of r and h maximize the tank's capacity?



MIXED REVIEW

PREVIEW Prepare for Lesson 8.4 in Exs. 37–45.

Factor the expression.

37. $x^2 - 64$ (p. 252)
38. $x^2 - 8x - 48$ (p. 252)
39. $18x^2 - 37x - 20$ (p. 259)
40. $12x^2 - 15x - 18$ (p. 259)
41. $5x^2 + 22x - 30$ (p. 259)
42. $5x^3 + 40$ (p. 353)
43. $x^3 - 4x^2 + 8x - 32$ (p. 353)
44. $x^3 + 2x^2 - 35x$ (p. 353)
45. $x^5 - 9x^3 - 36x$ (p. 353)

Simplify the expression. Tell which properties of exponents you used. (p. 330)

46. $\dfrac{x^5y}{x^2y^4}$
47. $\dfrac{48x^{-1}y^4}{6x^2y^3}$
48. $\left(\dfrac{x^2y^4}{xy^5}\right)^2$
49. $\left(\dfrac{72x^3y^{-1}}{12x^{-1}y^2}\right)^{-1}$
50. $\dfrac{6x^{-2}y^2}{36xy^{-3}}$
51. $\left(\dfrac{x^5y^4}{x^7y^8}\right)^{-2}$
52. $\left(\dfrac{90x^3y^{-1}}{18x^{-1}y}\right)^2$
53. $\left(\dfrac{y}{x^2y^5}\right)$

QUIZ for Lessons 8.1–8.3

The variables x and y vary inversely. Use the given values to write an equation relating x and y. Then find y when $x = -4$. (p. 551)

1. $x = 8, y = 3$
2. $x = 2, y = -9$
3. $x = -5, y = \frac{8}{3}$
4. $x = -\frac{1}{4}, y = -32$

Graph the function.

5. $y = \dfrac{3}{2x}$ (p. 558)
6. $y = \dfrac{4}{x-2} + 1$ (p. 558)
7. $f(x) = \dfrac{-2x}{3x-6}$ (p. 558)
8. $y = \dfrac{-8}{x^2-1}$ (p. 565)
9. $y = \dfrac{x^2-6}{x^2+2}$ (p. 565)
10. $g(x) = \dfrac{x^3-8}{2x^2}$ (p. 565)

11. **SOFTBALL** A pitcher throws 16 strikes in her first 38 pitches. The table shows how the pitcher's strike percentage changes if she throws x consecutive strikes after the first 38 pitches. Write a rational function for the strike percentage in terms of x. Graph the function. How many consecutive strikes must the pitcher throw to reach a strike percentage of 0.60? (p. 558)

x	Total strikes	Total pitches	Strike percentage
0	16	38	0.42
5	21	43	0.49
10	26	48	0.54
x	$x + 16$	$x + 38$	?

EFFICIENCY Manufacturers often package their products in a way that uses the least amount of packaging material. One measure of the efficiency of a package is the ratio of its surface area to its volume. The smaller the ratio, the more efficient the packaging.

EXAMPLE 2 Solve a multi-step problem

PACKAGING A company makes a tin to hold flavored popcorn. The tin is a rectangular prism with a square base. The company is designing a new tin with the same base and twice the height of the old tin.

- Find the surface area and volume of each tin.
- Calculate the ratio of surface area to volume for each tin.
- What do the ratios tell you about the efficiencies of the two tins?

Solution

	Old tin	New tin	
STEP 1	$S = 2s^2 + 4sh$	$S = 2s^2 + 4s(2h)$	**Find surface area, *S*.**
		$= 2s^2 + 8sh$	
	$V = s^2h$	$V = s^2(2h)$	**Find volume, *V*.**
		$= 2s^2h$	
STEP 2	$\frac{S}{V} = \frac{2s^2 + 4sh}{s^2h}$	$\frac{S}{V} = \frac{2s^2 + 8sh}{2s^2h}$	**Write ratio of S to *V*.**
	$= \frac{\cancel{s}(2s + 4h)}{\cancel{s}(sh)}$	$= \frac{\cancel{2s}(s + 4h)}{\cancel{2s}(sh)}$	**Divide out common factor.**
	$= \frac{2s + 4h}{sh}$	$= \frac{s + 4h}{sh}$	**Simplified form**

STEP 3 $\frac{2s + 4h}{sh} > \frac{s + 4h}{sh}$ because the left side of the inequality has a greater numerator than the right side and both have the same (positive) denominator. The ratio of surface area to volume is *greater* for the old tin than for the new tin. So, the old tin is *less* efficient than the new tin.

✓ GUIDED PRACTICE for Examples 1 and 2

Simplify the expression, if possible.

1. $\frac{2(x + 1)}{(x + 1)(x + 3)}$

2. $\frac{40x + 20}{10x + 30}$

3. $\frac{4}{x(x + 2)}$

4. $\frac{x + 4}{x^2 - 16}$

5. $\frac{x^2 - 2x - 3}{x^2 - x - 6}$

6. $\frac{2x^2 + 10x}{3x^2 + 16x + 5}$

7. **WHAT IF?** In Example 2, suppose the new popcorn tin is the same height as the old tin but has a base with sides twice as long. What is the ratio of surface area to volume for this tin?

EXAMPLE 7 Divide a rational expression by a polynomial

Divide: $\dfrac{6x^2 + x - 15}{4x^2} \div (3x^2 + 5x)$

$$\frac{6x^2 + x - 15}{4x^2} \div (3x^2 + 5x) = \frac{6x^2 + x - 15}{4x^2} \cdot \frac{1}{3x^2 + 5x}$$ **Multiply by reciprocal.**

$$= \frac{(3x + 5)(2x - 3)}{4x^2} \cdot \frac{1}{x(3x + 5)}$$ **Factor.**

$$= \frac{\cancel{(3x + 5)}(2x - 3)}{4x^2(x)\cancel{(3x + 5)}}$$ **Divide out common factors.**

$$= \frac{2x - 3}{4x^3}$$ **Simplified form**

✓ GUIDED PRACTICE for Examples 6 and 7

Divide the expressions. Simplify the result.

11. $\dfrac{4x}{5x - 20} \div \dfrac{x^2 - 2x}{x^2 - 6x + 8}$

12. $\dfrac{2x^2 + 3x - 5}{6x} \div (2x^2 + 5x)$

8.4 EXERCISES

HOMEWORK KEY

○ = **WORKED-OUT SOLUTIONS** on p. WS15 for Exs. 7, 25, and 49

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 20, 23, 50, and 52

SKILL PRACTICE

1. VOCABULARY Copy and complete: To divide one rational expression by another, multiply the first rational expression by the _?_ of the second rational expression.

2. ★ **WRITING** How do you know when a rational expression is simplified?

EXAMPLE 1 on p. 573 for Exs. 3–20

REASONING **Match the rational expression with its simplified form.**

3. $\dfrac{x^2 - 9x + 14}{x^2 - 5x - 14}$ **4.** $\dfrac{x^2 - 4}{x^2 + 9x + 14}$ **5.** $\dfrac{x^2 + 5x - 14}{x^2 - 4x + 4}$

A. $\dfrac{x - 2}{x + 7}$ **B.** $\dfrac{x - 2}{x + 2}$ **C.** $\dfrac{x + 7}{x - 2}$

SIMPLIFYING **Simplify the rational expression, if possible.**

6. $\dfrac{4x^2}{20x^2 - 12x}$ **7.** $\dfrac{x^2 - x - 20}{x^2 + 2x - 15}$ **8.** $\dfrac{x^2 + 2x - 24}{x^2 + 7x + 6}$ **9.** $\dfrac{x^2 - 11x + 24}{x^2 - 3x - 40}$

10. $\dfrac{x^2 + 4x + 4}{x^2 - 5x + 4}$ **11.** $\dfrac{2x^2 + 2x - 4}{x^2 - 5x - 14}$ **12.** $\dfrac{x - 4}{x^3 - 64}$ **13.** $\dfrac{x^2 - 36}{x^2 + 12x + 36}$

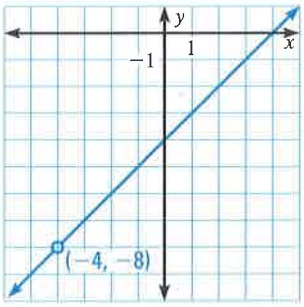
14. $\dfrac{3x^3 + 6x^2 + 12x}{x^3 - 8}$ **15.** $\dfrac{8x^2 + 10x - 3}{6x^2 + 13x + 6}$ **16.** $\dfrac{5x^2 + 18x - 8}{10x^2 - x - 2}$ **17.** $\dfrac{x^3 - 5x^2 - 3x + 15}{x^2 - 8x + 15}$

POINT DISCONTINUITY **In Exercises 44–46, use the following information.**

The graph of a rational function can have a hole in it, called a *point discontinuity*, where the function is undefined. An example is shown below.

$$y = \frac{x^2 - 16}{x + 4} = \frac{\cancel{(x+4)}(x-4)}{\cancel{x+4}} = x - 4$$

The graph of $y = \frac{x^2 - 16}{x + 4}$ is the same as the graph of $y = x - 4$ except that there is a hole at $(-4, -8)$ because the rational function is not defined when $x = -4$.



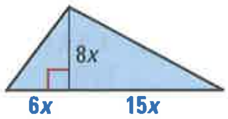
Graph the rational function. Use an open circle for a point discontinuity.

44. $y = \frac{x^2 + 10x + 21}{x + 3}$

45. $y = \frac{x^2 - 36}{x - 6}$

46. $y = \frac{2x^2 - x - 10}{x + 2}$

47. **CHALLENGE** Find the ratio of the perimeter to the area of the triangle shown at the right.



PROBLEM SOLVING

EXAMPLE 2
on p. 574 for
Exs. 48, 50–52

48. **GEOMETRY** Find the ratio of the volume of the square pyramid to the volume of the inscribed cone. Write your answer in simplified form.

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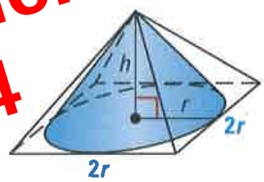

49. **ENTERTAINMENT** From 1992 to 2002, the gross ticket sales S (in millions of dollars) to Broadway shows and the total attendance A (in millions) at the shows can be modeled by

$$S = \frac{-6420t + 292{,}000}{6.02t^2 - 125t + 1000} \quad \text{and} \quad A = \frac{-407t + 7220}{5.92t^2 - 131t + 1000}$$

where t is the number of years since 1992. Write a model for the *average* dollar amount a person paid per ticket as a function of the year. What was the average amount a person paid per ticket in 1999?

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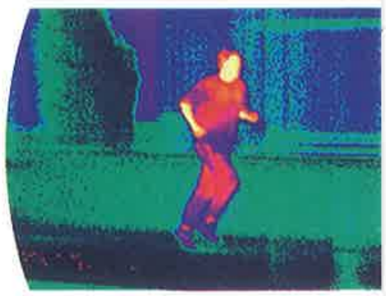
50. ★ **SHORT RESPONSE** Almost all of the energy generated by a long-distance runner is released in the form of heat. For a runner with height H and speed V, the rate h_g of heat generated and the rate h_r of heat released can be modeled by $h_g = k_1H^3V^2$ and $h_r = k_2H^2$ where k_1 and k_2 are constants.

a. Write the ratio of heat generated to heat released. Simplify the expression.

b. When the ratio of heat generated to heat released equals 1, how is speed related to height? Does a taller or shorter runner have the advantage? *Explain.*



Thermogram of runner

Graphing Calculator ACTIVITY *Use after Lesson 8.4*

@HomeTutor classzone.com Keystrokes

8.4 Verify Operations with Rational Expressions

QUESTION How can you use a graphing calculator to verify the results of operations on rational expressions?

EXAMPLE Check a simplified rational expression in two ways

Simplify $\frac{x^2 - x - 12}{x^2 - 9x + 20}$. Then verify the result numerically and graphically.

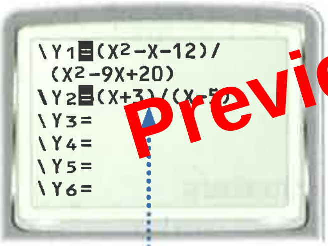
STEP 1 Simplify expression

Simplify the rational expression by factoring the numerator and denominator, then dividing out common factors.

$$\frac{x^2 - x - 12}{x^2 - 9x + 20} = \frac{\cancel{(x-4)}(x+3)}{\cancel{(x-4)}(x-5)} = \frac{x+3}{x-5}$$

STEP 2 Enter expressions

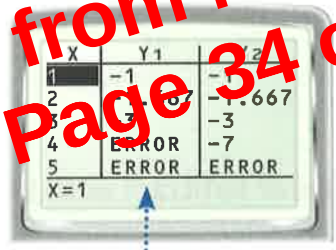
Enter the original expression as y_1 and the simplified result as y_2. Use the *thick* graph style for y_2.



Remember to use parentheses correctly.

STEP 3 Display table

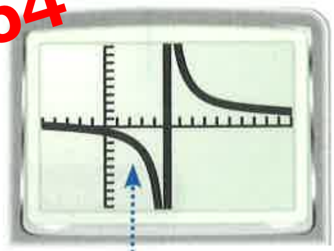
Use the *table* feature to examine corresponding values of the two expressions.



The values of y_1 and y_2 are the same, except that y_1 is undefined when $x = 4$ and $x = 5$, and y_2 is undefined only when $x = 5$.

STEP 4 Display graphs

Put your calculator in *connected* mode. Display the graphs in an appropriate viewing window.



By using the *thick* graph style for y_2, you can see the graph of y_2 being drawn over the graph of y_1. So, the graphs coincide.

PRACTICE

Simplify the expression. Verify your result numerically and graphically.

1. $\frac{x^2 - 5x}{x^2 - 7x + 10}$

2. $\frac{3x^2 + 6x}{x^2 - 2x - 8}$

3. $\frac{x^2 + 5x + 4}{x^2 + x - 12}$

Perform the indicated operation and simplify. Verify your result numerically and graphically.

4. $\frac{x+3}{5x^2} \cdot \frac{x-1}{x+3}$

5. $\frac{4x^2 - 8x}{5x + 15} \div \frac{x-2}{x+3}$

6. $\frac{x^2 - 3x - 10}{x^2 + 3x + 3} \cdot \frac{x^2 + 2x - 3}{x^2 + x - 2}$

PROBLEM SOLVING WORKSHOP
LESSON 8.6

Using ALTERNATIVE METHODS

Another Way to Solve Example 6, page 592

MULTIPLE REPRESENTATIONS In Example 6 on page 592, you solved a rational equation algebraically. You can also solve rational equations using tables and graphs.

PROBLEM

VIDEO GAME SALES From 1995 through 2003, the annual sales S (in billions of dollars) of entertainment software can be modeled by

$$S(t) = \frac{848t^2 + 3220}{115t^2 + 1000}, \quad 0 \le t \le 8$$

where t is the number of years since 1995. For which year were the total sales of entertainment software about $5.3 billion?

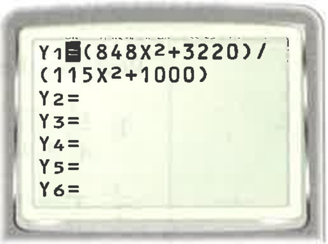
METHOD 1

Using a Table The problem requires solving the following rational equation:

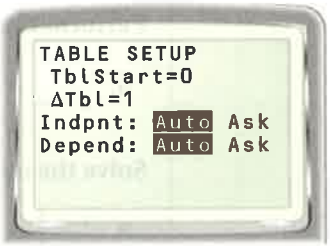
$$5.3 = \frac{848t^2 + 3220}{115t^2 + 1000}$$

One way to solve this equation is to make a table of values. You can use a graphing calculator to make the table.

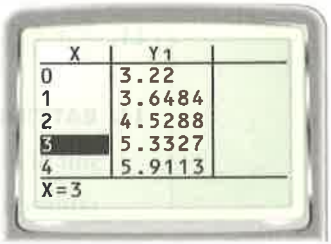
STEP 1 **Enter** the function $y = \frac{848x^2 + 3220}{115x^2 + 1000}$ into a graphing calculator.



STEP 2 **Set up** a table of values for the function. Start the table at zero so that the first several x-values in the table are in the domain of the function. The step value (ΔTbl) should represent one entire year.



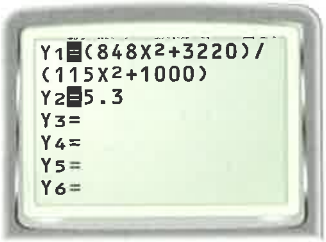
STEP 3 **Create** the table of values. You can see that $y \approx 5.3$ when $x = 3$.



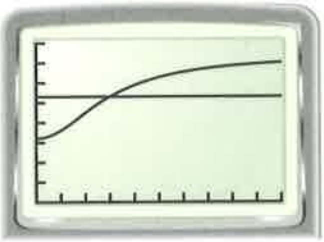
▶ Because $x = 3$ represents the number of years after 1995, total sales of entertainment software were about $5.3 billion in 1998.

METHOD 2 **Using a Graph** You can also use a graph to solve $5.3 = \dfrac{848t^2 + 3220}{115t^2 + 1000}$.

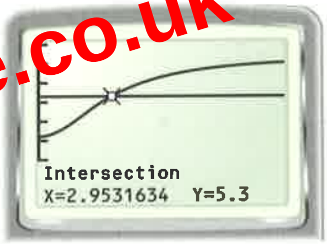
STEP 1 **Enter** the functions $y = \dfrac{848x^2 + 3220}{115x^2 + 1000}$ and $y = 5.3$ into a graphing calculator.



STEP 2 **Graph** the functions. Adjust the viewing window so that it shows the point in the first quadrant where the graphs intersect.



STEP 3 **Find** the intersection point of the graphs using the calculator's *intersect* feature. The graphs intersect at about (3.0, 5.3).



▶ Total sales of entertainment software were about \$5.3 billion 3 years after 1995, or in the year 1998.

PRACTICE

RATIONAL EQUATIONS **Solve the equation using a table and using a graph.**

1. $\dfrac{80x^2 + 300}{15x^2 + 200} = 4.2$

2. $\dfrac{5x + 5}{x^2 + 4} = 2$

3. $\dfrac{9x + 2}{x - 5} = 20.75$

4. $\dfrac{6x^2}{2x - 3} = 18$

5. $\dfrac{14x^2 + 60}{5x^2 + 7} = 3.5$

6. **WHAT IF?** In the problem on page 596, suppose you want to find the year when total sales of entertainment software were \$4.5 billion. Find this year using a table and using a graph.

7. **DIVING** The recommended percent p of oxygen (by volume) in the air that a diver breathes is given by $p = \dfrac{660}{d + 33}$ where d is the depth (in feet) of the diver.
   a. At what depth is air containing 5% oxygen recommended? Use a table to find the answer.
   b. At what depth is air containing 10% oxygen recommended? Use a graph to find the answer.

MIXED REVIEW *of Problem Solving*

STATE TEST PRACTICE
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Lessons 8.4–8.6

1. **MULTI-STEP PROBLEM** A cyclist travels 50 miles from her home to a state park at a speed of s miles per hour. On the return trip, she increases her speed by 5 miles per hour.

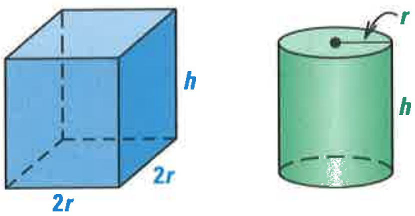
   a. Write an expression in terms of s for the time the cyclist takes to travel from her home to the state park.

   b. Write an expression in terms of s for the time the cyclist takes to return home from the state park.

   c. Write an expression in simplified form for the *total* time of the cyclist's round trip.

2. **SHORT RESPONSE** The speed of a river's current is 3 miles per hour. You travel 2 miles with the current and then return to where you started in a total time of 1.25 hours. What is your speed in still water?

3. **SHORT RESPONSE** A manufacturer of instant rice is considering two different styles of packaging. One is a rectangular container with a square base. The other is a cylinder.



   a. Find the ratio of surface area to volume for each container.

   b. Using the ratios, what can you determine about the efficiencies of the containers?

4. **OPEN-ENDED** Write two rational expressions $r(x)$ and $s(x)$ such that $r(x)$ and $s(x)$ each contain a quadratic polynomial and $r(x) \cdot s(x) = \frac{x-3}{x+4}$.

5. **MULTI-STEP PROBLEM** Brass is an alloy composed of 55% copper and 45% zinc by weight. You have 25 ounces of copper, and you want to determine how many ounces of zinc you need to make brass.

   a. Let x be the number of ounces of zinc you need. Write a verbal model and then a rational equation that you can use to find x.

   b. Solve the equation from part (a) to find the number of ounces of zinc you need to make brass.

   c. Consider the more general case where you have c ounces of copper. In terms of c, how many ounces of zinc must be added to make brass?

6. **EXTENDED RESPONSE** A car travels 120 miles in the same amount of time that it takes a truck to travel 100 miles. The car travels 10 miles per hour faster than the truck.

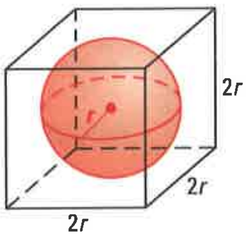
   a. Use the verbal model below to write an equation that relates the speeds of the vehicles.

$$\frac{\text{Distance for car}}{\text{Speed of car}} = \frac{\text{Distance for truck}}{\text{Speed of truck}}$$

   b. Solve the equation from part (a) to find the speeds of the car and the truck.

   c. How much time did the vehicles spend traveling? *Explain* your answer.

7. **GRIDDED ANSWER** Find the ratio of the volume of the sphere to the volume of the cube in the diagram below.



Use the formula $V = \frac{4}{3}\pi r^3$ for the volume of a sphere and the formula $V = s^3$ for the volume of a cube where r is the radius of the sphere and s is the side length of the cube. Write your answer as a decimal rounded to the nearest hundredth.

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Chapter Review Practice

8.4 Multiply and Divide Rational Expressions

pp. 573–580

EXAMPLE

Divide: $\dfrac{3x + 27}{6x - 48} \div \dfrac{x^2 + 9x}{x^2 - 4x - 32}$

$$\frac{3x + 27}{6x - 48} \div \frac{x^2 + 9x}{x^2 - 4x - 32} = \frac{3x + 27}{6x - 48} \cdot \frac{x^2 - 4x - 32}{x^2 + 9x} \qquad \text{Multiply by reciprocal.}$$

$$= \frac{3(x + 9)}{6(x - 8)} \cdot \frac{(x + 4)(x - 8)}{x(x + 9)} \qquad \text{Factor.}$$

$$= \frac{\cancel{3}\cancel{(x + 9)}(x + 4)\cancel{(x - 8)}}{2\cancel{(3)}\cancel{(x - 8)}(x)\cancel{(x + 9)}} \qquad \text{Divide out common factors.}$$

$$= \frac{x + 4}{2x} \qquad \text{Simplified form}$$

EXERCISES

EXAMPLES 3, 4, 6, and 7
on pp. 575–577 for Exs. 19–22

Perform the indicated operation. Simplify the result.

19. $\dfrac{80x^4}{y^3} \cdot \dfrac{xy}{5x^2}$

20. $\dfrac{x - 3}{2x - 8} \cdot \dfrac{6x^2 - 96}{x^2 - 9}$

21. $\dfrac{16x^2 - 8x + 1}{x^3 - 7x^2 + 12x} \div \dfrac{20x^2 - 5x}{15x^3}$

22. $\dfrac{x^2 - 13x + 40}{x^2 - 2x - 15} \div (x^2 - 5x - 24)$

8.5 Add and Subtract Rational Expressions

pp. 582–588

EXAMPLE

Add: $\dfrac{x}{6x + 24} + \dfrac{x + 2}{x^2 + 9x + 20}$

The denominators factor as $6(x + 4)$ and $(x + 4)(x + 5)$, so the LCD is $6(x + 4)(x + 5)$. Use this result to rewrite each expression with a common denominator, and then add.

$$\frac{x}{6x + 24} + \frac{x + 2}{x^2 + 9x + 20} = \frac{x}{6(x + 4)} + \frac{x + 2}{(x + 4)(x + 5)}$$

$$= \frac{x}{6(x + 4)} \cdot \frac{x + 5}{x + 5} + \frac{x + 2}{(x + 4)(x + 5)} \cdot \frac{6}{6}$$

$$= \frac{x^2 + 5x}{6(x + 4)(x + 5)} + \frac{6x + 12}{6(x + 4)(x + 5)}$$

$$= \frac{x^2 + 11x + 12}{6(x + 4)(x + 5)}$$

EXERCISES

EXAMPLES 3 and 4
on pp. 583–584 for Exs. 23–25

Perform the indicated operation and simplify.

23. $\dfrac{5}{6(x + 3)} + \dfrac{x + 4}{2x}$

24. $\dfrac{5x}{x + 8} + \dfrac{4x - 9}{x^2 + 5x - 24}$

25. $\dfrac{x + 2}{x^2 + 4x + 3} - \dfrac{5x}{x^2 - 9}$