1.6 Dynamics of rigid bodies

1.6.1 Moment of Inertia

The angular momentum in a moving coordinate system is given by:

$$\vec{L}' = I\vec{\omega} + \vec{L}'_n$$

where I is the *moment of inertia* with respect to a central axis, which is given by:

$$I = \sum_i m_i \vec{r}_i{}^2 \; ; \quad T' = W_{\text{rot}} = \tfrac{1}{2}\omega I_{ij}\vec{e}_i\vec{e}_j = \tfrac{1}{2}I\omega^2$$

or, in the continuous case:

$$I = \frac{m}{V}\int r'^2_n dV = \int r'^2_n dm$$

Further holds:

$$L_i = I^{ij}\omega_j \; ; \quad I_{ii} = I_i \; ; \quad I_{ij} = I_{ji} = -\sum_k m_k x'_i x'_j$$

Steiner's theorem is: $I_{\text{w.r.t.D}} = I_{\text{w.r.t.C}} + m(DM)^2$ if axis C $\parallel$ axis D.

Object	I	**Object**	I
Cavern cylinder	$I = mR^2$	Massive cylinder	$I = \frac{1}{2}mR^2$
Disc, axis in plane disc through m	$I = \frac{1}{4}mR^2$	Halter	$I = \frac{1}{2}\mu R^2$
Cavern sphere	$I = \frac{2}{3}mR^2$	Massive sphere	$I = \frac{2}{5}mR^2$
Bar, axis $\perp$ through c.o.m.	$I = \frac{1}{12}ml^2$	Bar, axis $\perp$ through end	$I = \frac{1}{3}ml^2$
Rectangle, axis $\perp$ plane thr. c.o.m.	$I = \frac{1}{12}m(a^2+b^2)$	Rectangle, axis $\parallel b$ thr. m	$I = ma^2$

1.6.2 Principal axes

Each rigid body has (at least) 3 principal axes which stand $\perp$ to each other. For a principal axis holds:

$$\frac{\partial I}{\partial \omega_x} = \frac{\partial I}{\partial \omega_y} = \frac{\partial I}{\partial \omega_z} = 0 \;\text{ so }\; L'_n = 0$$

The following holds: $\dot{\omega}_k = -a_{ijk}\omega_i\omega_j$ with $a_{ijk} = \dfrac{I_i - I_j}{I_k}$ if $I_1 \le I_2 \le I_3$.

1.6.3 Time dependence

For torque of force $\vec{\tau}$ holds:

$$\vec{\tau}' = I\ddot{\theta} \; ; \quad \frac{d''\vec{L}'}{dt} = \vec{\tau}' - \vec{\omega}\times\vec{L}'$$

The *torque* $\vec{T}$ is defined by: $\vec{T} = \vec{F}\times\vec{d}$.

1.7 Variational Calculus, Hamilton and Lagrange mechanics

1.7.1 Variational Calculus

Starting with:

$$\delta\int_a^b \mathcal{L}(q,\dot{q},t)dt = 0 \;\text{ with }\; \delta(a) = \delta(b) = 0 \;\text{ and }\; \delta\left(\frac{du}{dx}\right) = \frac{d}{dx}(\delta u)$$

the equations of Lagrange can be derived:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

When there are additional conditions applying to the variational problem $\delta J(u) = 0$ of the type $K(u)$ =constant, the new problem becomes: $\delta J(u) - \lambda \delta K(u) = 0$.

1.7.2 Hamilton mechanics

The *Lagrangian* is given by: $\mathcal{L} = \sum T(\dot{q}_i) - V(q_i)$. The *Hamiltonian* is given by: $H = \sum \dot{q}_i p_i - \mathcal{L}$. In 2 dimensions holds: $\mathcal{L} = T - U = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$.

If the used coordinates are *canonical* the Hamilton equations are the equations of motion for the system:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\ ; \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

Coordinates are canonical if the following holds: $\{q_i, q_j\} = 0,\ \{p_i, p_j\} = 0,\ \{q_i, p_j\} = \delta_{ij}$ where $\{,\}$ is the *Poisson bracket*:

$$\{A, B\} = \sum_i \left[\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right]$$

The Hamiltonian of a Harmonic oscillator is given by $H(x,p) = p^2/2m + \frac{1}{2}m\omega^2 x^2$. With new coordinates (θ, I), obtained by the canonical transformation $x = \sqrt{2I/m\omega}\cos(\theta)$ and $p = -\sqrt{2Im\omega}\sin(\theta)$, with inverse $\theta = \arctan(-p/m\omega x)$ and $I = p^2/2m\omega + \frac{1}{2}m\omega x^2$ it follows: $H(\theta, I) = \omega I$.

The Hamiltonian of a charged particle with charge q in an external electromagnetic field is given by:

$$H = \frac{1}{m}\left(p - qA\right)^2 + qV$$

This Hamiltonian can be derived from the Hamiltonian of a free particle $H = p^2/2m$ with the transformations $\vec{p} \to \vec{p} - q\vec{A}$ and $H \to H - qV$. This is elegant from a relativistic point of view: this is equivalent to the transformation of the momentum 4-vector, $p^\alpha \to p^\alpha - qA^\alpha$. A gauge transformation on the potentials A^α corresponds with a canonical transformation, which make the Hamilton equations the equations of motion for the system.

1.7.3 Motion around an equilibrium, linearization

For natural systems around equilibrium the following equations are valid:

$$\left(\frac{\partial V}{\partial q_i}\right)_0 = 0\ ; \quad V(q) = V(0) + V_{ik}q_i q_k \ \text{ with } \ V_{ik} = \left(\frac{\partial^2 V}{\partial q_i \partial q_k}\right)_0$$

With $T = \frac{1}{2}(M_{ik}\dot{q}_i\dot{q}_k)$ one receives the set of equations $M\ddot{q} + Vq = 0$. If $q_i(t) = a_i \exp(i\omega t)$ is substituted, this set of equations has solutions if $\det(V - \omega^2 M) = 0$. This leads to the eigenfrequencies of the problem: $\omega_k^2 = \dfrac{a_k^{\mathrm{T}} V a_k}{a_k^{\mathrm{T}} M a_k}$. If the equilibrium is stable holds: $\forall k$ that $\omega_k^2 > 0$. The general solution is a superposition if eigenvibrations.

1.7.4 Phase space, Liouville's equation

In phase space holds:

$$\nabla = \left(\sum_i \frac{\partial}{\partial q_i}, \sum_i \frac{\partial}{\partial p_i}\right) \ \text{ so } \ \nabla \cdot \vec{v} = \sum_i \left(\frac{\partial}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

$W^2 = m_0^2c^4 + p^2c^2$. $p = m_\mathrm{r}v = \gamma m_0 v = Wv/c^2$, and $pc = W\beta$ where $\beta = v/c$. The *force* is *defined* by $\vec{F} = d\vec{p}/dt$.

4-vectors have the property that their modulus is independent of the observer: their components *can* change after a coordinate transformation but not their modulus. The difference of two 4-vectors transforms also as a 4-vector. The 4-vector for the velocity is given by $U^\alpha = \dfrac{dx^\alpha}{d\tau}$. The relation with the "common" velocity $u^i := dx^i/dt$ is: $U^\alpha = (\gamma u^i, ic\gamma)$. For particles with nonzero restmass holds: $U^\alpha U_\alpha = -c^2$, for particles with zero restmass (so with $v = c$) holds: $U^\alpha U_\alpha = 0$. The 4-vector for energy and momentum is given by: $p^\alpha = m_0 U^\alpha = (\gamma p^i, iW/c)$. So: $p_\alpha p^\alpha = -m_0^2c^2 = p^2 - W^2/c^2$.

3.1.2 Red and blue shift

There are three causes of red and blue shifts:

1. Motion: with $\vec{e}_v \cdot \vec{e}_r = \cos(\varphi)$ follows: $\dfrac{f'}{f} = \gamma\left(1 - \dfrac{v\cos(\varphi)}{c}\right)$.
   This can give both red- and blueshift, also $\perp$ to the direction of motion.
2. Gravitational redshift: $\dfrac{\Delta f}{f} = \dfrac{\kappa M}{rc^2}$.
3. Redshift because the universe expands, resulting in e.g. the cosmic background radiation: $\dfrac{\lambda_0}{\lambda_1} = \dfrac{R_0}{R_1}$.

3.1.3 The stress-energy tensor and the field tensor

The stress-energy tensor is given by:

$$T_{\mu\nu} = (\varrho c^2 + p)\frac{v_\mu v_\nu}{c^2} + p g_{\mu\nu} + \frac{1}{c^2}\left(F_{\mu\alpha}F_\nu^\alpha + \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)$$

The conservation laws can than be written as: $\nabla_\nu T^\mu = 0$. The electromagnetic field tensor is given by:

$$\boxed{F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}}$$

with $A_\mu := (\vec{A}, iV/c)$ and $J_\mu := (\vec{J}, ic\rho)$. The Maxwell equations can than be written as:

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu\ ,\quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

The equations of motion for a charged particle in an EM field become with the field tensor:

$$\boxed{\frac{dp_\alpha}{d\tau} = qF_{\alpha\beta}u^\beta}$$

3.2 General relativity

3.2.1 Riemannian geometry, the Einstein tensor

The basic principles of general relativity are:

1. The geodesic postulate: free falling particles move along geodesics of space-time with the proper time τ or arc length s as parameter. For particles with zero rest mass (photons), the use of a free parameter is required because for them holds $ds = 0$. From $\delta\int ds = 0$ the equations of motion can be derived:

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0$$

2. The *principle of equivalence*: inertial mass $\equiv$ gravitational mass $\Rightarrow$ gravitation is equivalent with a curved space-time were particles move along geodesics.

3. By a proper choice of the coordinate system it is possible to make the metric locally flat in each point x_i: $g_{\alpha\beta}(x_i) = \eta_{\alpha\beta}$:=diag$(-1,1,1,1)$.

The *Riemann tensor* is defined as: $R^{\mu}_{\nu\alpha\beta}T^{\nu} := \nabla_\alpha\nabla_\beta T^\mu - \nabla_\beta\nabla_\alpha T^\mu$, where the covariant derivative is given by $\nabla_j a^i = \partial_j a^i + \Gamma^i_{jk}a^k$ and $\nabla_j a_i = \partial_j a_i - \Gamma^k_{ij}a_k$. Here,

$$\Gamma^i_{jk} = \frac{g^{il}}{2}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right), \text{ for Euclidean spaces this reduces to: } \Gamma^i_{jk} = \frac{\partial^2 \bar{x}^l}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial \bar{x}^l},$$

are the *Christoffel symbols*. For a second-order tensor holds: $[\nabla_\alpha, \nabla_\beta]T^\mu_\nu = R^\mu_{\sigma\alpha\beta}T^\sigma_\nu + R^\sigma_{\nu\alpha\beta}T^\mu_\sigma$, $\nabla_k a^i_j = \partial_k a^i_j - \Gamma^l_{kj}a^i_l + \Gamma^i_{kl}a^l_j$, $\nabla_k a_{ij} = \partial_k a_{ij} - \Gamma^l_{ki}a_{lj} - \Gamma^l_{kj}a_{jl}$ and $\nabla_k a^{ij} = \partial_k a^{ij} + \Gamma^i_{kl}a^{lj} + \Gamma^j_{kl}a^{il}$. The following holds: $R^\alpha_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$.

The *Ricci tensor* is a contraction of the Riemann tensor: $R_{\alpha\beta} := R^\mu_{\alpha\mu\beta}$, which is symmetric: $R_{\alpha\beta} = R_{\beta\alpha}$. The *Bianchi identities* are: $\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0$.

The *Einstein tensor* is given by: $G^{\alpha\beta} := R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$, where $R := R^\alpha_\alpha$ is the *Ricci scalar*, for which holds: $\nabla_\beta G_{\alpha\beta} = 0$. With the variational principle $\delta\int(\mathcal{L}(g_{\mu\nu}) - Rc^2/16\pi\kappa)\sqrt{|g|}d^4x = 0$ for variations $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ the *Einstein field equations* can be derived:

$$\boxed{G_{\alpha\beta} = \frac{8\pi\kappa}{c^2}T_{\alpha\beta}} \text{ , which can also be written as } R_{\alpha\beta} = \frac{8\pi\kappa}{c^2}(T_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}T^\mu_\mu)$$

For empty space this is equivalent to $R_{\alpha\beta} = 0$. The equation $R_{\alpha\beta\mu\nu} = 0$ has as only solution a flat space.

The Einstein equations are 10 independent equations which are of second order in $g_{\mu\nu}$. From this, the Laplace equation from Newtonian gravitation can be derived by stating: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h| \ll 1$. In the stationary case, this results in $\nabla^2 h_{00} = 8\pi\kappa\varrho/c^2$.

The most general form of the field equations is: $R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = \frac{8\pi\kappa}{c^2}T_{\alpha\beta}$

where Λ is the *cosmological constant*. This constant plays a role in inflatory models of the universe.

3.2.2 The line element

The *metric tensor* in an Euclidean space is given by: $g_{ij} = \sum_k \frac{\partial \bar{x}^k}{\partial x^i}\frac{\partial \bar{x}^k}{\partial x^j}$.

In general holds: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. In special relativity this becomes $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. This metric, $\eta_{\mu\nu}$:=diag$(-1,1,1,1)$, is called the *Minkowski metric*.

The *external Schwarzschild metric* applies in vacuum outside a spherical mass distribution, and is given by:

$$ds^2 = \left(-1 + \frac{2m}{r}\right)c^2dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2d\Omega^2$$

Here, $m := M\kappa/c^2$ is the *geometrical mass* of an object with mass M, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. This metric is singular for $r = 2m = 2\kappa M/c^2$. If an object is smaller than its event horizon $2m$, that implies that its escape velocity is $> c$, it is called a *black hole*. The Newtonian limit of this metric is given by:

$$ds^2 = -(1+2V)c^2dt^2 + (1-2V)(dx^2 + dy^2 + dz^2)$$

where $V = -\kappa M/r$ is the Newtonian gravitation potential. In general relativity, the components of $g_{\mu\nu}$ are associated with the potentials and the derivatives of $g_{\mu\nu}$ with the field strength.

The Kruskal-Szekeres coordinates are used to solve certain problems with the Schwarzschild metric near $r = 2m$. They are defined by:

1. Series connection: $V = IZ$,

$$Z_{\text{tot}} = \sum_i Z_i\ ,\ \ L_{\text{tot}} = \sum_i L_i\ ,\ \ \frac{1}{C_{\text{tot}}} = \sum_i \frac{1}{C_i}\ ,\ \ Q = \frac{Z_0}{R}\ ,\ \ Z = R(1 + iQ\delta)$$

2. parallel connection: $V = IZ$,

$$\frac{1}{Z_{\text{tot}}} = \sum_i \frac{1}{Z_i}\ ,\ \ \frac{1}{L_{\text{tot}}} = \sum_i \frac{1}{L_i}\ ,\ \ C_{\text{tot}} = \sum_i C_i\ ,\ \ Q = \frac{R}{Z_0}\ ,\ \ Z = \frac{R}{1 + iQ\delta}$$

Here, $Z_0 = \sqrt{\frac{L}{C}}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$.

The power given by a source is given by $P(t) = V(t) \cdot I(t)$, so $\langle P \rangle_t = \hat{V}_{\text{eff}} \hat{I}_{\text{eff}} \cos(\Delta\phi)$ $= \frac{1}{2}\hat{V}\hat{I}\cos(\phi_v - \phi_i) = \frac{1}{2}\hat{I}^2 \text{Re}(Z) = \frac{1}{2}\hat{V}^2 \text{Re}(1/Z)$, where $\cos(\Delta\phi)$ is the work factor.

4.4 Waves in long conductors

These cables are in use for signal transfer, e.g. coax cable. For them holds: $Z_0 = \sqrt{\frac{dL}{dx}\frac{dx}{dC}}$.

The transmission velocity is given by $v = \sqrt{\frac{dx}{dL}\frac{dx}{dC}}$.

4.5 Coupled conductors and transformers

For two coils enclosing each others flux holds: if Φ_{11} is the part of the flux originating from I_2 through coil 2 which is enclosed by coil 1, than holds $\Phi_{12} = M_{12} I_2$, $\Phi_{21} = M_{21} I_1$. For the coefficients of mutual induction M_{ij} holds:

$$M_{12} = M_{21} := M = k\sqrt{L_1 L_2} = \frac{N_1 \Phi_1}{I_2} = \frac{N_2 \Phi_2}{I_1} \sim N_1 N_2$$

where $0 \leq k \leq 1$ is the *coupling factor*. For a transformer is $k \approx 1$. At full load holds:

$$\frac{V_1}{V_2} = \frac{I_2}{I_1} = -\frac{i\omega M}{i\omega L_2 + R_{\text{load}}} \approx -\sqrt{\frac{L_1}{L_2}} = -\frac{N_1}{N_2}$$

4.6 Pendulums

The oscillation time $T = 1/f$, and for different types of pendulums is given by:

- Oscillating spring: $T = 2\pi\sqrt{m/C}$ if the spring force is given by $F = C \cdot \Delta l$.
- Physical pendulum: $T = 2\pi\sqrt{I/\tau}$ with τ the moment of force and I the moment of inertia.
- Torsion pendulum: $T = 2\pi\sqrt{I/\kappa}$ with $\kappa = \frac{2lm}{\pi r^4 \Delta\varphi}$ the constant of torsion and I the moment of inertia.
- Mathematical pendulum: $T = 2\pi\sqrt{l/g}$ with g the acceleration of gravity and l the length of the pendulum.

3. E_z and B_z are zero everywhere: the Transversal electromagnetic mode (TEM). Than holds: $k = \pm\omega\sqrt{\varepsilon\mu}$ and $v_\mathrm{f} = v_\mathrm{g}$, just as if here were no waveguide. Further $k \in I\!R$, so there exists no cut-off frequency.

In a rectangular, 3 dimensional resonating cavity with edges a, b and c the possible wave numbers are given by: $k_x = \dfrac{n_1\pi}{a}$, $k_y = \dfrac{n_2\pi}{b}$, $k_z = \dfrac{n_3\pi}{c}$ This results in the possible frequencies $f = vk/2\pi$ in the cavity:

$$f = \frac{v}{2}\sqrt{\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}}$$

For a cubic cavity, with $a = b = c$, the possible number of oscillating modes N_L for longitudinal waves is given by:

$$N_\mathrm{L} = \frac{4\pi a^3 f^3}{3v^3}$$

Because transversal waves have two possible polarizations holds for them: $N_\mathrm{T} = 2N_\mathrm{L}$.

5.6 Non-linear wave equations

The *Van der Pol* equation is given by:

$$\frac{d^2x}{dt^2} - \varepsilon\omega_0(1-\beta x^2)\frac{dx}{dt} + \omega_0^2 x = 0$$

βx^2 can be ignored for very small values of the amplitude. Substitution of $x \sim \mathrm{e}^{i\omega t}$ gives: $\omega = \frac{1}{2}\omega_0(i\varepsilon \pm 2\sqrt{1-\frac{1}{2}\varepsilon^2})$. The lowest-order instabilities grow as $\frac{1}{2}\varepsilon\omega_0$. While x is growing, the 2nd term becomes larger and diminishes the growth. Oscillations on a time scale $\sim \omega_0^{-1}$ can exist. If x is expanded as $x = x^{(0)} + \varepsilon x^{(1)} + \varepsilon^2 x^{(2)} + \cdots$ and this is substituted one obtains, besides periodic, *secular terms* $\sim \varepsilon t$. If it is assumed that there exist timescales τ_n, $0 \le n \le N$ with $\partial\tau_n/\partial t = \varepsilon^n$ and if the secular terms are put 0 one obtains:

$$\frac{d}{dt}\left\{\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\omega_0^2 x^2\right\} = \varepsilon\omega_0(1-\beta x^2)\left(\frac{dx}{dt}\right)^2$$

This is an energy equation. Energy is conserved if the left-hand side is 0. If $x^2 > 1/\beta$, the right-hand side changes sign and an increase in energy changes into a decrease of energy. This mechanism limits the growth of oscillations.

The *Korteweg-De Vries* equation is given by:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \underbrace{au\frac{\partial u}{\partial x}}_{\mathrm{non-lin}} + \underbrace{b^2\frac{\partial^3 u}{\partial x^3}}_{\mathrm{dispersive}} = 0$$

This equation is for example a model for ion-acoustic waves in a plasma. For this equation, soliton solutions of the following form exist:

$$u(x-ct) = \frac{-d}{\cosh^2(e(x-ct))}$$

with $c = 1 + \frac{1}{3}ad$ and $e^2 = ad/(12b^2)$.

Chapter 7

Statistical physics

7.1 Degrees of freedom

A molecule consisting of n atoms has $s = 3n$ degrees of freedom. There are 3 translational degrees of freedom, a linear molecule has $s = 3n - 5$ vibrational degrees of freedom and a non-linear molecule $s = 3n - 6$. A linear molecule has 2 rotational degrees of freedom and a non-linear molecule 3.

Because vibrational degrees of freedom account for both kinetic and potential energy they count double. So, for linear molecules this results in a total of $s = 6n - 5$. For non-linear molecules this gives $s = 6n - 6$. The average energy of a molecule in thermodynamic equilibrium is $\langle E_{\text{tot}} \rangle = \frac{1}{2} skT$. Each degree of freedom of a molecule has in principle the same energy: the *principle of equipartition*.

The rotational and vibrational energy of a molecule are:

$$W_{\text{rot}} = \frac{\hbar^2}{2I} l(l+1) = Bl(l+1)\,, \quad W_{\text{vib}} = (v + \tfrac{1}{2})\hbar\omega_0$$

The vibrational levels are excited if $kT \approx \hbar\omega$, the rotational levels of a not homonuclear molecule are excited if $kT \approx 2B$. For homonuclear molecules additional selection rules apply so the rotational levels are well coupled if $kT \approx 6B$.

7.2 The energy distribution function

The general form of the equilibrium velocity distribution function is $P(v_x, v_y, v_z)dv_x dv_y dv_z = P(v_x)dv_x \cdot P(v_y)dv_y \cdot P(v_z)dv_z$ with

$$P(v_i)dv_i = \frac{1}{\alpha\sqrt{\pi}} \exp\left(-\frac{v_i^2}{\alpha^2}\right) dv_i$$

where $\alpha = \sqrt{2kT/m}$ is the *most probable velocity* of a particle. The average velocity is given by $\langle v \rangle = 2\alpha/\sqrt{\pi}$, and $\langle v^2 \rangle = \frac{3}{2}\alpha^2$. The distribution as a function of the absolute value of the velocity is given by:

$$\frac{dN}{dv} = \frac{4N}{\alpha^3\sqrt{\pi}}\, v^2 \exp\left(-\frac{mv^2}{2kT}\right)$$

The general form of the energy distribution function then becomes:

$$P(E)dE = \frac{c(s)}{kT}\left(\frac{E}{kT}\right)^{\frac{1}{2}s-1} \exp\left(-\frac{E}{kT}\right) dE$$

where $c(s)$ is a normalization constant, given by:

1. Even s: $s = 2l$: $c(s) = \dfrac{1}{(l-1)!}$
2. Odd s: $s = 2l + 1$: $c(s) = \dfrac{2^l}{\sqrt{\pi}(2l-1)!!}$

7.3 Pressure on a wall

The number of molecules that collides with a wall with surface A within a time τ is given by:

$$\iiint d^3N = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} nAv\tau\cos(\theta)P(v,\theta,\varphi)dvd\theta d\varphi$$

From this follows for the particle flux on the wall: $\Phi = \frac{1}{4}n\langle v\rangle$. For the pressure on the wall then follows:

$$d^3p = \frac{2mv\cos(\theta)d^3N}{A\tau}\ ,\ \text{so}\ \ p = \frac{2}{3}n\langle E\rangle$$

7.4 The equation of state

If intermolecular forces and the volume of the molecules can be neglected then for gases from $p = \frac{2}{3}n\langle E\rangle$ and $\langle E\rangle = \frac{3}{2}kT$ can be derived:

$$pV = n_sRT = \frac{1}{3}Nm\langle v^2\rangle$$

Here, n_s is the number of *moles* particles and N is the total number of particles within volume V. If the own volume and the intermolecular forces cannot be neglected the *Van der Waals* equation can be derived:

$$\left(p + \frac{an_s^2}{V^2}\right)(V - bn_s) = n_sRT$$

There is an isotherme with a horizontal point of inflection. In the Van der Waals equation this corresponds with the *critical temperature*, *pressure* and *volume* of the gas. This is the upper limit of the area of coexistence between liquid and vapour. From $dp/dV = 0$ and $d^2p/dV^2 = 0$ follows:

$$T_{\rm cr} = \frac{8a}{27bR}\ ,\ \ p_{\rm cr} = \frac{a}{27b^2}\ ,\ \ V_{\rm cr} = 3bn_s$$

For the critical point holds: $p_{\rm cr}V_{m,\rm cr}/RT_{\rm cr} = \frac{3}{8}$, which differs from the value of 1 which follows from the general gas law.

Scaled on the critical quantities, with $p^* := p/p_{\rm cr}$, $T^* = T/T_{\rm cr}$ and $V_m^* = V_m/V_{m,\rm cr}$ with $V_m := V/n_s$ holds:

$$\left(p^* + \frac{3}{(V_m^*)^2}\right)\left(V_m^* - \tfrac{1}{3}\right) = \tfrac{8}{3}T^*$$

Gases behave the same for equal values of the reduced quantities: the *law of the corresponding states*. A *virial expansion* is used for even more accurate views:

$$p(T,V_m) = RT\left(\frac{1}{V_m} + \frac{B(T)}{V_m^2} + \frac{C(T)}{V_m^3} + \cdots\right)$$

The *Boyle temperature* $T_{\rm B}$ is the temperature for which the 2nd virial coefficient is 0. In a Van der Waals gas, this happens at $T_{\rm B} = a/Rb$. The *inversion temperature* $T_{\rm i} = 2T_{\rm B}$.

The equation of state for solids and liquids is given by:

$$\frac{V}{V_0} = 1 + \gamma_p\Delta T - \kappa_T\Delta p = 1 + \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p\Delta T + \frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T\Delta p$$

7.5 Collisions between molecules

The collision probability of a particle in a gas that is translated over a distance dx is given by $n\sigma dx$, where σ is the *cross section*. The mean free path is given by $\ell = \dfrac{v_1}{nu\sigma}$ with $u = \sqrt{v_1^2 + v_2^2}$ the relative velocity between the particles. If $m_1 \ll m_2$ holds: $\dfrac{u}{v_1} = \sqrt{1 + \dfrac{m_1}{m_2}}$, so $\ell = \dfrac{1}{n\sigma}$. If $m_1 = m_2$ holds: $\ell = \dfrac{1}{n\sigma\sqrt{2}}$. This means that the average time between two collisions is given by $\tau = \dfrac{1}{n\sigma v}$. If the molecules are approximated by hard spheres the cross section is: $\sigma = \frac{1}{4}\pi(D_1^2 + D_2^2)$. The average distance between two molecules is $0.55n^{-1/3}$. Collisions between molecules and small particles in a solution result in the *Brownian motion*. For the average motion of a particle with radius R can be derived: $\langle x_i^2 \rangle = \frac{1}{3}\langle r^2 \rangle = kTt/3\pi\eta R$.

A gas is called a *Knudsen gas* if $\ell \gg$ the dimensions of the gas, something that can easily occur at low pressures. The equilibrium condition for a vessel which has a hole with surface A in it for which holds that $\ell \gg \sqrt{A/\pi}$ is: $n_1\sqrt{T_1} = n_2\sqrt{T_2}$. Together with the general gas law follows: $p_1/\sqrt{T_1} = p_2/\sqrt{T_2}$.

If two plates move along each other at a distance d with velocity w_x the *viscosity* η is given by: $F_x = \eta\dfrac{Aw_x}{d}$. The velocity profile between the plates is in that case given by $w(z) = zw_x/d$. It can be derived that $\eta = \frac{1}{3}\varrho\ell\langle v \rangle$ where v is the *thermal velocity*.

The heat conductance in a non-moving gas is described by: $\dfrac{dQ}{dt} = \kappa A\left(\dfrac{T_2 - T_1}{d}\right)$, which results in a temperature profile $T(z) = T_1 + z(T_2 - T_1)/d$. It can be derived that $\kappa = \frac{1}{3}C_{mV}n\ell\langle v \rangle / N_\mathrm{A}$. Also holds: $\kappa = C_V\eta$. A better expression for κ can be obtained with the *Eucken correction*: $\kappa = (1 + 9R/4C_{mV})C_V\eta$ with an error <5%.

7.6 Interaction between molecules

For dipole interaction between molecules can be derived that $U \sim -1/r^6$. If the distance between two molecules approaches the molecular diameter D a repulsing force between the electron clouds appears. This force can be described by $U_\mathrm{rep} \sim \exp(-r/\rho)$ or $U_\mathrm{rep} = +C_s/r^s$ with $12 \leq s \leq 20$. This results in the *Lennard-Jones* potential for intermolecular forces:

$$U_\mathrm{LJ} = 4\epsilon\left[\left(\frac{D}{r}\right)^{12} - \left(\frac{D}{r}\right)^{6}\right]$$

with a minimum ϵ at $r = r_\mathrm{m}$. The following holds: $D \approx 0.89r_\mathrm{m}$. For the Van der Waals coefficients a and b and the critical quantities holds: $a = 5.275N_\mathrm{A}^2D^3\epsilon$, $b = 1.3N_\mathrm{A}D^3$, $kT_\mathrm{kr} = 1.2\epsilon$ and $V_\mathrm{m,kr} = 3.9N_\mathrm{A}D^3$.

A more simple model for intermolecular forces assumes a potential $U(r) = \infty$ for $r < D$, $U(r) = U_\mathrm{LJ}$ for $D \leq r \leq 3D$ and $U(r) = 0$ for $r \geq 3D$. This gives for the potential energy of one molecule: $E_\mathrm{pot} = \displaystyle\int_D^{3D} U(r)F(r)dr$.

with $F(r)$ the spatial distribution function in spherical coordinates, which for a homogeneous distribution is given by: $F(r)dr = 4n\pi r^2 dr$.

Some useful mathematical relations are:

$$\int_0^\infty x^n \mathrm{e}^{-x}dx = n! \quad , \quad \int_0^\infty x^{2n}\mathrm{e}^{-x^2}dx = \frac{(2n)!\sqrt{\pi}}{n!2^{2n+1}} \quad , \quad \int_0^\infty x^{2n+1}\mathrm{e}^{-x^2}dx = \tfrac{1}{2}n!$$

From this one can derive Maxwell's relations:

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V , \quad \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p , \quad \left(\frac{\partial p}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T , \quad \left(\frac{\partial V}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T$$

From the total differential and the definitions of C_V and C_p it can be derived that:

$$TdS = C_V dT + T\left(\frac{\partial p}{\partial T}\right)_V dV \text{ and } TdS = C_p dT - T\left(\frac{\partial V}{\partial T}\right)_p dp$$

For an ideal gas also holds:

$$S_m = C_V \ln\left(\frac{T}{T_0}\right) + R\ln\left(\frac{V}{V_0}\right) + S_{m0} \text{ and } S_m = C_p \ln\left(\frac{T}{T_0}\right) - R\ln\left(\frac{p}{p_0}\right) + S'_{m0}$$

Helmholtz' equations are:

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial p}{\partial T}\right)_V - p , \quad \left(\frac{\partial H}{\partial p}\right)_T = V - T\left(\frac{\partial V}{\partial T}\right)_p$$

for an enlarged surface holds: $đW_{\text{rev}} = -\gamma dA$, with γ the surface tension. From this follows:

$$\gamma = \left(\frac{\partial U}{\partial A}\right)_S = \left(\frac{\partial F}{\partial A}\right)_T$$

8.6 Processes

The *efficiency* η of a process is given by: $\eta = \dfrac{\text{Work done}}{\text{Heat added}}$

The *Cold factor* ξ of a cooling down process is given by: $\xi = \dfrac{\text{Cold delivered}}{\text{Work added}}$

Reversible adiabatic processes

For adiabatic processes holds: $W = U_1 - U_2$. For reversible adiabatic processes holds Poisson's equation: with $\gamma = C_p/C_V$ one gets that pV^γ =constant. Also holds: $TV^{\gamma-1}$ =constant and $T^\gamma p^{1-\gamma}$ =constant. Adiabatics exhibit a greater steepness p-V diagram than isothermics because $\gamma > 1$.

Isobaric processes

Here holds: $H_2 - H_1 = \int_1^2 C_p dT$. For a reversible isobaric process holds: $H_2 - H_1 = Q_{\text{rev}}$.

The throttle process

This is also called the *Joule-Kelvin* effect and is an adiabatic expansion of a gas through a porous material or a small opening. Here H is a conserved quantity, and $dS > 0$. In general this is accompanied with a change in temperature. The quantity which is important here is the *throttle coefficient*:

$$\alpha_H = \left(\frac{\partial T}{\partial p}\right)_H = \frac{1}{C_p}\left[T\left(\frac{\partial V}{\partial T}\right)_p - V\right]$$

The *inversion temperature* is the temperature where an adiabatically expanding gas keeps the same temperature. If $T > T_\text{i}$ the gas heats up, if $T < T_\text{i}$ the gas cools down. $T_\text{i} = 2T_\text{B}$, with for T_B: $[\partial(pV)/\partial p]_T = 0$. The throttle process is e.g. applied in refridgerators.

The Carnotprocess

The system undergoes a reversible cycle with 2 isothemics and 2 adiabatics:

1. Isothermic expansion at T_1. The system absorbs a heat Q_1 from the reservoir.
2. Adiabatic expansion with a temperature drop to T_2.

This is expressed as: $f_\kappa^{(j)}$ is the part of F that transforms according to the κ^{th} row of $\Gamma^{(j)}$.

F can also be expressed in base functions φ: $F = \sum_{aj\kappa} c_{aj\kappa}\varphi_\kappa^{(aj)}$. The functions $f_\kappa^{(j)}$ are in general not transformed into each other by elements of the group. However, this does happen if $c_{ja\kappa} = c_{ja}$.

Theorem: Two wavefunctions transforming according to non-equivalent unitary representations or according to different rows of an unitary irreducible representation are orthogonal:
$\langle\varphi_\kappa^{(i)}|\psi_\lambda^{(j)}\rangle \sim \delta_{ij}\delta_{\kappa\lambda}$, and $\langle\varphi_\kappa^{(i)}|\psi_\kappa^{(i)}\rangle$ is independent of κ.

13.3.4 The direct product of representations

Consider a physical system existing of two subsystems. The subspace $D^{(i)}$ of the system transforms according to $\Gamma^{(i)}$. Basefunctions are $\varphi_\kappa^{(i)}(\vec{x}_i)$, $1 \le \kappa \le \ell_i$. Now form all $\ell_1 \times \ell_2$ products $\varphi_\kappa^{(1)}(\vec{x}_1)\varphi_\lambda^{(2)}(\vec{x}_2)$. These define a space $D^{(1)} \otimes D^{(2)}$.

These product functions transform as:

$$P_R(\varphi_\kappa^{(1)}(\vec{x}_1)\varphi_\lambda^{(2)}(\vec{x}_2)) = (P_R\varphi_\kappa^{(1)}(\vec{x}_1))(P_R\varphi_\lambda^{(2)}(\vec{x}_2))$$

In general the space $D^{(1)} \otimes D^{(2)}$ can be split up in a number of invariant subspaces:

$$\Gamma^{(1)} \otimes \Gamma^{(2)} = \sum_i n_i \Gamma^{(i)}$$

A useful tool for this reduction is that for the characters hold:

$$\chi^{(1)}(R)\chi^{(2)}(R) = \sum_i n_i \chi^{(i)}(R)$$

13.3.5 Clebsch-Gordan coefficients

With the reduction of the direct-product matrix w.r.t. the basis $\varphi_\kappa^{(i)}\varphi_\lambda^{(j)}$ one uses a new basis $\varphi_\mu^{(a\kappa)}$. These base functions yield subspaces $D^{(ak)}$. The unitary base transformation is given by:

$$\varphi_\mu^{(ak)} = \sum_{\kappa\lambda} \varphi_\kappa^{(i)}\varphi_\lambda^{(j)}(i\kappa j\lambda|ak\mu)$$

and the inverse transformation by: $\varphi_\kappa^{(i)}\varphi_\lambda^{(j)} = \sum_{ak\mu} \varphi_\mu^{(a\kappa)}(ak\mu|i\kappa j\lambda)$

In essence the Clebsch-Gordan coefficients are dot products: $(i\kappa j\lambda|ak\mu) := \langle\varphi_k^{(i)}\varphi_\lambda^{(j)}|\varphi_\mu^{(ak)}\rangle$

13.3.6 Symmetric transformations of operators, irreducible tensor operators

Observables (operators) transform as follows under symmetry transformations: $A' = P_R A P_R^{-1}$. If a set of operators $A_\kappa^{(j)}$ with $0 \le \kappa \le \ell_j$ transform into each other under the transformations of $\mathcal{G}$ holds:

$$P_R A_\kappa^{(j)} P_R^{-1} = \sum_\nu A_\nu^{(j)} \Gamma_{\nu\kappa}^{(j)}(R)$$

If $\Gamma^{(j)}$ is irreducible they are called *irreducible tensor operators* $A^{(j)}$ with components $A_\kappa^{(j)}$.

An operator can also be decomposed into symmetry types: $A = \sum_{jk} a_k^{(j)}$, with:

$$a_\kappa^{(j)} = \left(\frac{\ell_j}{h}\sum_{R\in\mathcal{G}} \Gamma_{\kappa\kappa}^{(j)*}(R)\right)(P_R A P_R^{-1})$$

substates which exist independently for protons and neutrons. This gives rise to the so called *magical numbers*: nuclei where each state in the outermost level are filled are particulary stable. This is the case if N or $Z \in \{2, 8, 20, 28, 50, 82, 126\}$.

14.2 The shape of the nucleus

A nucleus is to first approximation spherical with a radius of $R = R_0 A^{1/3}$. Here, $R_0 \approx 1.4 \cdot 10^{-15}$ m, constant for all nuclei. If the nuclear radius is measured including the charge distribution one obtains $R_0 \approx 1.2 \cdot 10^{-15}$ m. The shape of oscillating nuclei can be described by spherical harmonics:

$$R = R_0 \left[1 + \sum_{lm} a_{lm} Y_l^m(\theta, \varphi) \right]$$

$l = 0$ gives rise to monopole vibrations, density vibrations, which can be applied to the theory of neutron stars. $l = 1$ gives dipole vibrations, $l = 2$ quadrupole, with $a_{2,0} = \beta \cos\gamma$ and $a_{2,\pm 2} = \frac{1}{2}\sqrt{2}\beta \sin\gamma$ where β is the deformation factor and γ the shape parameter. The multipole moment is given by $\mu_l = Zer^l Y_l^m(\theta, \varphi)$. The parity of the electric moment is $\Pi_E = (-1)^l$, of the magnetic moment $\Pi_M = (-1)^{l+1}$.

There are 2 contributions to the magnetic moment: $\vec{M}_L = \dfrac{e}{2m_\mathrm{p}} \vec{L}$ and $\vec{M}_S = g_S \dfrac{e}{2m_\mathrm{p}} \vec{S}$.

where g_S is the *spin-gyromagnetic ratio*. For protons holds $g_S = 5.5855$ and for neutrons $g_S = -3.8263$. The z-components of the magnetic moment are given by $M_{L,z} = \mu_\mathrm{N} m_l$ and $M_{S,z} = g_S \mu_\mathrm{N} m_S$. The resulting magnetic moment is related to the nuclear spin I according to $\vec{M} = g_I (e/2m_\mathrm{p}) \vec{I}$. The z-component is then $M_z = \mu_\mathrm{N} g_I m_I$.

14.3 Radioactive decay

The number of nuclei decaying is proportional to the number of nuclei: $\dot{N} = -\lambda N$. This gives for the number of nuclei N: $N(t) = N_0 \exp(-\lambda t)$. The *half life time* follows from $\tau_{\frac{1}{2}} \lambda = \ln(2)$. The average life time of a nucleus is $\tau = 1/\lambda$. The probability that N nuclei decay within a time interval is given by a Poisson distribution:

$$P(N)dt = N_0 \frac{\lambda^N \mathrm{e}^{-\lambda}}{N!} dt$$

If a nucleus can decay into more final states then holds: $\lambda = \sum \lambda_i$. So the fraction decaying into state i is $\lambda_i / \sum \lambda_i$. There are 5 types of natural radioactive decay:

1. α-decay: the nucleus emits a He^{2+} nucleus. Because nucleons tend to order themselves in groups of 2p+2n this can be considered as a tunneling of a He^{2+} nucleus through a potential barrier. The tunnel probability P is

   $$P = \frac{\text{incoming amplitude}}{\text{outgoing amplitude}} = \mathrm{e}^{-2G} \text{ with } G = \frac{1}{\hbar}\sqrt{2m \int [V(r) - E] dr}$$

   G is called the *Gamow factor*.

2. β-decay. Here a proton changes into a neutron or vice versa:
   $\mathrm{p}^+ \rightarrow \mathrm{n}^0 + \mathrm{W}^+ \rightarrow \mathrm{n}^0 + \mathrm{e}^+ + \nu_\mathrm{e}$, and $\mathrm{n}^0 \rightarrow \mathrm{p}^+ + \mathrm{W}^- \rightarrow \mathrm{p}^+ + \mathrm{e}^- + \overline{\nu}_\mathrm{e}$.

3. Electron capture: here, a proton in the nucleus captures an electron (usually from the K-shell).

4. Spontaneous fission: a nucleus breaks apart.

5. γ-decay: here the nucleus emits a high-energetic photon. The decay constant is given by

   $$\lambda = \frac{P(l)}{\hbar\omega} \sim \frac{E_\gamma}{(\hbar c)^2} \left(\frac{E_\gamma R}{\hbar c} \right)^{2l} \sim 10^{-4l}$$

where l is the quantum number for the angular momentum and P the radiated power. Usually the decay constant of electric multipole moments is larger than the one of magnetic multipole moments. The energy of the photon is $E_\gamma = E_i - E_f - T_\mathrm{R}$, with $T_\mathrm{R} = E_\gamma^2/2mc^2$ the recoil energy, which can usually be neglected. The parity of the emitted radiation is $\Pi^l = \Pi^i \cdot \Pi^f$. With I the quantum number of angular momentum of the nucleus, $L = \hbar\sqrt{I(I+1)}$, holds the following selection rule: $|\vec{I}_i - \vec{I}_f| \leq \Delta l \leq |\vec{I}_i + \vec{I}_f|$.

14.4 Scattering and nuclear reactions

14.4.1 Kinetic model

If a beam with intensity I hits a target with density n and length x (Rutherford scattering) the number of scatterings R per unit of time is equal to $R = Inx\sigma$. From this follows that the intensity of the beam decreases as $-dI = In\sigma dx$. This results in $I = I_0\mathrm{e}^{-n\sigma x} = I_0\mathrm{e}^{-\mu x}$.

Because $dR = R(\theta,\varphi)d\Omega/4\pi = Inxd\sigma$ it follows: $\dfrac{d\sigma}{d\Omega} = \dfrac{R(\theta,\varphi)}{4\pi nxI}$

If N particles are scattered in a material with density n then holds: $\dfrac{\Delta N}{N} = n\dfrac{d\sigma}{d\Omega}\Delta\Omega\Delta x$

For Coulomb collisions holds: $\left.\dfrac{d\sigma}{d\Omega}\right|_\mathrm{C} = \dfrac{Z_1Z_2e^2}{8\pi\varepsilon_0\mu v_0^2}\dfrac{1}{\sin^4(\frac{1}{2}\theta)}$

14.4.2 Quantum mechanical model for n-p scattering

The initial state is a beam of neutrons moving along the z-axis with wavefunction $\psi_\mathrm{init} = \mathrm{e}^{ikz}$ and current density $J_\mathrm{init} = v|\psi_\mathrm{init}|^2 = v$. At large distances from the scattering point they have approximately a spherical wavefunction $\psi_\mathrm{scat} = f(\theta)\mathrm{e}^{ikr}/r$, where $f(\theta)$ is the *scattering amplitude*. The total wavefunction is then given by

$$\psi = \psi_\mathrm{init} + \psi_\mathrm{scat} = \mathrm{e}^{ikz} + f(\theta)\frac{\mathrm{e}^{ikr}}{r}$$

The particle flux of the scattered particles is $v|\psi_\mathrm{scat}|^2 = v|f(\theta)|^2 d\Omega$. From this it follows that $\sigma(\theta) = |f(\theta)|^2$. The wavefunction of the incoming particles can be expressed as a sum of angular momentum wavefunctions:

$$\psi_\mathrm{init} = \mathrm{e}^{ikz} = \sum_l \psi_l$$

The impact parameter is related to the angular momentum with $L = bp = b\hbar k$, so $bk \approx l$. At very low energy only particles with $l = 0$ are scattered, so

$$\psi = \psi_0' + \sum_{l>0}\psi_l \quad\text{and}\quad \psi_0 = \frac{\sin(kr)}{kr}$$

If the potential is approximately rectangular holds: $\psi_0' = C\dfrac{\sin(kr+\delta_0)}{kr}$

The cross section is then $\sigma(\theta) = \dfrac{\sin^2(\delta_0)}{k^2}$ so $\sigma = \displaystyle\int \sigma(\theta)d\Omega = \frac{4\pi\sin^2(\delta_0)}{k^2}$

At very low energies holds: $\sin^2(\delta_0) = \dfrac{\hbar^2k^2/2m}{W_0 + W}$

with W_0 the depth of the potential well. At higher energies holds: $\sigma = \dfrac{4\pi}{k^2}\displaystyle\sum_l \sin^2(\delta_l)$

The probability to find a final state with CP= -1 is $\frac{1}{2}|\langle \mathrm{K}_2^0|\mathrm{K}^0\rangle|^2$, the probability of CP=+1 decay is $\frac{1}{2}|\langle \mathrm{K}_1^0|\mathrm{K}^0\rangle|^2$.

The relation between the mass eigenvalues of the quarks (unaccented) and the fields arising in the weak currents (accented) is $(u', c', t') = (u, c, t)$, and:

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_2 & \sin\theta_2 \\ 0 & -\sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{i\delta} \end{pmatrix} \begin{pmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 \\ 0 & -\sin\theta_3 & \cos\theta_3 \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$\theta_1 \equiv \theta_\mathrm{C}$ is the *Cabibbo angle*: $\sin(\theta_\mathrm{C}) \approx 0.23 \pm 0.01$.

15.13 The standard model

When one wants to make the Lagrange density which describes a field invariant for local gauge transformations from a certain group, one has to perform the transformation

$$\frac{\partial}{\partial x_\mu} \to \frac{D}{Dx_\mu} = \frac{\partial}{\partial x_\mu} - i\frac{g}{\hbar} L_k A_\mu^k$$

Here the L_k are the generators of the gauge group (the "charges") and the A_μ^k are the gauge fields. g is the matching coupling constant. The Lagrange density for a scalar field becomes:

$$\mathcal{L} = -\tfrac{1}{2}(D_\mu \Phi^* D^\mu \Phi + M^2 \Phi^* \Phi) - \tfrac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}$$

and the field tensors are given by: $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g c_{lm}^a A_\mu^l A_\nu^m$.

15.13.1 The electroweak theory

The electroweak interaction arises from the necessity to keep the Lagrange density invariant for local gauge transformations of the group $SU(2)_L \otimes U(1)_Y$. Right- and left-handed spin states are treated different because the weak interaction does not conserve parity. If a fifth Dirac matrix is defined by:

$$\gamma_5 := \gamma_1\gamma_2\gamma_3\gamma_4 = -\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

the left- and right- handed solutions of the Dirac equation for neutrino's are given by:

$$\psi_\mathrm{L} = \tfrac{1}{2}(1+\gamma_5)\psi \quad \text{and} \quad \psi_\mathrm{R} = \tfrac{1}{2}(1-\gamma_5)\psi$$

It appears that neutrino's are always left-handed while antineutrino's are always right-handed. The *hypercharge* Y, for quarks given by $Y = \mathrm{B} + \mathrm{S} + \mathrm{C} + \mathrm{B}^* + \mathrm{T}'$, is defined by:

$$Q = \tfrac{1}{2}Y + T_3$$

so $[Y, T_k] = 0$. The group $\mathrm{U}(1)_Y \otimes \mathrm{SU}(2)_T$ is taken as symmetry group for the electroweak interaction because the generators of this group commute. The multiplets are classified as follows:

	e_R^-	ν_eL e_L^-	u_L d'_L	u_R	d_R
T	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0
T_3	0	$\frac{1}{2}$ $-\frac{1}{2}$	$\frac{1}{2}$ $-\frac{1}{2}$	0	0
Y	-2	-1	$\frac{1}{3}$	$\frac{4}{3}$	$-\frac{2}{3}$

Now, 1 field $B_\mu(x)$ is connected with gauge group U(1) and 3 gauge fields $\vec{A}_\mu(x)$ are connected with SU(2). The total Lagrange density (minus the fieldterms) for the electron-fermion field now becomes:

$$\begin{aligned}\mathcal{L}_{0,\mathrm{EW}} &= -(\overline{\psi_{\nu\mathrm{e,L}}}, \overline{\psi_{\mathrm{eL}}})\gamma^\mu \left(\partial_\mu - i\frac{g}{\hbar}\vec{A}_\mu \cdot (\tfrac{1}{2}\vec{\sigma}) - \tfrac{1}{2}i\frac{g'}{\hbar}B_\mu \cdot (-1)\right)\begin{pmatrix}\psi_{\nu\mathrm{e,L}}\\ \psi_{\mathrm{eL}}\end{pmatrix} - \\ &\quad \overline{\psi_{\mathrm{eR}}}\gamma^\mu \left(\partial_\mu - \tfrac{1}{2}i\frac{g'}{\hbar}(-2)B_\mu\right)\psi_{\mathrm{eR}}\end{aligned}$$

Here, $\frac{1}{2}\vec{\sigma}$ are the generators of T and -1 and -2 the generators of Y.

15.13.2 Spontaneous symmetry breaking: the Higgs mechanism

All leptons are massless in the equations above. Their mass is probably generated by *spontaneous symmetry breaking*. This means that the dynamic equations which describe the system have a symmetry which the ground state does not have. It is assumed that there exists an isospin-doublet of scalar fields Φ with electrical charges +1 and 0 and potential $V(\Phi) = -\mu^2\Phi^*\Phi + \lambda(\Phi^*\Phi)^2$. Their antiparticles have charges -1 and 0. The extra terms in $\mathcal{L}$ arising from these fields, $\mathcal{L}_H = (D_{L\mu}\Phi)^*(D_L^\mu\Phi) - V(\Phi)$, are globally U(1)⊗SU(2) symmetric. Hence the state with the lowest energy corresponds with the state $\Phi^*(x)\Phi(x) = v = \mu^2/2\lambda$ =constant. The field can be written (were $\omega^\pm$ and z are Nambu-Goldstone bosons which can be transformed away, and $m_\phi = \mu\sqrt{2}$) as:

$$\Phi = \begin{pmatrix}\Phi^+\\ \Phi^0\end{pmatrix} = \begin{pmatrix}i\omega^+\\ (v+\phi-iz)/\sqrt{2}\end{pmatrix} \text{ and } \langle 0|\Phi|0\rangle = \begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix}$$

Because this expectation value $\neq 0$ the SU(2) symmetry is broken but the U(1) symmetry is not. When the gauge fields in the resulting Lagrange density are separated one obtains:

$$\begin{aligned}W_\mu^- &= \tfrac{1}{2}\sqrt{2}(A_\mu^1 + iA_\mu^2) \qquad W_\mu^+ = \tfrac{1}{2}\sqrt{2}(A_\mu^1 - iA_\mu^2)\\ Z_\mu &= \frac{gA_\mu^3 - g'B_\mu}{\sqrt{g^2+g'^2}} \equiv A_\mu^3\cos(\theta_\mathrm{W}) - B_\mu\sin(\theta_\mathrm{W})\\ A_\mu &= \frac{g'A_\mu^3 + gB_\mu}{\sqrt{g^2+g'^2}} \equiv A_\mu^3\sin(\theta_\mathrm{W}) + B_\mu\cos(\theta_\mathrm{W})\end{aligned}$$

where θ_W is called the *Weinberg angle*. For this angle holds: $\sin^2(\theta_\mathrm{W}) = 0.255 \pm 0.010$. Relations for the masses of the field quanta can be obtained from the remaining terms: $M_W = \frac{1}{2}vg$ and $M_Z = \frac{1}{2}v\sqrt{g^2+g'^2}$, and for the elementary charge holds: $e = \dfrac{gg'}{\sqrt{g^2+g'^2}} = g'\cos(\theta_\mathrm{W}) = g\sin(\theta_\mathrm{W})$

Experimentally it is found that $M_W = 80.022 \pm 0.26$ GeV/c^2 and $M_Z = 91.187 \pm 0.007$ GeV/c^2. According to the weak theory this should be: $M_W = 83.0 \pm 0.24$ GeV/c^2 and $M_Z = 93.8 \pm 2.0$ GeV/c^2.

15.13.3 Quantumchromodynamics

Coloured particles interact because the Lagrange density is invariant for the transformations of the group SU(3) of the colour interaction. A distinction can be made between two types of particles:

1. "White" particles: they have no colour charge, the generator $\vec{T} = 0$.
2. "Coloured" particles: the generators $\vec{T}$ are 8 3×3 matrices. There exist three colours and three anti-colours.

The Lagrange density for coloured particles is given by

$$\mathcal{L}_\mathrm{QCD} = i\sum_k \overline{\Psi_k}\gamma^\mu D_\mu\Psi_k + \sum_{k,l}\overline{\Psi_k}M_{kl}\Psi_l - \tfrac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu}$$

The gluons remain massless because this Lagrange density does not contain spinless particles. Because left- and right- handed quarks now belong to the same multiplet a mass term can be introduced. This term can be brought in the form $M_{kl} = m_k\delta_{kl}$.

The ∇-operator

In cartesian coordinates (x, y, z) holds:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \;\;,\;\; \mathrm{grad} f = \vec{\nabla} f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\mathrm{div}\,\vec{a} = \vec{\nabla}\cdot\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \;\;,\;\; \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\mathrm{rot}\,\vec{a} = \vec{\nabla}\times\vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\vec{e}_z$$

In cylinder coordinates (r, φ, z) holds:

$$\vec{\nabla} = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \varphi}\vec{e}_\varphi + \frac{\partial}{\partial z}\vec{e}_z \;\;,\;\; \mathrm{grad} f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi + \frac{\partial f}{\partial z}\vec{e}_z$$

$$\mathrm{div}\,\vec{a} = \frac{\partial a_r}{\partial r} + \frac{a_r}{r} + \frac{1}{r}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z} \;\;,\;\; \nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\mathrm{rot}\,\vec{a} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \varphi} - \frac{\partial a_\varphi}{\partial z}\right)\vec{e}_r + \left(\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r}\right)\vec{e}_\varphi + \left(\frac{\partial a_\varphi}{\partial r} + \frac{a_\varphi}{r} - \frac{1}{r}\frac{\partial a_r}{\partial \varphi}\right)\vec{e}_z$$

In spherical coordinates (r, θ, φ) holds:

$$\begin{aligned}
\vec{\nabla} &= \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\vec{e}_\varphi \\
\mathrm{grad} f &= \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi \\
\mathrm{div}\,\vec{a} &= \frac{\partial a_r}{\partial r} + \frac{2a_r}{r} + \frac{1}{r}\frac{\partial a_\theta}{\partial \theta} + \frac{a_\theta}{r\tan\theta} + \frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial \varphi} \\
\mathrm{rot}\,\vec{a} &= \left(\frac{1}{r\sin\theta}\frac{\partial a_\varphi}{\partial \theta} + \frac{a_\theta}{r\tan\theta} - \frac{1}{r\sin\theta}\frac{\partial a_\theta}{\partial \varphi}\right)\vec{e}_r + \left(\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial \varphi} - \frac{\partial a_\varphi}{\partial r} - \frac{a_\varphi}{r}\right)\vec{e}_\theta + \\
&\quad \left(\frac{\partial a_\theta}{\partial r} + \frac{a_\theta}{r} - \frac{1}{r}\frac{\partial a_r}{\partial \theta}\right)\vec{e}_\varphi \\
\nabla^2 f &= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\tan\theta}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2}
\end{aligned}$$

General orthonormal curvelinear coordinates (u, v, w) can be obtained from cartesian coordinates by the transformation $\vec{x} = \vec{x}(u, v, w)$. The unit vectors are then given by:

$$\vec{e}_u = \frac{1}{h_1}\frac{\partial \vec{x}}{\partial u} \;,\;\; \vec{e}_v = \frac{1}{h_2}\frac{\partial \vec{x}}{\partial v} \;,\;\; \vec{e}_w = \frac{1}{h_3}\frac{\partial \vec{x}}{\partial w}$$

where the factors h_i set the norm to 1. Then holds:

$$\begin{aligned}
\mathrm{grad} f &= \frac{1}{h_1}\frac{\partial f}{\partial u}\vec{e}_u + \frac{1}{h_2}\frac{\partial f}{\partial v}\vec{e}_v + \frac{1}{h_3}\frac{\partial f}{\partial w}\vec{e}_w \\
\mathrm{div}\,\vec{a} &= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u}(h_2 h_3 a_u) + \frac{\partial}{\partial v}(h_3 h_1 a_v) + \frac{\partial}{\partial w}(h_1 h_2 a_w)\right) \\
\mathrm{rot}\,\vec{a} &= \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 a_w)}{\partial v} - \frac{\partial(h_2 a_v)}{\partial w}\right)\vec{e}_u + \frac{1}{h_3 h_1}\left(\frac{\partial(h_1 a_u)}{\partial w} - \frac{\partial(h_3 a_w)}{\partial u}\right)\vec{e}_v + \\
&\quad \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 a_v)}{\partial u} - \frac{\partial(h_1 a_u)}{\partial v}\right)\vec{e}_w \\
\nabla^2 f &= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial w}\right)\right]
\end{aligned}$$