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Cover picture by Pat Keck (Untitled, 1992)

absolute significance of relative magnitude *only* if it has the power-law form

$$Q = \alpha A^a B^b C^c \ldots \tag{2.5}$$

where A, B, C, etc. are numerical values of base quantities and the coefficient α and exponents a, b, c, etc. are real numbers whose values distinguish one type of derived quantity from another. *All monomial derived quantities have this power-law form*; no other form represents a physical quantity.

A derived quantity of the first kind is defined in terms of a numerical value, which depends on the choice of base units. A derived quantity does not necessarily represent something tangible in the same sense as a base quantity, although it may. The square root of time, for example, is a derived quantity because it has the required power-law form, but we cannot point to any physical thing that "is" the square root of time.

To avoid talking of "units" for quantities that may have no physical representation, but whose numerical values nevertheless depend on the choice of base units, we introduce the concept of *dimension*. Each type of base quantity has by definition its own dimension. If A is the numerical value of a length, we say it has the dimension "of length", and write this as $[A]=L$ where the square brackets imply "the dimension of" and L symbolizes the concept of length. By this we mean simply that if the length unit size is increased by a factor n, the numerical value A will increase by a factor n^{-1}.

The dimension of a derived quantity conveys the same information in generalized form, for derived as well as base quantities. Consider a quantity defined by the formula

$$Q = \alpha L_1^{l_1} L_2^{l_2} \ldots M_1^{m_1} M_2^{m_2} \ldots t_1^{\tau_1} t_2^{\tau_2} \ldots \tag{2.6}$$

where the L_i's are numerical values of certain lengths, M_i's of certain masses, and t_i's of certain times, and α and all exponents are real numbers. If the length unit is changed by a factor n_L, the mass unit by a factor n_m, and the time unit by a factor n_t, it follows from equations (2.4) and (2.6) that the value of Q changes to

$$Q' = n^{-1} Q \tag{2.7a}$$

This same transformation could also have been obtained by arguing that equation (2.12), being an expression of a general physical law, *must*, according to Bridgman's principle of absolute significance of relative magnitude, be dimensionally homogeneous, and therefore should properly have been written

$$x = ct^2 \qquad (c = 4.91 ms^{-2}) \tag{2.14}$$

This form makes clear that the coefficient c is a physical quantity rather than a numerical coefficient. The units of c indicate its dimension and show that a change of the length unit from meters to feet, with the time unit remaining invariant, changes c by the factor 3.28, the inverse of the factor by which the length unit is changed. This gives c=16.1 ft s^{-2}, as in equation (2.13).

Equation (2.14) is the correct way of representing the data of equation (2.11). It is dimensionally homogeneous, and makes the transformation to different base units straightforward.

Every correct physical equation—that is, every equation that expresses a physically significant relationship between numerical values of physical quantities—must be dimensionally homogeneous. A fitting formula derived from correct empirical data may at first sight appear dimensionally non-homogeneous because it is intended for particular base units. Such formulas can always be rewritten in general, homogeneous form by the following procedure (Bridgman, 1931):

(1) Replace all the numerical coefficients in the equation with unknown dimensional constants.
(2) Determine the dimensions of these constants by requiring that the new equation be dimensionally homogeneous.
(3) Determine the numerical values of the constants by matching them with those in the original equation when the units are the same.

This is of course how equation (2.14) was derived from equation (2.12).

Another example serves to reinforce this point. Suppose it is found that the pressure distribution in the earth's atmosphere over much of the United States can be represented (approximately) by the formula

$$p = 1.01x10^5 e^{-0.00012z} \tag{2.15}$$

where p is the pressure in Nm^{-2} and z is the altitude in meters. This expression applies only with the cited units. The correct, dimensionally homogeneous form of this equation is

$$p = ae^{-bz} \quad (a = 1.01 \times 10^5 Nm^{-2},\ b = 0.00012m^{-1}) \tag{2.16}$$

where a and b are physical quantities. In this form the equation is valid independent of the chosen base units. The dimensions of a and b indicate how these quantities change when units are changed.

The two quantities a and b in equation (2.16) are *physical constants* in the sense that their values are fixed once the system of units is chosen. In this case the constants characterize a particular environment¾the pressure distribution in the earth's atmosphere over the US. Similarly, the acceleration of gravity g in equation (2.11) is a physical constant that characterizes the (local!) gravitational force field at the earth's surface.

The basic laws of physics also contain a number of *universal physical constants* whose magnitudes are the same in all problems once the system of units is chosen: the speed of light in vacuum c, the universal gravitational constant G, Planck's constant h, Boltzmann's constant k_B, and many others.

2.6 Derived quantities of the second kind

The classification of quantities as base or derived is not unique. There exist general laws that bind different kinds of quantities together in certain relationships, and these laws can be used to transform base quantities into derived ones. Such transformations are useful because they reduce the number of units that must be chosen arbitrarily, and simplify the forms of physical laws.

Area, for example, may be taken as a base quantity with its own comparison and addition operations, and measured in terms of an arbitrarily chosen (base) unit: a certain postage stamp, say, to use an absurd example. The floor area of a room may be measured by covering the floor with copies of this postage stamp and parts thereof, and counting the number of whole stamps required. If we adopt this practice we will eventually find that, regardless of the unit we have chosen for measuring the base units taken are the foot, the second, the pound-mass (lbm), and the pound-force (lbf), and the constant in Newton's law has the value c=1/32.2 lbf s^2 lbm^{-1} ft^{-1}.

Table 2.2 illustrates the fact that, while an actual *physical* quantity like force is the same regardless of the (arbitrary) choice of the system of units, its dimension depends on that choice. What is more, depending on how derived quantities are defined, a given physical law may contain a dimensional physical constant the value of which must be specified (as in F=cma), or it may contain no physical constant (as in F=ma).

An interesting point to note is that only a few of the available universal laws are usually "used up" to make base quantities into derived ones of the second kind. There are many laws left with universal dimensional physical constants that could in principle be set equal to unity: the gravitational constant G, Planck's constant h, Boltzmann's constant k_B, the speed of light in vacuum c, etc. This leaves us with some interesting possibilities. For example, it is possible to define systems of units that have no base quantities at all (see Bridgman, 1931). In such systems all units of measurement are related to some of the universal constants that describe our universe. In effect, there exist in the universe "natural" units for all base quantities, based on universal constants such as the speed of light, the quantum of energy, etc. Unfortunately, the choice of such “natural” systems of units turns out to be far from unique, which renders futile any attempt to endow any one of them with unique significance.

2.8 Recapitulation

1. A *base quantity* is a property that is defined in physical terms by two operations: a comparison operation, and an addition operation. The comparison operation is a physical procedure for establishing whether two samples of the quantity are equal or unequal; the addition operation defines what is meant by the sum of two samples of that property.

2. Base quantities are properties for which the following concepts are defined in terms of physical operations: equality, addition, subtraction, multiplication by a pure number, and division by a pure number. Not defined in terms of physical operations are: product, ratio, power, and logarithmic, exponential, trigonometric and other special functions of physical quantities.

Step 3: Dimensionless variables

We now define *dimensionless forms* of the n-k remaining independent variables by dividing each one with the product of powers of $Q_1...Q_k$ which has the same dimension,

$$\Pi_i = \frac{Q_{k+i}}{Q_1^{N_{(k+i)1}} Q_2^{N_{(k+i)2}} ... Q_k^{N_{(k+i)k}}} \quad , \tag{3.5}$$

where i=1, 2,..., n-k, and a dimensionless form of the dependent variable Q_0,

$$\Pi_0 = \frac{Q_0}{Q_1^{N_{01}} Q_2^{N_{02}} ... Q_k^{N_{0k}}} \quad . \tag{3.6}$$

Step 4: The end game and Buckingham's π-theorem

An alternative form of equation (3.1) is

$$\Pi_0 = f(Q_1, Q_2, ..., Q_k, \Pi_1, \Pi_2, ..., \Pi_{n-k}) \tag{3.7}$$

in which all quantities are dimensionless except $Q_1...Q_k$. The values of the dimensionless quantities are independent of the sizes of the base units. The values of $Q_1...Q_k$, on the other hand, do depend on base unit size. They *cannot* be put into dimensionless form since they are (by definition) dimensionally independent of each other. From the principle that any physically meaningful equation must be dimensionally homogeneous, that is, valid independent of the sizes of the base units, it follows that $Q_1...Q_k$ must in fact be *absent* from equation (3.7), that is,

$$\Pi_0 = f(\Pi_1, \Pi_2, ..., \Pi_{n-k}) \quad . \tag{3.8}$$

This equation is the final result of the dimensional analysis, and contains

Buckingham's π-theorem:

> When a complete relationship between dimensional physical quantities is expressed in dimensionless form, the number of independent quantities that appear in it is reduced from the original *n* to *n-k*, where *k* is the maximum number of the original n that are dimensionally independent.

The theorem derives its name from Buckingham's use of the symbol Π for the dimensionless variables in his original 1914 paper. The π-theorem tells us that, because all complete physical equations must be dimensionally homogeneous, a restatement of any such equation in an appropriate dimensionless form will reduce the number of independent quantities in the problem by *k*. This can simplify the problem enormously, as will be evident from the example that follows.

The π−theorem itself merely tells us the *number* of dimensionless quantities that affect the value of a particular dimensionless dependent variable. It does not tell us the forms of the dimensionless variables. That has to be discovered in the third and fourth steps described above. Nor does the π-theorem, or for that matter dimensional analysis as such, say anything about the form of the functional relationship expressed by equation (3.7). That form has to be discovered by experimentation or by solving the problem theoretically.

3.2 An example: Deformation of an elastic ball striking a wall

Suppose we wish to investigate the deformation that occurs in elastic balls when they impact on a wall. We might be interested, for example, in finding out what determines the diameter *d* of the circular imprint left on the wall after a freshly dyed ball has rebounded from it (figure 3.1).

Step 1: The independent variables

The first step is to identify a complete set of independent quantities that determine the imprint radius *d*. We begin by specifying the problem more clearly. We agree to restrict our attention to (initially) spherical, homogeneous balls made of perfectly elastic material, to impacts at perpendicular to the wall, and to walls that are perfectly smooth and flat

The result is independent of how one chooses a dimensionally independent subset

Suppose we had chosen the dimensionally independent subset *V*, *E*, and ρ instead of *V*, *D* and ρ. Non-dimensionalizing *d* and *D* with combinations of *V*, *E* and ρ, we might have obtained the result

$$\frac{d}{(\rho V^2/E)^{1/3}} = F\left(\frac{D}{(mV^2/E)^{1/3}}, \gamma\right) \qquad (3.18)$$

This can, however, be rewritten as

$$\frac{d}{D} = \left(\frac{mV^2}{ED^3}\right)^{1/3} F\left(\left(\frac{ED^3}{mV^2}\right)^{1/3}, \gamma\right) = f\left(\frac{ED^3}{mV^2}, \gamma\right) \qquad (3.19)$$

where *F* and *f* are different functions of their arguments. Equation (3.19) is of course identical to equation (3.13).

The result is independent of the type of system of units

The choice of system of units may affect the dimensions of physical quantities as well as the values of the physical constants that appear in the underlying physical laws. What effect, if any, does this have on dimensional analysis? Reason dictates there should be no effect on the "bottom line", since the observer (the analyst) is free to choose or make up whatever system of units he wants, and his arbitrary choice should not affect the laws of physics.

Consider our example of the dyed ball, but viewed in terms of a system of units like the British Engineering System (type3 in table 2.2), where mass, length, time *and* force are taken as base units. In such a system Newton's law reads $F = cma$, where *c* is a physical constant with dimension $Ft^2m^{-1}L^{-1}$. This affects the very first step of the analysis. Since the impact process is controlled by Newton's law, which now contains the constant *c*, the value of which must be specified, we now have

$$d = d(V, D, E, m, \gamma, c). \qquad (3.20)$$

density are essentially the same in all the applications that interest us. The similarity law equation (4.1) cannot be simplified.

Simplification occurs only when some of the fixed quantities are dimensionally dependent on the rest.