1 VECTOR ANALYSIS 3
1.1 VECTOR ALGEBRA 3
1.1.1 Scalars and Vectors 3
1.1.2 Unit Vector 4
1.1.3 Vector Addition and Subtraction 5
1.1.4 Position and Distance Vectors 6
1.1.5 Multiplication of a vector by a scalar 9
1.1.6 Vector Multiplication 9
1.1.6.1 Dot product 10
1.1.6.2 Cross product 11
1.1.6.3 Scalar Triple Product 13
1.1.6.4 Vector Triple Product 13
1.2 COORDINATE SYSTEMS AND TRANSFORMATION 17
1.2.1 Introduction 17
1.2.2 Cartesian Coordinates 17
1.2.3 Circular Cylindrical Coordinates 18
1.2.4 Spherical Coordinates 20
1.2.5 Distance between two points 23
1.3 VECTOR CALCULUS 33
1.3.1 Introduction 33
1.3.2 Differential Length, Area, and Volume 33
1.3.2.1 Cartesian coordinates 33
1.3.2.2 Cylindrical coordinates 34
1.3.2.3 Spherical coordinates 36
1.3.3 Line, Surface, and Volume Integrals 39
1.3.3.1 Line integral 39
1.3.3.2 Surface integral 40
1.3.3.3 Volume integral 41
1.3.4 Del Operator 46
1.3.5 Gradient of a Scalar 47
1.3.6 Divergence of a Vector and Divergence Theorem 48
1.3.7 Curl of a Vector and Stokes's Theorem 50
1.3.8 Laplacian of a Scalar 54
1.3.9 Phasors 55

$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y \qquad \textbf{(1. 29)}$$

Example 1. 5:

Verify Lagrange's identity, which states that if **A** and **B** are two arbitrary vectors, then $|\mathbf{A} \times \mathbf{B}|^2 \; = \; \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A}.\mathbf{B})^2$

Solution:

From the definition of the cross product of two vectors, we have

$$\begin{aligned}
|\mathbf{A} \times \mathbf{B}|^2 \; &= \; \mathbf{A}^2\mathbf{B}^2 \sin^2\theta \\
&= \mathbf{A}^2\mathbf{B}^2(1 \; - \; \cos^2\theta) \\
&= \mathbf{A}^2\mathbf{B}^2 - \mathbf{A}^2\mathbf{B}^2\cos^2\theta = \mathbf{A}^2\mathbf{B}^2 - (\mathbf{A}.\mathbf{B})^2
\end{aligned}$$

Just as multiplication of two vectors gives a scalar or vector result, multiplication of three vectors **A**, **B**, and **C** gives a scalar or vector result depending on how the vectors are multiplied. Thus we have scalar or vector triple product.

1.1.6.3 Scalar Triple Product

Given three vectors **A**, **B**, and **C**, we define the scalar triple product as,

$$\mathbf{A}.(\mathbf{B} \times \mathbf{C}) = \mathbf{B}.(\mathbf{C} \times \mathbf{A}) = \mathbf{C}.(\mathbf{A} \times \mathbf{B}) \qquad \textbf{(1. 30)}$$

This is obtained using a cycle permutation. If $\mathbf{A} = (A_x, A_y, A_z)$, $\mathbf{B} = (B_x, B_y, B_z)$ and $\mathbf{C} = (C_x, C_y, C_z)$ then $\mathbf{A}.(\mathbf{B} \times \mathbf{C})$ is the volume of a parallelepiped having **A**, **B**, and **C** as edges and is easily obtained by finding the determinant of the 3 × 3matrix formed by **A**, **B**, and **C**; that is,

$$\mathbf{A}.(\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \qquad \textbf{(1. 31)}$$

Since the result of this vector multiplication is scalar these two equations are called the scalar triple product.

1.1.6.4 Vector Triple Product

For vectors **A**, **B**, and **C**, we define the vector triple product as

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}.\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B}) \qquad \textbf{(1. 32)}$$

This is obtained using the “bac-cab” rule.

Example 1. 6:

Given vectors $\mathbf{A} = 3\mathbf{a}_x + 4\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_y - 5\mathbf{a}_z$, find the angle between **A** and **B**.

Solution:

(A_x, A_y, A_z) or $A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$	**(1. 33)**

Where $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$ are unit vectors along the x,y, and z directions.

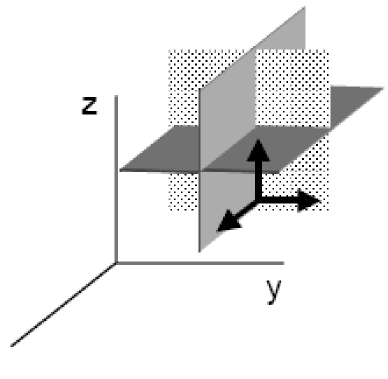


Figure 1. 7: A point in Cartesian coordinates is defined by the intersection of the three planes: x = constant, y = constant, z = constant. The three unit vectors are normal to each of the three surfaces.

The ranges of the variables are:

$-\infty \leq x \leq +\infty$	
$-\infty \leq y \leq +\infty$	**(1. 34)**
$-\infty \leq z \leq +\infty$	

1.2.3 Circular Cylindrical Coordinates

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point *P* in cylindrical coordinates is represented as (ρ, ϕ, z) and is as shown in Figure 1.8. Observe Figure 1.8 closely and note how we define each space variable: ρ is the radius of the cylinder passing through *P* or the radial distance from the z-axis: ϕ, called the azimuthal angle, is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are:

$0 \leq \rho \leq \infty$	
$0 \leq \phi \leq 2\pi$	**(1. 35)**
$-\infty \leq z \leq +\infty$	

A vector **A** in cylindrical coordinates can be written as

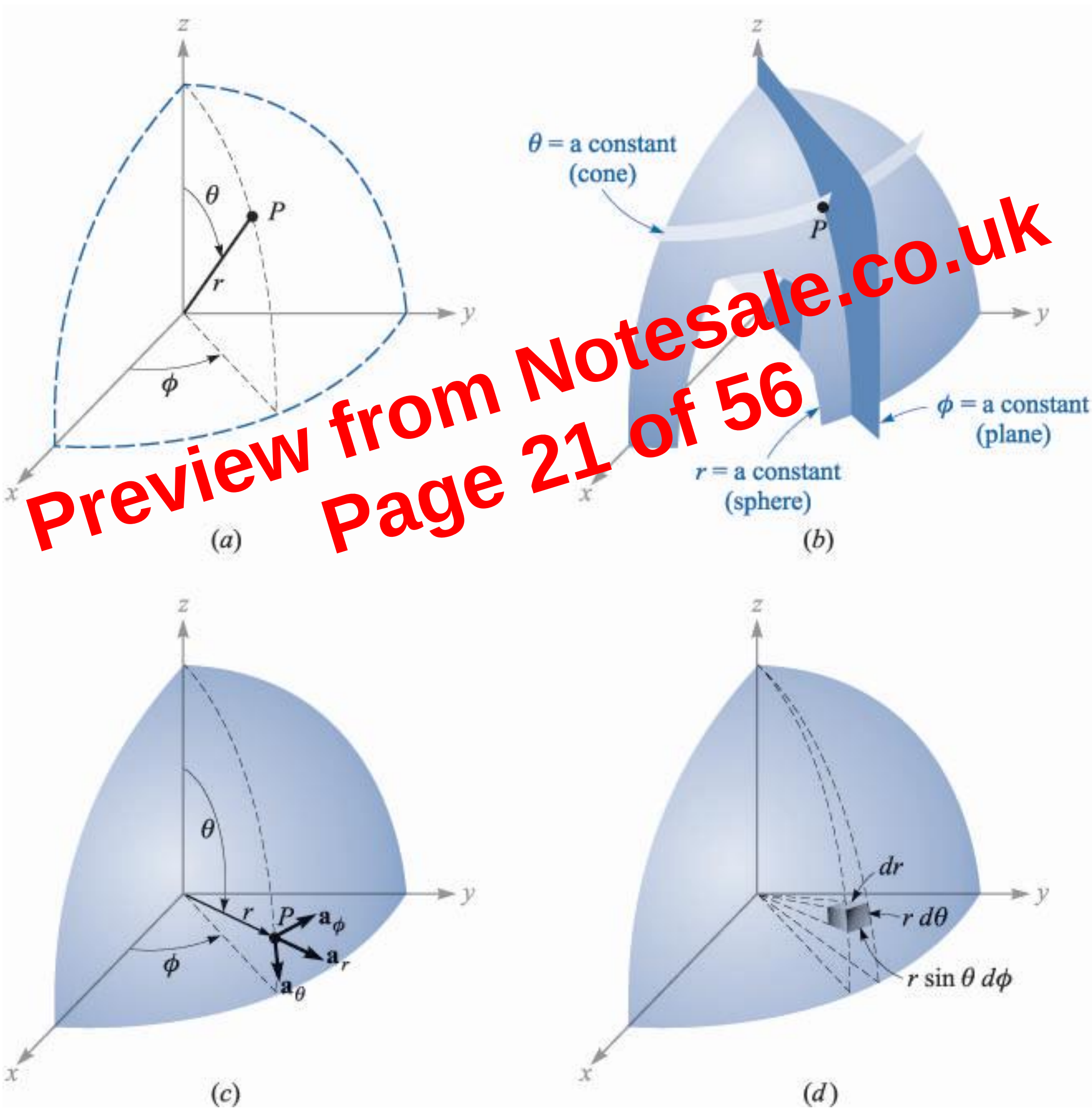
(A_ρ, A_ϕ, A_z) or $A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$	**(1. 36)**

(called the colatitudes) is the angle between the z-axis and the position vector of P; and ϕ is measured from the x-axis (the same azimuthal angle in cylindrical coordinates). According to these definitions, the ranges of the variables are

$0 \leq r \leq \infty$	**(1. 48)**
$0 \leq \theta \leq \pi$	
$0 \leq \phi \leq 2\pi$	

A vector **A** in spherical coordinates can be written as

(A_r, A_θ, A_ϕ) or $A_r\mathbf{a}_r + A_\theta\mathbf{a}_\theta + A_\phi\mathbf{a}_\phi$	**(1. 49)**



(a)	**(b)**

Calculate the work ΔW required to move the cart along the circular path from point A to point B if the force field is

$$\mathbf{F} = 3xy\mathbf{a}_x + 4x\mathbf{a}_y$$

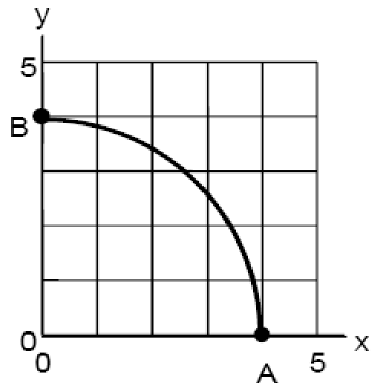


Figure 1. 21: For this example.

Solution:

The integral can be performed in Cartesian coordinates or in cylindrical coordinates. In Cartesian coordinates, we write

$$\begin{aligned}\mathbf{F}.d\mathbf{l} &= (3xy\mathbf{a}_x + 4x\mathbf{a}_y).(dx\,\mathbf{a}_x + dy\,\mathbf{a}_y)\\ &= (3xy\,dx + 4x\,dy)\end{aligned}$$

The equation of a circle is $x^2 + y^2 = 4^2$. Hence

$$\begin{aligned}\int_A^B \mathbf{F}.d\mathbf{l} &= \int_4^0 3x\sqrt{16-x^2}dx + \int_0^4 4\sqrt{16-y^2}\,dy\\ &= \left|-(16-x^2)\right|_4^0 + \left|4\left(\frac{y}{2}\sqrt{16-y^2} + 8\sin^{-1}\left(\frac{y}{4}\right)\right)\right|_0^4\\ &= -64 + 16\pi\end{aligned}$$

In cylindrical coordinates, we write

$$\mathbf{F}.d\mathbf{l} = (3xy\mathbf{a}_x + 4x\mathbf{a}_y).(dr\,\mathbf{a}_r + rd\varphi\,\mathbf{a}_\varphi + dz\mathbf{a}_z)$$

Since the integral is to be performed along the indicated path where only the angle φ is changing, we have $dr = 0$ and $dz = 0$. Also $r = 4$. Therefore

$$\mathbf{F}.d\mathbf{l} = (3xy\mathbf{a}_x + 4x\mathbf{a}_y).(4d\varphi\,\mathbf{a}_\varphi)$$

$$\mathbf{a}_x\mathbf{a}_\varphi = -\sin\varphi$$

$$\mathbf{a}_y\mathbf{a}_\varphi = \cos\varphi$$

The integral becomes

$$\int_3 \mathbf{F}.d\mathbf{l} = \int_0^1 (x^2 - 1)dx = \left|\frac{x^3}{3} - \mathrm{x}\right|_0^1 = -\frac{2}{3}$$

For segment 4, x= 1, $= \mathbf{a}_x - z\mathbf{a}_y - y^2\,\mathbf{a}_z$, $d\mathbf{l} = dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$,so

$$\int_4 \mathbf{F}.d\mathbf{l} = \int (-z\,dy - y^2 dz)$$

But on 4, $z = y$; that is, $dz = dy$. Hence,

$$\int_3 \mathbf{F}.d\mathbf{l} = \int_1^0 (-y - y^2)dy = \left|-\frac{y^2}{2} - \frac{y^3}{3}\right|_1^0 = \frac{5}{6}$$

By putting all these together, we obtain

$$\int_L \mathbf{F}.d\mathbf{l} = -\frac{1}{3} + 0 - \frac{2}{3} + \frac{5}{6} = -\frac{1}{6}$$

Example 1. 27:

Calculate the circulation of $A = \rho\,cos\,\varphi\,\mathbf{a}_\rho + z\,sin\,\varphi\,\mathbf{a}_z$ around the edge L of the wedge defined by $0 < \rho < 2, 0 \leq t \leq 60°, z = 0$ and shown in Figure 1.22.

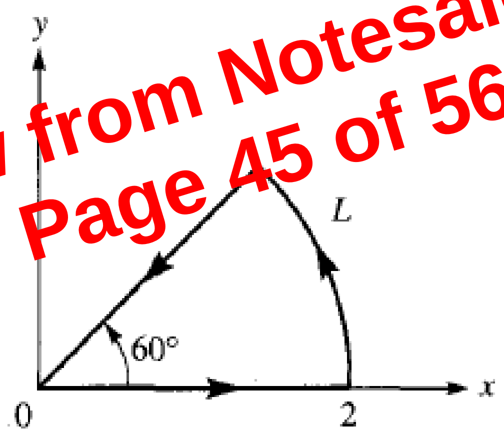


Figure 1. 23: For this example.

Solution :

Answer: 1

Example 1. 28:

Find the volume of a cylinder that has a radius a and *a* length *L*.

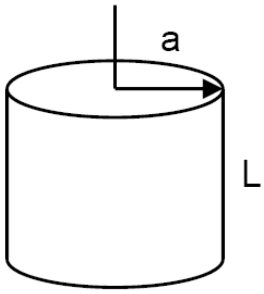


Figure 1. 24: For this example.

Solution:

The volume of a cylinder is calculated to be

$$\Delta v = \int dv = \int_{z=0}^{z=L} \int_{\varphi=0}^{\varphi=2\pi} \int_{r=0}^{r=3} r dr\, d\varphi\, dz = \pi a^2 L$$

1.3.4 DEL OPERATOR

The del operator, written ∇, is the vector differential operator. In Cartesian coordinates,

$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$	**(1.81)**

This vector differential operator, otherwise known as the *gradient operator,* is not a vector in itself, but when it operates on a scalar function, for example, a vector ensues. The operator is useful in defining

1. The gradient of a scalar V, written, as ∇V.

2. The divergence of a vector $\mathbf{A}$, written as $\nabla . \mathbf{A}$.

3. The curl of a vector $\mathbf{A}$, written as $\nabla \times \mathbf{A}$.

4. The Laplacian of a scalar V, written as $\nabla^2 V$.

Each of these will be denned in detail in the subsequent sections. Before we do that, it is appropriate to obtain expressions for the del operator ∇ in cylindrical and spherical coordinates.

Cartesian coordinates	$\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z$	**(1. 82)**
Cylindrical coordinates	$\nabla = \frac{\partial}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \phi}\mathbf{a}_\phi + \frac{\partial}{\partial z}\mathbf{a}_z$	**(1. 83)**
Spherical coordinates	$\nabla = \frac{\partial}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\ \sin\theta}\frac{\partial}{\partial \phi}\mathbf{a}_\phi$	**(1. 84)**

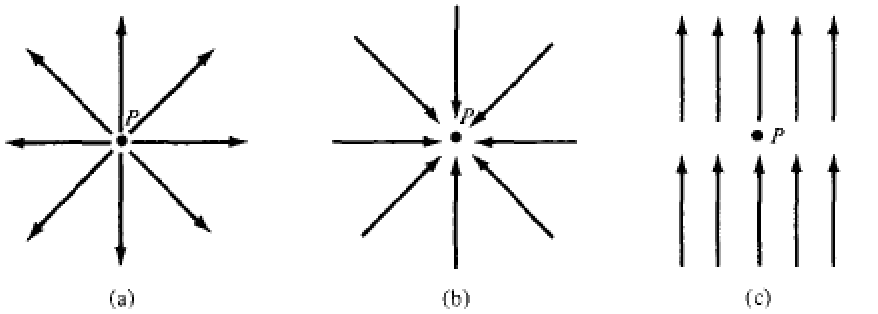


Figure 1. 25: Illustration of the divergence of a vector field at P; (a) positive divergence, (b) negative divergence, (c) zero divergence.

The divergence of **A** at point *P* is given by:

Cartesian coordinates	$\nabla.\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	**(1. 92)**
Cylindrical coordinates	$\nabla.\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$	**(1. 93)**
Spherical coordinates	$\nabla.\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\ \sin\theta}\frac{\partial}{\partial \theta}(A_\theta\ \sin\theta) + \frac{1}{r\ \sin\theta}\frac{\partial A_\phi}{\partial \phi}$	**(1. 94)**

From the definition of the divergence of **A**, it is not difficult to expect that.

$$\oint_S \mathbf{A}.d\mathbf{S} = \int_v \nabla.\mathbf{A}\ dv$$

This is called the divergence theorem, otherwise known as the Gauss-Ostrogradsky theorem.

The divergence theorem states that the total outward flux of a vector field A through the closed surface *S* is the same as the volume integral of the divergence of A.

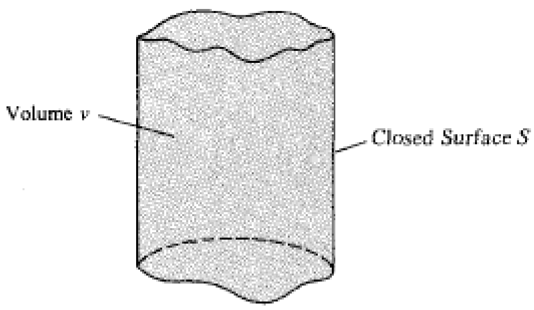


Figure 1. 26: Volume v enclosed by surface S.

Example 1. 31:

Determine the divergence of these vector fields:

Cylindrical coordinates	$\nabla \times \mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$	**(1. 97)**
Spherical coordinates	$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \mathbf{a_r} & r\,\mathbf{a}_\theta & r\,\sin\theta\,\mathbf{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\,\sin\theta\;A_\phi \end{vmatrix}$	**(1. 98)**

The physical significance of the curl of a vector field is evident; the curl provides the maximum value of the circulation of the field per unit area (or circulation density) and indicates the direction along which this maximum value occurs. The curl of a vector field **A** at a point *P* may be regarded as a measure of the circulation or how much the field curls around *P*. For example, Figure 1.26(a) shows that the curl of a vector field around *P* is directed out of the page. Figure 1.26(b) shows a vector field with zero curl.

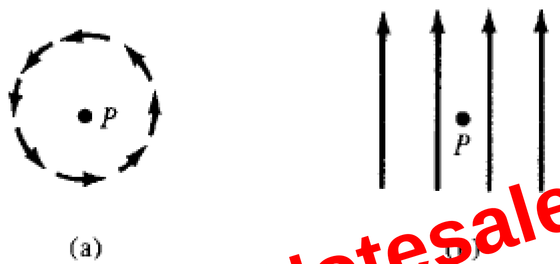


Figure 1. 27: Illustration of a curl: (a) curl at P points out of the page; (b) curl at P is zero.

From the definition of the curl of a vector we can obtain Stokes' theorem that relates a closed line integral to a surface integral. We can write

$$\oint_L \mathbf{A}.\,d\mathbf{l} = \int_S (\nabla \times \mathbf{A})\,.\,d\mathbf{S}$$

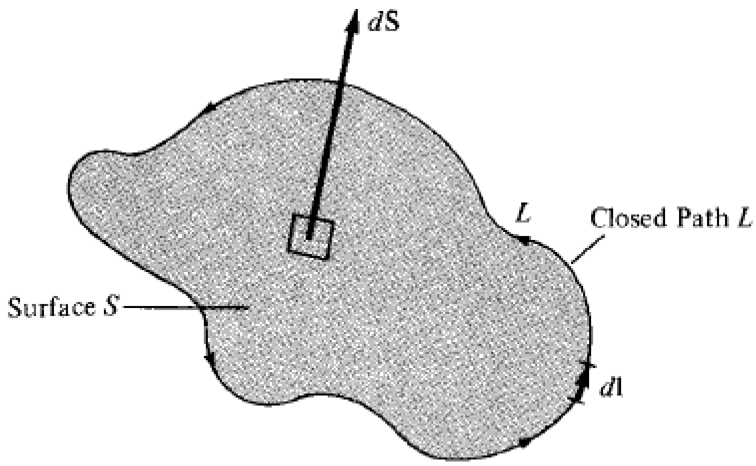


Figure 1. 28: Determining the sense of dl and dS involved in Stokes's theorem.