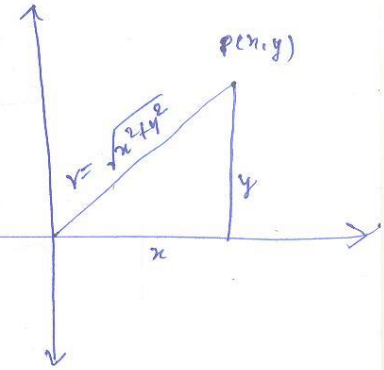
* Polar form of a complex number :-

$z = x+iy$, $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow \quad r^2 = x^2+y^2$

$\Rightarrow \quad r = \sqrt{x^2+y^2}$

$\Rightarrow |z| = |x+iy| = \sqrt{x^2+y^2} = r$



$z = x+iy = r\cos\theta + i\sin\theta$

$\Rightarrow \quad z = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \boxed{z = re^{i\theta}}$

$\boxed{\theta = \tan^{-1}(y/x)}$

* Properties of the modulus :-

If $z = x+iy$ (x, y real), then $|z| = (x^2+y^2)^{1/2}$

(a) $|\bar{z}| = |z|$

(b) $z\bar{z} = |z|^2$

(c) $|z_1 . z_2| = |z_1||z_2|$

(d) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(e) the distance between two points z_1 & z_2 is

$|z_1 - z_2| = |z_2 - z_1|$

Now $f(z+\Delta z) = f(x+\Delta x, y+\Delta y)$

$$= u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)$$

$$= \left[u(x,y) + \left\{\Delta x \frac{\partial u}{\partial x} + \Delta y \frac{\partial u}{\partial y}\right\} + \frac{1}{2!}\left\{(\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + (\Delta y)^2 \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial^2 u}{\partial x \partial y}\right\} + \cdots\right]$$

$$+ i\left[v(x,y) + \left(\Delta x \frac{\partial v}{\partial x} + \Delta y \frac{\partial v}{\partial y}\right) + \frac{1}{2!} \cdots\right]$$

$$= [u(x,y) + i\,v(x,y)] + \Delta x\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + \Delta y\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right] + \cdots$$

neglecting higher order derivative...

$$= f(z) + \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$$

$$\Rightarrow f(z+\Delta z) - f(z) = \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \Delta y\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)$$ (by C-R eqns)

$$= \Delta x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\Delta y\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$= (\Delta x + i\Delta y)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = \Delta z\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$\Rightarrow \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u + iv) = \frac{\partial}{\partial x}(f)$$

taking limit both sides as $\Delta z \to 0$, we have

$$\Rightarrow \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{\partial f}{\partial x}$$

$$f'(z) = \frac{\partial f}{\partial x}$$

similarly $f'(z) = \frac{\partial f}{\partial y}$

and also given $u_x = v_y$ & $u_y = -v_x$ and these are contn in D.

$\therefore f(z)$ is analytic in D.

(iii) To show that :- $f'(0)$ does not exist.

Now
$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$
$$= \lim_{z\to 0} \frac{f(z)}{z}$$

$$f'(0) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)} \quad \text{———— (A)}$$

(a) along x-axis : then $y=0$ in (A) we get
$$f'(0) = \lim_{x\to 0} \frac{x^3+ix^3}{x^3} = 1+i$$

(b) along y-axis : then $x=0$ in (A) we get
$$f'(0) = \lim_{y\to 0} \frac{-y^3+iy^3}{iy^3} = \frac{i-1}{i} = 1+i$$

(c) along ~~steught~~ staight line: then $y=x$ in (A)
$$f'(0) = \lim_{x\to 0} \frac{(x^3-x^3)+i(x^3+x^3)}{(x^2+x^2)(x+ix)} = \lim_{x\to 0} \frac{2ix^3}{2x^3(1+i)} = \frac{i}{1+i}$$

$\therefore$ value of $f'(z)$ is not exist or unique along different paths

Hence $f(z)$ is not analytic at $z=0$

Problem :- If $u-v = (x-y)(x^2+4xy+y^2)$ and $f(z)=u+iv$ is an analytic funⁿ of $z=x+iy$, find $f(z)$ in terms of z.

Solⁿ :- Given
$$f(z) = u+iv \quad \text{———— (1)}$$
$$if(z) = iu-v \quad \text{———— (2)}$$

adding (1) and (2), we get
$$(1+i)f(z) = (u-v)+i(u+v)$$
$$F(z) = U+iV \quad \text{———— (3)}$$