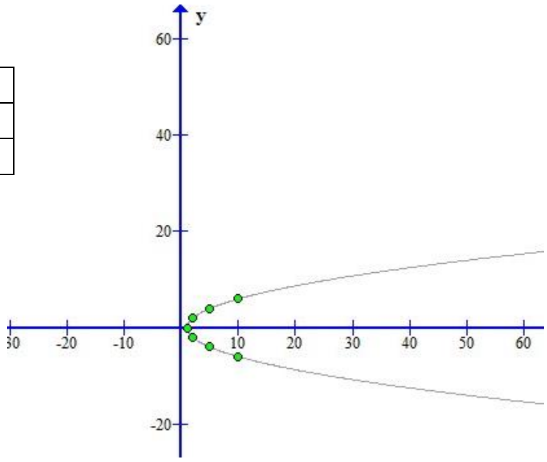
Example 1.1

Draw the curve given by the parametric equations $x = 1 + t^2$, $y = 2t$, $t \in \mathbb{R}$

t	-3	-2	-1	0	1	2	3
x	10	5	2	1	2	5	1
y	-6	-4	-2	0	2	4	6



Example 1.2

Find the Cartesian equation of the line with parametric equations $x = 3t - 1$ $y = 2t + 2$

$x = 3t - 1 \Rightarrow x + 1 = 3t \Rightarrow t = \frac{x+1}{3}$

$y = 2t + 2 = 2\left(\frac{x+1}{3}\right) + 2 \Rightarrow$ $\boxed{y = \frac{2x}{3} + \frac{8}{3}}$

Example 1.3

Find the Cartesian equation of the line with parametric equations $x = 1 + t^2$ and $y = 2t$

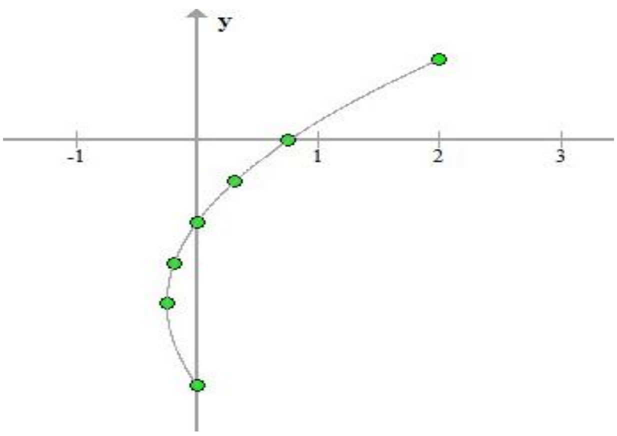
$y = 2t \Rightarrow t = \frac{y}{2}$

$x = 1 + t^2 \Rightarrow x = 1 + \left(\frac{y}{2}\right)^2 \Rightarrow x = 1 + \frac{y^2}{4} \Rightarrow$ $\boxed{y^2 = 4x - 1}$

Example 1.4

Draw the curve given by the parametric equations $x = t + t^2$, $y = 2t - 1$, $1 \le t \le 1$. Also find the Cartesian equation of the curve.

t	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1
x	0	$-\frac{1}{4}$	$-\frac{3}{16}$	0	$\frac{5}{16}$	$\frac{3}{4}$	2
y	-3	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	1



$\frac{3}{4} - \frac{3}{4}x + \frac{3}{4}x^2 - \frac{3}{4}x^3 + \frac{9}{4} + \frac{27}{4}x + \frac{81}{4}x^2 + \frac{243}{4}x^3$

$$\boxed{= 3 + 6x + 21x^2 + 60x^3 + \cdots}$$

Method 2 (Directly)

$\frac{3}{(x+1)(1-3x)} = 3[(x+1)(1-3x)]^{-1} = 3\,(x+1)^{-1}(1-3x)^{-1}$

$= 3\left(1 - x + \frac{-1(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!}x^3 + \cdots\right)\left(1 + 3x + \frac{-1(-2)}{2!}(-3x)^2 + \frac{(-1)(-2)(-3)}{3!}(-3x)^3 \ldots\right)$

$3(1 + 3x + 9x^2 + 27x^3 - x - 3x^2 - 9x^3 + x^2 + 3x^3 - x^3) = 3(1 + 2x + 7x^2 + 20x^3)$

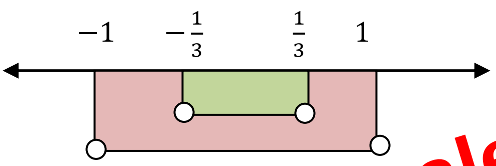
$$\boxed{3 + 6x + 21x^2 + 60x^3 + \cdots}$$

Since there is more than one expansion, for the overall fraction to be valid we have to find the validity for both.

The expansion is valid if

$|x| < 1 \;\Rightarrow\; -1 < x < 1$

$|3x| < 1 \;\Rightarrow\; -\frac{1}{3} < x < \frac{1}{3}$



$\therefore$ the expansion is valid for $-\frac{1}{3} < x < \frac{1}{3}$

<u>Example 47</u>

Find the first four terms in the binomial expansion of $(5 + 3x)^{1/3}$. State the range in values of x for which the expansion is valid

$$(8 + 3x)^{1/3} = 8^{\frac{1}{3}}\left(1 + \frac{3}{8}x\right)^{\frac{1}{3}} = 2\left(1 + \frac{1}{3}\left(\frac{3}{8}x\right) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{3}{8}x\right)^2 + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}\left(\frac{3}{8}x\right)^3 + \cdots\right)$$

$$1 + \frac{1}{8}x - \frac{1}{64}x^2 + \frac{5}{1536}x^3 + \cdots$$

The series is valid for $\left|\frac{3}{8}x\right| < 1 \Rightarrow\; -\frac{8}{3} < x < \frac{8}{3}$

Integration using trigonometric identities

<u>Example 5.11</u>

$$\int \sin^2 x \; dx = \int \frac{1-\cos 2x}{2}\, dx = \int \frac{1}{2} - \frac{\cos 2x}{2} dx = \int \frac{1}{2} dx - \int \frac{\cos 2x}{2} dx = \frac{1}{2}x - \frac{\sin 2x}{4} + C$$

<u>Example 5.12</u>

$\int \tan^2 x \, dx$

$$\int \tan^2 x \, dx = \int \sec^2 x - 1 \; dx = \int \sec^2 x \; dx - \int dx = \tan x - x + C$$

Exercise 5.5

1. Integrate the Following

a) $\int \cos^2 x \; dx$

b) $\int \tan^2 3x \; dx$

c) $\int \frac{dx}{\sin^2 x \cos^2 x}$

d) $\int \sin^3 x \; dx$

e) $\int \cos^3 x \; dx$

f) $\int \tan^3 x \; dx$

g) $\int \sec^4 x \; dx$

h) $\int \cos^2 3x \; dx$

i) $\int \cot^2 x \; dx$

j) $\int \frac{\tan x}{\sec^3 x} \, dx$

k) $\int \sin^2 x \cos^3 x \; dx$

l) $\int \cot^6 x + \cot^8 x \; dx$

m) $\int \sqrt{1 + \sin 2x} \; dx$

n) $\int \sqrt{10 + 10\cos 10x} \; dx$

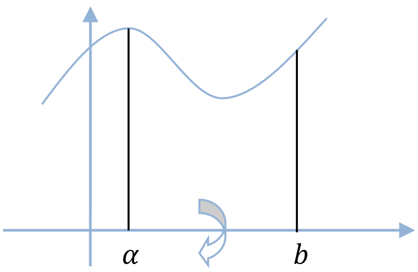
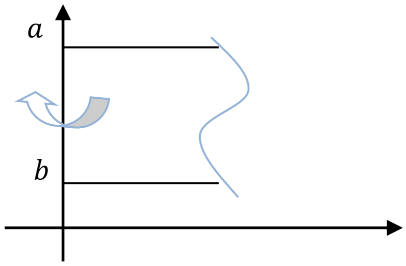
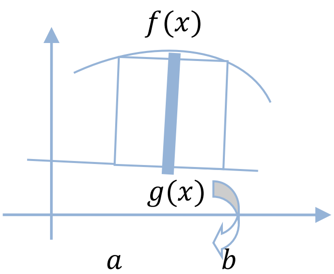
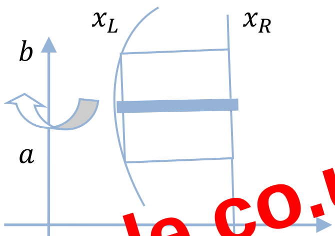
o) $\int \cot^3 x \; dx$

p) $\int \frac{x^2}{\sqrt{4-x^2}} \, dx$ [sub: $x = 2\sin\theta$]

q) $\int \sqrt{9 - x^2} \, dx$ [sub: $x = 3\sin\theta$]

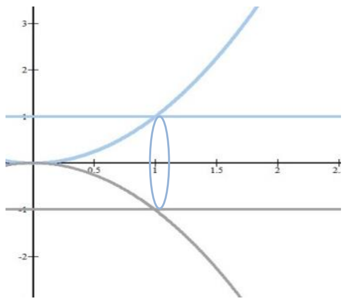
r) $\int \frac{dx}{\sqrt{x^2-16}}$ [sub: $x = 4\sec\theta$]

Volumes

1) $V=\pi\int_a^b y^2\,dx$	2) $V=\pi\int_a^b x^2\,dy$
	
3) $V=\pi\int_a^b f^2(x)-g^2(x)\,dx$	4) $V=\pi\int_a^b x_R^2-x_L^2\ dy$
	

Example 5.18

Calculate the volume of revolution of the graph generated by the curve $y=x^2$ the Ox axis and the line $x=1$ when is rotated 2π by the x axis.



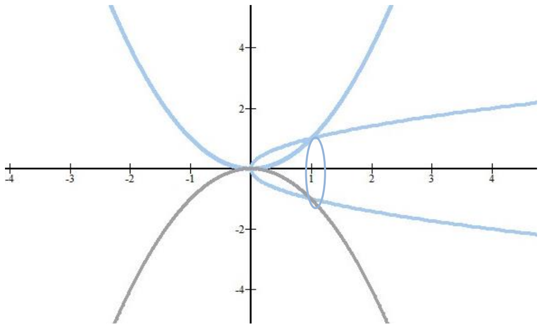
$V=\pi\int_a^b y^2\,dx=\pi\int_0^1 (x^2)^{\,2}\ dx=\pi\int_0^1 x^4\ dx=\pi\left[\frac{x^5}{5}\right]_0^1=\frac{\pi}{5}\ c.u.$

Example 5.19

Calculate the volume generated when we rotate the curves $y^2=8x$, $y=x^2$. From 0<x<2

$V=\pi\int_a^b f^2(x)-g^2(x)\,dx=\pi\int_0^2 8x-x^4\,dx$

$V=\pi\left[\frac{8x^2}{2}-\frac{x^5}{5}\right]_0^2=\pi\left[16-\frac{32}{5}\right]=\frac{48}{5}\pi\ c.u.$



Exercise 5.7

1. Find the area of the region bounded by the curves $y = e^x$, $y = x$ from $x = 0$ to $x = 1$.

2. Find the area enclosed by the curves $y = x^2$ and $y = 2x - x^2$

3. Find the area enclosed by the curves $y = x - 1$ and $y^2 = 2x + 6$

4. Find the area enclosed by the curves $y = \tan^2 x$, $y = \sqrt{x}$

5. Find the area enclosed by the curves $y = \cos x$, $y = x + 2\sin^4 x$

6. Find the area enclosed by the curves $y = \cos \pi x$, $y = 4x^2 - 1$

7. Find the area enclosed by the curves $y = |x|$, $y = x^2 - 2$

8. Suppose that $0 < c < \frac{\pi}{2}$. For what values of c is the area of the region enclosed by the curves $y = \cos x$, $y = \cos(x - c)$, and $x = 0$ equal to the area of the region enclosed by the curves $y = \cos(x - c)$, $x = \pi$ and $y = 0$

9. Find the volume of the solid obtained by rotating about the x axis the area under the curve $y = \sqrt{x}$ from $0 \le x \le 1$

10. Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y axis.

11. Find the volume of the solid obtained by rotating the region bounded by $x = \sqrt{y}$, $x = 0$ and $y = 9$ about the y axis.

12. Area volume lvkns;lns; differentia l

First Order Differential equations

Method of separable variable

Suppose we have the differential equation $g(y)y' = f(x)$. We can simply solve it by:

$$g(y)\frac{dy}{dx} = f(x) \Rightarrow g(y)dy = f(x)dx \Rightarrow \int g(y)\,dy = \int f(x)\,dx + C$$

Example 5.23

Solve the differential equation $y' = \frac{3x^2-1}{3+2y}$

$$\frac{dy}{dx} = \frac{3x^2-1}{3+2y} \Rightarrow (3+2y)dy = (3x^2-1)dx \Rightarrow \int (3+2y)dy = \int (3x^2-1)dx$$

$$\Rightarrow 3y + y^2 = x^3 - x + c$$

Example 5.24

Solve the differential equation $y' = 1 + y^2$

$$\frac{dy}{dx} = 1 + y^2 \Rightarrow \frac{1}{1+y^2}dy = 1\,dx \Rightarrow \int \frac{1}{1+y^2}dy = \int 1\,dx \Rightarrow \arctan y = x + c \Rightarrow y = \tan(x+c)$$

Example 5.25

Consider a mouse population that reproduces at a rate proportional to the current population, with a rate equal to 0.5 births/mouse-month (assuming no owls present).

When owls are present, they eat the mice. Suppose that the owls eat 15 per day (on average).

a) Write a differential equation describing the mouse population in the presence of owls. (Assume that there are 30 days in a month.)
b) Hence solve the differential equation

a) $\frac{dp}{dt} = 0.5p - 450$
b)

$$\frac{dp}{dt} = 0.5(p-900) \Rightarrow \int \frac{1}{p-900}dp = \int 0.5\,dt \Rightarrow \ln|p-900| = \frac{1}{2}t + c$$

$$\Rightarrow p - 900 = e^{0.5t+c} \Rightarrow p = e^{0.5t+c} + 900$$