CONTENT

Textbook and References

For further reading and more detailed information about the course, the following materials are recommended:

Adams, R.A. (2006). *Calculus: A Complete Course* (6^{th} edition): Pearson Education Inc., Toronto, Ontairo, Canada.

Chinang, A.C. & Wainwright, K. (2005). *Fundamental Methods of mathematical Economics* (4^{th}edition): McGraw-Hill/Irwin, New York, NY, USA.

Dowling, E. T. (2001). *Introduction to Mathematical Economics* (3^{rd}): Schaum's outline Series, McGraw-Hill, USA.

Ekanem, O.T. (2000). *Mathematics for Economics and Business* (2^{nd}): Uniben Press, Benin City, Nigeria.

Lial, M.L., Greenwill, R.N, & Ritchey, N.P. (2005).*Calculus with Applications* (8^{th} Edition): Pearson Education, Inc., Boston, Massachusetts, USA.

Sydsaeter, K.,& Hammond, P. (2002). *Essential Mathematics for Economic Analysis:* Pearson education ltd, Edinburgh gate, England.

Assignment File

Assignment files and marking scheme will be made available to you. This file presents you with details of the work you must submit to your tutor for marking. The marks you obtain from these assignments shall form part of your final mark for this course. Additional information on assignments will be found in the assignment file and later in this Course Guide in the section on assessment.

There are three assignments in this course. The three course assignments will cover:
Assignment 1 - All TMAs' question in Units 1 – 4 (Modules 1)
Assignment 2 - All TMAs' question in Units 5 – 8 (Module 2)
Assignment 3 - All TMAs' question in Units 9 – 12 (Module 3)

Presentation Schedule

The presentation schedule included in your course materials gives you the important dates in this year for the completion of tutor-marking assignments and attending tutorials. Remember, you are required to submit all your assignments by the due dates. You are to guide against falling behind in your work.

marked assignments and comments on them before the examination. The final examination covers information from all parts of the course.

Course Marking Scheme

The Table presented below indicates the total marks (100%) allocation.

Assignment	**Marks**
Assignments (Best three assignments out of four that is marked)	30%
Final Examination	70%
Total	**100%**

Course Overview

The Table presented below indicates the units, number of weeks and assignments to be taken by you to successfully complete the course - Mathematics for Economists II (ECO 256).

Units	**Title of Work**	**Week's Activities**	**Assessment (end of unit)**
	Course Guide		
Module 1Calculus			
1	Derivative I	Week 1	Assignment 2
2	Derivative II	Week 1	Assignment 2
3	Integration	Week 2	Assignment 3
4	Differentiation and Integration in Economics	Week 2	Assignment 2
Module 2Optimization			
1	Introduction to Optimization	Week 3	Assignment 2
2	Function of Several Variables	Week 4	Assignment 3
3	Optimization with Constrains	Week 5	Assignment 3
4.	Differentials	Week 6	Assignment 3
	Module 3Linear Algebra		
1	Matrix	Week 7	Assignment 3
2	Matrix Operations	Week 8	Assignment 2
3	Matrix Inversion	Week 9	Assignment 3
4	Economic applications of matrix	Week 10	Assignment 2

MODULE 1: CALCULUS

UNIT 1 DERIVATIVE I

CONTENTS

1.0 INTRODUCTION

Mathematics for Economists I that is, ECO 255 is the foundation course upon which this course (Mathematics for Economists II) will be built. In fact, it is a prerequisite course for ECO 256. This means as a student of ECO 256 you are required to have registered for the ECO 255, read it, understand it and passed it before you are allowed to register for this advanced version, ECO 256.Mathematics for economists I would have introduced you to what you are expected to study in this advanced version at the rounding-off of the course (ECO 255).Meanwhile, in the prerequisite course, that is, Mathematics for economists I, you are expected to learn number system, exponents and roots, equations, logarithms, and a lot more. In this aspect (Mathematics for economists II), you will be studying mathematical economics at a level higher than what you have studied in the foundation class. In this advanced version, you will be discussing basically calculus and its periphery. We shall be treating topics like derivatives, integration, optimization with constraint, functions of variables, linear algebra, and etcetera. In the very first unit of this module, we shall be looking at the concept of derivatives, notations of derivatives, differentiation rules, and many others and its application to economics.It is important to make it clear to students that, mathematical economics is not pure mathematics but an application of mathematical concepts in explaining economic issues. All these we shall be discussing in this version.

in the exogenous variables.All these will be discussed in this section. Differentiation is the process of determining the derivative of a function or a mathematical model. It is about applying some basic rulesto a given function or mathematical model. In discussing the rules of differentiation, authors or writers also use functions such as *g(x)* and *h(x)*, where *g* and *h* are both unspecified functions of *x*.

The variables *f, g* and *h* are used in any mathematical model to indicate the functional relationship between left hand variable(s) and right hand variable(s). That is, any change that is observed in the left hand variable(s)is dependent on the behaviours of the variable(s) on the right hand of the model. For instance in the function $y = f(x)$, the variable *y* is dependent on the variable *x* which is on the right hand side of the model or function. The dependability of the *y* on *x* can be determined using differential calculus called derivative as already explained. To do this effectively, you differentiate the function base on certain guarding rules. These rules are regard as rules for a function of one variable.

Rule 1: The derivative of a Constant

The derivative of a constant function like $p = k$, *or* $f(t) = k$ is zero. Note that *k* is constant because it is a numerate standing alone without any variable attached to it, or multiplying it.

If $p = f(t) = k$, and *k* is a constant, the derivative is

$$\frac{dp}{dt} = \frac{dk}{dt} = 0 \quad \text{or} \quad f'(t) = 0$$

Example 3: Find the derivative of the followings if:

i) $p = f(t) = 12$
ii) $q = 4^2$
iii) $f(x) = \pi$

Solution: i) $\frac{dp}{dt} = 0$ or $f'(t) = 0$

ii) $\frac{dq}{dk} = 0$

iii) $D_x\,[f(x)] = 0$ or $f'(x) = 0$

Rule 2: The Linear Function Rule

A linear function is a function that is to the power of one. If we have a model or a function $g(x) = mx + d$which is a linear function, the derivative will be equal to *m*(the coefficient of *x*). Note, the derivative of any variable to the power of one is all the time

Before now, we have been dealing with functions in which one variable is directly dependent on the other. In a situation where we have an inverse of a function, it indicates the direct oppose of the formal.

Recall that, $y = f(x)$ is a function which represents one on one mapping. That is, any change in variable $y(\Delta y)$ is a result of change in x (Δx). In this instance, the relationship shows in this function is a direct one. However, where $x = f(y)$ or $y = f^{-1}(x)$, this is an inverse function of the just considered function. The derivative of an inverse function as already given is the reciprocal of the derivative of the direct function given.

That is, $\frac{dy}{dx} = \frac{1}{dy/dx}$.

Example 4: $q = 5p + 45$, find the derivative of q^{-1}

Solution: $f'(p) = 5$. But $q^{-1} = \frac{1}{f'(p)}$

$\therefore \quad q^{-1} = \frac{1}{5}$.

Also, find the derivative of the inverse function of q, if $q = p^5 + p$.

Therefore, $\frac{dq}{dp} = 5p^4 + 1 = \frac{1}{5p^4 - 1}$.

Rule 4: The Chain rule

This rule comes up under the discussion of the derivative of a function of a function. It is also known as the composite rule. In this, there is more than one function, and the functions are dependent on one another. This is why it is referred to as a function of a function. A good instance is where y is function of u and u in turn is a function of z so,

$y = f(u)$ and $u = h(z)$, then

$y = f[h(z)]$.

The derivate of y with respect to z is equal to the differentiation of the 1st function with respect to u multiply by the differentiation of the 2nd with respect to z.

$$\therefore \quad \frac{dy}{dz} = \frac{dy}{du}.\frac{du}{dz} = f'(y)h'(z)$$

Example 5: If $y = (x^2 + 5x)^2$ where $y = u^2$ and $u = x^2 + 5x$. Differentiate the function using chain rule.

derivative is vital in the discussion optimization in calculus. See discussion on optimization in module two.

5.0 SUMMARY

In the unit you have leant about the followings:

i) Other rules of differentiations such as product rule, quotient rule, a function of a function rule, and many more that were perceived more complicated than the ones discussed in unit one.

ii) Higher-order derivatives. In this topic, you were exposed to the fact that an equation can be differentiated more than one if circumstance surrounding it demands it.

6.0 TUTOR MARKED ASSIGNMENT

- Compute $\frac{d}{dx}(3x^5 + \frac{x^{60}}{60})$
- Compute $\frac{d}{dx}(\frac{C(x)}{x})$ using quotient rule.
- Find the derivative of the function $q = \sqrt{x^3 - 2}$ using thechain rule.
- If $y = Ak^a$, compute $\frac{d^3y}{dk^3}$.

7.0 REFERENCES AND FURTHER READINGS

Adams, R.A. (2006). *Calculus: A Complete Course (6th edition): Pearson Education Inc., Toronto, Ontairo, Canada.*

Chiang, A.C. & Wainwright, K. (2005). *Fundamental Methods of mathematical Economics* (4thedition): McGraw-Hill/Irwin, New York, NY, USA.

Dowling, E.T. (2001). *Introduction to Mathematical Economics* (3rd): Schaum's outline series, McGraw-Hill, USA.

Ekanem, O.T. (2000). *Mathematics for Economics and Business* (2nd): Uniben Press, Benin City, Nigeria.

Lial, M.L, Greenwill, R.N, & Ritchey, N.P. (2005*). Calculus with Applications (8th Edition): Pearson Education, Inc.,* Boston, Massachusetts, USA.

Sydsaeter, K & Hammond, P. (2002). *Essential Mathematics for Economic Analysis:* Pearson education ltd, Edinburgh gate, England.

In this case, outcome is constant value multiplied by the integral of the function.

Example 3:determine the initial functions of the followings:

i) $\int 3x^2 dx$

ii) $\int 2a^8\, da$

Solution: In applying rule 3 to solve this problem, we have to first identify the constant value and the function. *3*and x^2 are the constant value and functional variable respectively. Therefore:

$\int 3x^2\, dx = 3\int x^2\, dx$, we now apply rule 1 at this point and determine the initial function

$$= 3\left(\frac{1}{3}x^3 + C\right)$$

$$= x^3 + C.$$

ii) Applying the method in solving above (i), we will have: $\int 2a^8 da = 2\int a^8 da$.

Therefore, we have $9\left(\frac{1}{9}a^9 + C\right) = a^9 + C$.

Rule 4: The integral of x^{-1} rule

This is also known as logarithmic rule of integration. The integrand being considered is *1/x* which is same as x^{-1}. You will recall that under the power rule of integration i.e. rule 1, n = -1 is not accepted. However,under this rule, n = -1is welcomed based on the law of indices.

$$\int \frac{1}{x}dx = \ln x + C \qquad (x > 0)$$

The rule stated is applicable where *x* is positive that is $x > 0$. Where it is otherwise, that is *x* is not equal to zero $(x \neq 0)$, which mean that *x* can take a negative value, we use

$$\int \frac{1}{x}dx = \ln|x| + C\ (x \neq 0)$$

Note that as a matter of notation, the integral of $\int \frac{1}{x}dx$ can sometimes be stated thus $\int \frac{dx}{x}$. Also, under this same rule, we have

$$\int \frac{f'(x)}{f(x)}dx = \ln f(x) + C$$,where *f(x)* is positive or

$= \ln|f(x)| + C$, in this case, *f(x)* not positive. (where | *f(x)*| means the absolute value of the *f(x)*).

Example 4: integrate thefunction $\int \frac{6}{x}dx$.

Solution: Note that the function is same as $\int 6\frac{1}{x}dx$. Therefore, applying rule 3 and 4, we will have

$$\int \frac{6}{x}dx = 6\int \frac{1}{x}dx = 6 \ln x + c.$$

Rule 5: The exponential rule

To find the integral of an exponential function using exponential rule, it is advisable to understand derivative of an exponential function. Without much ado, the derivative of exponential e^x is the e^x itself. Thus,

$\int e^x dx = e^x + C$. More generally, (where e = exponential)

$\int e^{ax} dx = \frac{1}{a}e^{ax} + C$. Where $a$is a positive.

Rule 6: The sum or difference of two functions rule

The integral of sum or difference of two or more functions is the sum or difference of the individual integral.

$$\int [f(x) + h(x)]dx = \int f(x)dx + \int h(x)dx$$ Or

$$\int [h(x) - g(x)]dx = \int h(x)dx - \int g(x)dx$$

Example 6:compute the $\int (3x^3 - x + 2)dx$.

Solution: To resolve this problem, we need the combined application of the rules we have discussed thus far. In this case, we will combine rules 3 and 6, therefore,

$$\int (3x^3 - x + 2)dx = 3\int x^3 dx - \int xdx + \int 2dx \quad \rightarrow \quad \text{rule 6}$$

We shall then apply rule 1and have,

Also at $Q = 9, MR = 15 – 2(9) = -3$.

You will recall that the primary aim of any businessman is to make profit. Meanwhile, all other cost incurred would have been adjusted for to arrive at the expected profit. That gain is what is referred to as profit. In economics, the gain/profit that accrued to the business owner is sales made at the prevailing market price (total revenue *TR*) less all expenses incurred (total cost *TC*). We can from this, formulate a function called profit function. The profit function is a combination of the total functions (i.e. total revenue and total cost).Note, that the notation for profit in economics is represented by the sign π, and the function is expressed thus

$$\pi = \pi(Q) = TR - TC \text{ or } PQ - CQ$$

What we have seen in the profit function is that, the amount any businessman would gain or make as profit is dependent on the volume of sales or volume of production *(Q)*. Having known the total profit, we may decide to probe further mathematically the profit made per unit item as the production progress. This brings us to the issue of additional or marginal gain/profit. This (marginal profit) is an expression of the derivative of the profit function. Note that, *P* and *C*in the model $PQ - CQ$ are per unit price and per unit cost respectively.

Example 2: Determine and evaluate the marginal gain function of the profit function $\pi = Q^2 - 16Q + 50$ at $Q = 4$ *and* $Q = 6$.

Solution: Given the profit function, the marginal gain function is an expression of the derivative of the total profit/profit function, thus we have,

$M\pi = \frac{d\pi}{dQ} = 2Q - 16$. Note that the notation $M\pi$ as used in the expression denotes marginal profit/gain. To evaluate the function as required based on the outputs given, substitute the output one after the other and you will arrive at the answers. That is

At $Q = 4, M\pi = 2(4) - 16 = -8$

Also at $Q = 6, M\pi = 2(6) - 16 = -4$.

What we have seen with marginal concept is to estimate the additional cost or gain per unit of any product produced in any firm. However, there is the average concept. This concept estimate the total function to get either the cost of producing a unit of product or the revenue per a unit of good sold. To determine the average function, divide the total cost or total revenue function including the constant term with *Q*. That is, if we are to determine for instance the average revenue (*AR*), we will have:

- Find the consumption function *(C)* if marginal propensity to consume (*MPC*) is *0.6*, and consumption is *70* with income*(Y)*equal tozero.
- Determine the capital function (*K*) if the rate of net investment (*I*) is $20t^{3/5}$, and stock of capital at *t*equal to zero is *50*.

4.0 CONCLUSION

In this unit, what we have done is to practically apply the principles of differentiation and integration to issues as they relate to economics. So, what was seen done in this part of the module, confirms that these two principles are important in economic analysis of certain issues.

5.0 SUMMARY

This unit is fundamentally about the application of differentiation and integration into economics. You will recall that, in unit three of this module we stated that integral calculus is the direct opposite of differential calculus (derivative). All you have learnt in this unit is basically about the application of differentiation and integration in resolving some basic economic problems. We looked at marginal concept. It was discovered that, given total cost/revenue function, the marginal cost/revenue can be ascertained applying the principle of derivative. However, in a reverse manner, the total cost/revenue can be determined from the marginal cost/revenue using integration.

6.0 TUTOR MARKED ASSIGNMENTS

- Find the total revenue function and the per unit price given that marginal revenue (MR) is $40 - 4Q - Q^3$
- If marginal cost (MC) is $24e^{0.5Q}$, and the fixed cost is 50, ascertain the total cost function.
- Find the marginal revenue (MR) function of the demand function $Q = 72 - 4p$.
- Assuming we have a consumption function (*C*) which is $600 + 0.4Yd$, where *Yd* is $Y - T$, and *T* is *200*, find marginal propensity to consume (*MPC*).
  Hints: note that the demand function is same as the price function.

7.0 REFERENCES AND FURTHER READINGS

Adams, R.A. (2006). *Calculus: A Complete Course (6th edition):* Pearson Education Inc., Toronto, Ontairo, Canada.

Chiang, A.C. & Wainwright, K. (2005). *Fundamental Methods of mathematical Economics* (4th edition): McGraw-Hill/Irwin, New York, NY, USA.

Dowling, E .T. (2001). *Introduction to Mathematical Economics* (3rd): Schaum's outline series, McGraw-Hill, USA.

Ekanem, O.T. (2000). *Mathematics for Economics and Business* (2nd): Uniben Press, Benin City, Nigeria.

Lial, M.L, Greenwill, R.N, & Ritchey, N.P (2005). *Calculus with Applications (8th edition): Pearson Education, Inc.*, Boston, Massachusetts, USA.

Sydsaeter, K & Hammond, P. (2002). *Essential Mathematics for Economic Analysis:* Pearson education ltd, Edinburgh gate, England.

MODULE 2: OPTIMIZATION

Unit 1: Introduction to Optimization

Unit 2: Function of Variables

Unit 3: Optimization with Constraints

indicates that from that table-like form, the function is concave when the curve bends downward in the case of relative maximum and convex when it bends upward in the case of minimum.

c) The product of the 2nd-order partialsestimated at the critical point shouldsurpass the product of the crossed partials also calculated at the critical point. This isrequired to rule out an inflection point. In summary:

Relative maximum	Relative minimum
1. $g_x,\ g_y = 0$	1. $g_x, g_y = 0$
2. $g_{xx}, g_{yy} < 0$	2. $g_{xx}, g_{yy} > 0$
3. $g_{xx} \cdot g_{yy} > (g_{xy})^2$	3. $g_{xx} \cdot g_{yy} > (g_{xy})^2$

d) If $g_{xx} \cdot g_{yy} < (g_{xy})^2$, when g_{xx} and g_{yy} have the same signs, the function is at an inflection point; when g_{xx} and g_{yy} have different signs, the function is at an inflection point.

e) If $g_{xx} \cdot g_{yy} = (g_{xy})^2$, the test is inconclusive.

f) If the function is strictly concave (convex) in *x* and *y*, there will be only one maximum (minimum), called an absolute or global maximum (minimum). If the function is simply concave (convex) in *x* and *y* on an interval, the critical point is a relative or local maximum (minimum).

SELF ASSESSMENT EXERCISE 3

Determine i) the 1st -order, ii) the 2nd -order, and iii) the crossed partial derivatives for the equation

$q = 4p^2i^3$

Also, determine the critical values and if the function below is at a relative maximum or minimum, given that,

$q = 2i^3 - p^3 + 147p - 54i + 12.$

4.0 CONCLUSION

We stated earlier at the starting of this unit that, what we studied in unit one was a case of one endogenous parameter being impacted by just a single exogenous parameter too. However, there are extreme cases where a single endogenous parameter is affected by more than one exogenous parameter. That had led us to the study of two-

4.0 Conclusion
5.0 Summary
6.0 Tutor Marked Assignment
7.0 Reference and Further Readings

1.0 INTRODUCTION

In this section, we shall still continue our discussion on the concept of optimization, but with a difference. In the preceding sections (that is, units one and two) of this module, we studied optimization where we determined the relative extrema of an objective function of two or more choice variables. One basic feature of this form of optimization is that all the choice variables were independent of one another. Where this form of undependability is found among the choice variables in optimization, this form is generally referred to as free or unconstrained optimization.

However, in the field of economics, certain problems needed to be optimized. In some cases, variables involved are often required to satisfy certain constraints. For instance, the amounts of different items demanded by a buyer must fulfill the budget constraint of the buyer (in this case, the buyer's income can be seen as a constraint). This unit will introduce us to optimization with constraint.In particular, the method of Lagrange multipliers will be studied to understand how problems in optimization are resolved using it.

2.0 OBJECTIVES

At the end of this unit, the students should be able to:

- Know the difference between unconstrained optimization and the constrained type.
- Understand the module operands pertaining to optimization with constraint.
- Study the usefulness of Lagrange multipliers in optimization discussion.

3.0 MAIN CONTENT

3.1 Constraint: It's Effect

You will recall that, we have mentioned at the starting of this module that, optimization denotes the quest for the best. No matter how bad any economic activity may look, there will always be an optimum point, if we apply the concept of optimization. Basically in economics, every economic activity is all about limiting factors which hinder or constraint the ability or power of any economic agent to do certain things. So, in concept of optimization these limiting factors are duly recognized as constraints.

pragmatism. In economics, indeed human wants are many, but, the resources to meet them are limited (the limiting factors). With what we have studied thus far about Lagrange multiplier, one can submit that, Lagrange multiplier is indeed a vital mathematical tool useful in economics, and it is a measure of marginal impact in applied economics.

5.0 SUMMARY

In sum, we have in this unit, considered the followings:

- We have studied constraint and its effect. That, constraints are the limiting factors that would not allow a consumer or buyer to purchase all he/she desire to get. For instance, the price of an item, the consumer's income, and a lot more constitute constraints.
- Also, we have discussed constrained optimization. What we have studied before this very unit were cases of free or unconstrained optimization, so as to ascertain the relative extrema. However, we have found out that that is not practicable. So, with our discussion on constrained optimization, we have seen how real life economic issues can be resolved.
- Lastly, we studied a method often used by Economists to resolve constrained optimization problems. The method is Lagrange multiplier. It is about subjecting the objective function to constant constraint, and by this mode, the variables are made mutually independent with one another. This method helps Economists to ascertain the marginal effect of a phenomenon.

6.0 TUTOR MARKED ASSIGNMENT

- Optimize $U(x, y) = xy$ subject to $2x + y = 50$
- State the Lagrange function of $f(K, L)$ subject to$rK + wL = I$

7.0 REFERENCES AND FURTHER READINGS

Adams, R.A. (2006). *Calculus: A Complete Course (6th edition):* Pearson Education Inc., Toronto, Ontairo, Canada.

Chiang, A.C. & Wainwright, K. (2005). *Fundamental Methods of mathematical Economics* (4th edition): McGraw-Hill/Irwin, New York, NY, USA.

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Ekanem, O.T. (2000). *Mathematics for Economics and Business* (2nd): Uniben Press, Benin City, Nigeria.

$a = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and $b = [2 \quad 4 \quad 6]$, going by the principle of multiplication, we have matrix *a* times matrix *b*. we therefore have,

$$ab = \begin{bmatrix} 5(2) & 5(4) & 5(6) \\ 2(2) & 2(4) & 2(6) \end{bmatrix} = \begin{bmatrix} 10 & 20 & 30 \\ 4 & 8 & 12 \end{bmatrix}$$

Now, the dimension of outcome *ab* is a *2 by 3* matrix. Remember that matrix *a* is a*2 by 1*, whilematrix *b*is a *1 by 3* matrix.When we compare these matrices dimensions with the outcome's dimension, we can see a sort of resemblance in the dimensions. In matrix multiplication, the dimension/scope/magnitude of the resultant matrix is a combination of the rows *(m)* and columns *(n)* of the individual matrix. Notice that in vector operation, a *1 by n* row vector *a* and an *n by 1* column vector *b*, the product *ab* will produce a matrix of dimension *1 by 1*.

It is important to note that for matrices to be conformable, the number of columns in the *lead matrix* (matrix that comes before the other matrix in any matrix operation, e.g. Matrix*a* as in the example above) must be equal to the number of rows in the *lag matrix* (a matrix that comes after the lead matrix, e.g. matrix *b*).

Supposing we have

a = $[5 \quad 2]$ and b = $\begin{bmatrix} 6 \\ 9 \end{bmatrix}$ by matrix multiplication, the resultant matrix will be

ab = [5(6) + 2(9)]=[48]. The scope of this resultant matrix is *1 by 1* as already indicated. This outcome (that is, the *1 by1*), is a good example of a scalar matrix that has just magnitude only (matrix *ab* is known as singleton).

SELF ASSESSMENT EXERCISE 2

- What are vector matrices?
- Distinguished between a vector matrix and a scalar matrix.

3.4 Laws in Matrix

In matrix, there are laws that guide its operations. These are universally referred to as laws in matrix. These laws are commutative, associative, and distributive in nature. Let us understand how these laws work. It is important to note that both multiplication and addition in matrix are done in line with commutative, associative, and distributive laws. Firstly, with commutative law, matrix addition is ($a + b = b + a$). However, since the addition is merely the summation of the corresponding elements of the matrices involved, the order of their summation is immaterial. Still discussing commutative law, matrix multiplication with vector is not commutative (that is, $ab \neq ba$), just with few

3.0 MAIN CONTENT

3.1 Linear Dependence

In matrix operation, a set of vectors x_1 to $x_n$can be said to be *linearly dependent* if and only if any of the set of vectors can be expressed as a linear combination of the other vectors; if not, the set of vectors will belinearly independent. In other words, if two or more vectors (either row or column) are examined mathematically to be equal to a particular vector matrix, it means that there is linear dependence among the set of vectors. See theexample below:

Supposing we have three vector matrices *x*, *y* and *z*, where one say *z* is a linear combination of *x* and *y*. if:

$x = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$, $y = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, and $z = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$. Matrix *z* is linearly dependent on the remaining two matrices *x* and *y*. I know for sure someone will ask how:

$2x - y = \begin{bmatrix} 8 \\ 16 \end{bmatrix} - \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} = z$. to further verify our result, we can again work on these matrices to have a zero or null matrix which shows that these matrices are linearly dependent. How?

$2x - y - z = 0$ that is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This is a null or zero vector matrix.

Also, we can have row vector matrices say two that can be linearly dependent. Assuming,

$a = [2 \quad 5]$ and $b = [8 \quad 20]$, these vector matrices are linearly dependent because

$$4a = 4[2 \quad 5] = [8 \quad 20] = b.$$

The concept of linear dependence has a simple interpretation. Two vectors *a* and *4a* –one being a multiple of the other –are obviously dependent. When more than two vectors in the 2-space are considered, there emerges this significance conclusion: once we have found two linearly independentvectors in the 2-space (say, *x* and *y*) all the other vectors in the space will be expressed as a linear combination of these (*x* and *y*). Furthermore, by extending, shortening, and reversing the given vectors *x*and *y*and then combining these into various parallelograms, we can generate avast number of new vectors, which will exhaust the set of all 2-vectors. Because of this, any set of three or more 2-vectors (three or more vectors in a 2-space) must be linearly dependent. Two of them can be independent, but then the third must be a linear combination of the first two.

UNIT 3 MATRIX INVERSION

CONTENTS

1.0 INTRODUCTION

In unit two, we discussed basically the mathematical operations (addition, subtraction, and multiplication) involved in matrices. We have seen the intricacies involved in matrix operations. For instance, we saw the case of the special forms of matrix (that is, the identity and null matrices) where their inclusion or exclusion from matrix operation make the product matrix unchanged or changed. Still treating linear algebra, we shall continue our discussion by looking at "matrix inversion".

Matrix inversion in linear algebra is primarily about Determinants and Nonsingularity of matrices. We do know that matrix inversion or inverse matrix is the reciprocal of the matrix in question. And it is (i.e. matrix inversion) required to put mathematical operations right in linear algebra for easy application. For example in economics, some models are required to be formulated by means of matrix inversionbefore any matrix operation can be fully applied.

2.0 OBJECTIVE

At the end of this unit, the students should be able to:

- Define and explain determinants
- Also discuss what nonsingularity means in matrix
- Explain other terms in matrix such as minor, cofactor, adjoint, and many more.
- Lastly, apply Cramer's rule to matrix operations.

SELF ASSESSMENT EXERCISE 2

- Define a cofactor and a minor; state the main difference between the two.

3.3 Cofactor and Adjoint Matrices

In section 3.2, we looked at the concepts of minor and cofactor as they relate to matrix determinant. At the end of the section, we saw how a cofactor was worked from a minor. There is just a tiny line of difference between the two concepts. Still we continue our study on matric inversion; we shall be looking at cofactor and adjoint matrices. A *cofactor matrix* is a matrix in which every entriesx_{ij} is substituted with its cofactor$\left|\mathbf{C}_{ij}\right|$. If it happens that this matrix (the cofactor matrix) is transposed, whereby the rows are transformed to columns, and the columns to rows, the new matrix so formed is known as*adjoint matrix*. See example below:

$$\mathbf{C} = \begin{bmatrix} |\mathbf{C}_{11}| & |\mathbf{C}_{12}| & |\mathbf{C}_{13}| \\ |\mathbf{C}_{21}| & |\mathbf{C}_{22}| & |\mathbf{C}_{23}| \\ |\mathbf{C}_{31}| & |\mathbf{C}_{32}| & |\mathbf{C}_{33}| \end{bmatrix} \qquad \text{Adj } \mathbf{X} = \boldsymbol{C}' = \begin{bmatrix} |\mathbf{C}_{11}| & |\mathbf{C}_{21}| & |\mathbf{C}_{31}| \\ |\mathbf{C}_{12}| & |\mathbf{C}_{22}| & |\mathbf{C}_{32}| \\ |\mathbf{C}_{13}| & |\mathbf{C}_{32}| & |\mathbf{C}_{33}| \end{bmatrix}$$

Example 1: Given matrix X, estimate the cofactor and the adjoint of the matrix.

$$\mathbf{X} = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 2 & 4 \end{bmatrix}$$

Solution:

Hints:

In find the cofactor of a matrix, the allocation of the signs is very important. Meanwhile, see the section on minor and cofactor concerning signs determination and allocation. There is a short cut as regards signs distribution in a cofactor matrix. Once the minors (M_{ij}) are determined using the first row of the matrix, and the signs of these minors are known via the cofactor, needless calculating the minors of the remaining rows. The sign already determined will be used. Supposing we used for the first row + - +, the second row will definitely be, - + -, and so on.

Therefore, replacing the elements with their cofactors according to the laws of cofactors,

$$\mathbf{C} = \begin{bmatrix} \begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 3 & 2 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 14 & -5 & -8 \\ -8 & -2 & 7 \\ -5 & 3 & -2 \end{bmatrix}$$

dimension or order of matrix A should be $A_{mxn,}$ where $m = n$ (that is, the number of rows equal the number of columns in matrix A). We have seen a case of necessary but not sufficient condition for the existence of an inverse matrix. However, there is a necessary and sufficient condition for the existence of an inverse matrixrequires that a matrix must be squared, and that the set of vectors that make up the matrix must be linearly independent.

When a matrix has no inverse, the matrix is referred to as a singular matrix. Recall that, for singular matrix, the determinant is always zero. The formula for deriving the inverse of a matrix is

$$A^{-1} = \frac{1}{|A|} Adj \text{ A}.$$

Note that if the determinant of an inverse matrix is zero, that is $|A| = 0$. It therefore shows that a singular matrix has no inverse.The inverse of matrix A, is represented by A^{-1}, is defined only if A is a square matrix, in which case the inverse is the matrix that satisfies the condition.

$$AA^{-1} = A^{-1} A = I$$

That is, whether A is pre- or postmultiplied by A^{-1}, the product will be the same identity matrix I. This is another exception to the rule that matrix multiplication is not commutative.

About four (4) steps are involved in the estimation of an inverse matrix. These steps are as follows:

- Estimate the determinant of the given matrix, such as matrix $|A|$
- Evaluate the minors and cofactors of the matrix in question
- Based on the outcome of step two above, get the Adjoint matrix from the cofactor matrix.
- Finally, divide the Adjoint matrix so derived in step three by the determinant in step one to arrive at the inverse of the matrix in question (i.e. $\frac{adjA}{|A|} = A^{-1}$).

Let us see one or two worked examples.

Example 1: Assuming $Z = \begin{bmatrix} 4 & 2 & -2 \\ 0 & 2 & 3 \\ 2 & 0 & 6 \end{bmatrix}$ determine Z^{-1}

Solution: What we shall be doing here is to determining the inverse of the matrix by following the steps stated above:

(i) $|Z| = 4\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix}$ $-2\begin{bmatrix} 0 & 3 \\ 2 & 6 \end{bmatrix}$ $-2\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$