2. First Order Dif. Eq.

$$\frac{dy}{dt} = f(t,y) \qquad (1)$$

2.1 Linear Equations; (Method of Integrating Factors)

If in (1), f depends linearly on y, it is called a first order linear eq.

$$\frac{dy}{dt} = -ay + b \quad , \ a, b \text{ are const.}$$

To get the most general form, we replace const. a and b by func. of t, (p, g are given).

$$\frac{dy}{dt} + p(t)y = g(t) \longrightarrow P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

Ex: $(4+t^2)\frac{dy}{dt} + 2ty = 4t.$

$$\Rightarrow \frac{d}{dt}\left[(4+t^2)y\right] = 4t$$

(integrating)

$$\Rightarrow (4+t^2)\,y = \int 4t \, dt$$

$$(4+t^2)y = 2t^2 + C$$

$$y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2} \quad \text{General sol.}$$

We'll consider an eq. of the form

$$\frac{dy}{dt} + ay = g(t) \qquad (2) \quad (a \text{ is a given const.}, g \text{ is given f.})$$

Multiply eq. 2 by $\mu(t)$, which is not determined yet!

$$\mu(t)\frac{dy}{dt} + \mu(t)ay = \mu(t).g(t)$$

Want to choose $\mu(t)$ such that the left hand side is a total der. of $\mu(t).y$. But $\frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y$

So, $\frac{d\mu(t)}{dt} = \mu(t).a \iff \frac{d\mu(t)}{dt} \Big/ \mu(t) = a.$

$$\frac{d\mu(t)}{dt} = \mu(t).a$$

$$\int \frac{\frac{d\mu(t)}{dt}}{\mu(t)}.dt = \int a.dt$$

$$\Rightarrow \ln(\mu(t)) = at + c_1$$

$$\boxed{\mu(t) = c.e^{at}} \quad (e^{at}.e^{c} = e^{at+c})$$

We will choose $\mu(t) = e^{at}.$

Note 3: Usually ~~it is better to leave~~ the sol. in implicit form rather than trying to solve it explicitly. Thus for non-linear eq. "Solve the following eq" means find the solution explicitly if possible, but otherwise implicitly.

Homogenous Equations.

If the right side can be express as a func. of the ratio y/x only, then the eq. is called Homogenous. Such eq can always be transport into seperate forms by a change of the dependent var.

Ex $\dfrac{dy}{dx} = \dfrac{y-4x}{x-y} \Rightarrow \dfrac{dy}{dx} = \dfrac{\frac{y}{x}-4}{1-\frac{y}{x}} \Rightarrow$ Homogenous Eq.

Let $v = y/x \Rightarrow y = x \cdot v(x)$

$$\frac{dy}{dx} = \frac{d}{dx}(x \cdot v(x)) = v(x) + x\frac{dv}{dx}$$

$$v + x \cdot \frac{dv}{dx} = \frac{v-4}{1-v} \Rightarrow \int \frac{1-v}{v^2-4}\,dv = \int \frac{dx}{x} \quad \text{Seperable}$$

$$x \cdot \frac{dv}{dx} = \frac{v^2-4}{1-v}$$

$$\int \frac{1-v}{v^2-4}\,dv = \int \frac{dx}{x} \Rightarrow \int \frac{-1}{4(v-2)}\,dv + \int \frac{-3\,dv}{4(v+2)} = \ln|x|$$

$$\Rightarrow \frac{-1}{4}\ln|v-2| + \frac{-3}{4}\ln|v+2| = \ln|x| + c$$

$$\Rightarrow -\ln|v-2||v+2|^3 = \ln|x|^4 + c$$

$$\Rightarrow \ln|x|^4 + \ln|v-2||v+2|^3 = \ln c$$

$$x^4|v-2||v+2|^3 = c$$

$$x^4\left|\tfrac{y}{x}-2\right|\left|\tfrac{y}{x}+2\right|^3 = c.$$

$$\Rightarrow |y-2x||y+2x|^3 = c$$

General Sol. in implicit form.

Lec 5
Oct. 5

C3. Second Order Linear eq.

3.1 Homogeneus eq. with const. coefficents:

A second order ODE has the form

$$\boxed{\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)} \quad (1)$$

where f is given func. Eq (1) is linear if f has the form

$$\boxed{f\left(t, y, \frac{dy}{dt}\right) = g(t) - p(t)\frac{dy}{dt} - q(t)y \quad (2)}$$

that is, if f is linear in y and $\frac{dy}{dt}$. In (2), g, p, q are given func. that do not depend on y

We usually write

$$\boxed{y'' + p(t)y' + q(t)y = g(t)} \quad (3)$$

We might also see

$$\boxed{P(t)y'' + Q(t)y' + R(t)y = G(t)} \quad (4)$$

which can be turned into the form of (3) if $P(t) \neq 0$

$$p(t) = \frac{Q(t)}{P(t)}, \quad q(t) = \frac{R(t)}{P(t)}, \quad g(t) = \frac{G(t)}{P(t)} \quad (5)$$

↳ If Eq. (1) is not of the form 3 or 4, it is called nonlinear.

↳ An initial-value problem consists of a dif eq. like 1, 3, 4 and a pair of initial cond.

$$y(t_0) = y_0, \quad y'(t_0) = \tilde{y}_0$$

where $y_0, \tilde{y}_0$ are given. We need two initial cond. cause two integrations are required. And each integ. introduces an arbitrary cons.

The next question is whether all sol. are included in this infinite family. We first look at whether c_1 and c_2 can be chosen to satisfy the ic; $y(t_0)=y_0$, $y'(t_0)=\tilde{y}_0$

$$(2)\ \begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = \tilde{y}_0 \end{cases} \Longrightarrow$$ The det. of the system is;

$$W(y_1,y_2)(t_0)\ \boxed{W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} = y_1(t_0)\,y_2'(t_0) - y_2(t_0)\,y_1'(t_0)$$

If $W \neq 0$, the the system 2 has a unique sol. (c_1, c_2) determined regardless of y_0 and $\tilde{y}_0$;

$$(3)\qquad c_1 = \frac{y_0\,y_2'(t_0) - \tilde{y}_0\,y_2(t_0)}{y_1(t_0)\,y_2'(t_0) - y_1'(t_0)\,y_2(t_0)}, \quad c_2 = \frac{-y_0\,y_1'(t_0) + \tilde{y}_0\,y_1(t_0)}{y_1(t_0)\,y_2'(t_0) - y_1'(t_0)\,y_2(t_0)}$$

If $W=0$, then system (2) has no sol. on this y_0 and $\tilde{y}_0$ have values that also make the numerators in (3) equal to 0

So if $W=0$, there are many initial cond. that cannot be satisfied no matter how c_1 and c_2 are chosen

The det. W is called Wronskian of sol. y_1 and y_2. We denote $W(y_1, y_2)(t_0)$

Theorem 3.2.3 Suppose that y_1 and y_2 are two sol. of

$$L[y] = y'' + p(t)y' + q(t)y = 0, \qquad (4)$$

and that the initial cond. is given by

$$y(t_0) = y_0 \quad y'(t_0) = \tilde{y}_0$$ are assigned.

Then it is always pos. to choose const. c_1 and c_2 so that $\boxed{y = c_1 y_1(t) + c_2 y_2(t)}$ satisfies (4) and the above initial cond. if and only if

$W = y_1 y_2' - y_1' y_2$ is not zero at t_0.

Theorem 3.2.7 (Abel's Theorem)

If y_1 and y_2 are sol. of dif. eq.

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where p and q are cont. on an open interval I, then the $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \cdot e^{-\int p(t)\,dt}$$ c is depending on y_1 and y_2

Further; W is either zero $\forall t$ in I (if $c=0$) or else is never 0 in I

Proof: $y_1'' + p(t)y_1' + q(t)y_1 = 0 \rightarrow$ multiply y by $-y_2$

$y_2'' + p(t)y_2' + q(t)y_2 = 0 \rightarrow$ multiply y by y_1

$$\Rightarrow (y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) = 0.$$ So we can

Let $W(t) = W(y_1, y_2)(t)$; $W' = y_1 y_2'' - y_1'' y_2$

write (3) as

$$W' + p(t)W = 0,$$ which is first order linear eq. (also seperable)

$$\Rightarrow W(t) = c \exp\left(-\int p(t)\,dt\right)$$ W is not zero unless $c = 0$.

Since $\exp$ is never zero $W \neq 0$ for $\forall t$.

otherwise if $\underline{c=0} \Rightarrow \underline{W=0}$

Ex: Previously, we showed $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are sol. of

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0.$$

$$W(y_1, y_2)(t) = -\frac{3}{2} t^{-3/2}.$$

Rewriting the dif. eq.

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0 \Rightarrow p(t) = 3/2t$$

$$\Rightarrow W(y_1, y_2)(t) = c \cdot e^{-\int \frac{3}{2t}\,dt} = c\, t^{-3/2}$$

This gives the Wronskian of any pair of sol. For the particular sol. c is given $-3/2$.

Ex: Solve the dif. eq.

$$y'' + 4y' + 4y = 0$$

$$y = v(t)e^{-2t} \Rightarrow v'e^{-2t} - 2ve^{-2t} = y'$$

$$y'' = v''e^{-2t} - 4v'e^{-2t} + 4ve^{-2t}$$

$$r^2 + 4r + 4 = 0$$

$$r_1 = r_2 = -2 \Rightarrow y_1(t) = e^{-2t}$$

$$\left[v''(t) - \cancel{4v'(t)} + \cancel{4v'(t)} + \cancel{4v(t)} - \cancel{8v(t)} + \cancel{4v(t)}\right] e^{-2t} = 0$$

$$v''(t) \cdot e^{-2t} = 0.$$

$$v''(t) = 0$$

$$v'(t) = c_2$$

$$\boxed{v(t) = c_1 t + c_2.}$$

$$y = c_1 \underbrace{te^{-2t}}_{y_2} + c_2 \underbrace{e^{-2t}}_{y_1}$$

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t} \neq 0$$

Lec 7.
Oct, 12.

Reduction of Order

Suppose that we know one solution $y_1(t)$, not everywhere zero, of

$$y'' + p(t)y' + q(t)y = 0 \quad (1)$$

To find a second sol., let $y = v(t)y_1(t)$

$$y' = v'(t)y_1(t) + v(t)y_1'(t)$$

$$y'' = v''(t)y_1(t) + \underbrace{v'(t)y_1'(t) + v'(t)y_1'(t)}_{2v'(t)y_1'(t)} + v(t)y_1''(t)$$

Sub. into (1)

$$y_1 v'' + (2y_1' + py_1)v'(t) + \underbrace{(y_1'' + py_1' + qy_1)}_{0 \text{ cause } y_1(t) \text{ is a sol.}} v(t) = 0$$

$\Rightarrow y_1 v'' + (2y_1' + py_1)v' = 0$ which is a first order eq. for v'.

Once v' is found, v is found by integration and finally y is determined.

This procedure is called the reduction of order.

Remark: Thr. 3.5.2 states that, to solve the nonhomogeneous eq. 1. we must do three things

1. Find the gen. sol. of the corresponding homogeneous eq. $c_1y_1+c_2y_2$, which is usually called the complementary sol. and denoted by $y_c(t)$.
2. Find a specific sol. $Y(t)$ to the nonhomogeneous eq. often it's referred as particular sol.
3. Form the sum of the func. in Step 1 and Step 2.

The Method Of Undetermined Coefficients:

To find particular sol. of $ay''+by'+cy=g(t)$, we use the following table:

$g_i(t)$	$Y_i(t)$
$P_n(t)=a_0t^n+a_1t^{n-1}+\dots+a_n$	$t^s(A_0t^n+A_1t^{n-1}+\dots+A_n)$
$P_n(t)e^{\alpha t}$	$t^s(A_0t^n+A_1t^{n-1}+\dots+A_n)e^{\alpha t}$
$P_n(t)e^{\alpha t}\begin{cases}\sin\beta t\\ \cos\beta t\end{cases}$	$t^s[(A_0t^n+\dots+A_n)e^{\alpha t}\cos\beta t+(B_0t^n+\dots+B_n)e^{\alpha t}\sin\beta t]$

Here s is the smallest nonnegative integer ($s=0,1$ or 2) that will ensure that no term in $Y_i(t)$ is a sol. of the corresponding homogeneous eq.

1. Find the gen. sol. of the corresponding homogeneous eq.
2. Make sure $g(t)$ is one of the following: exponential, sines, cosines, polynomials or sums or products of such func.
3. If $g(t)=g_1(t)+g_2(t)+\dots+g_n(t)$, form n subproblems for each $g_i(t)$
4. For the i-th subproblem assume $Y_i(t)$ according to the table
5. Find a particular sol. $Y_i(t)$ for each subproblem, then sum $Y_1(t)+\dots+Y_n(t)$ as the particular sol. of the full non homogeneous p.

Finally, we need to impose the cond. that Y is a sol. to (1)
By. dif $Y^{(n-1)}$ we obtain

$$\boxed{Y^{(n)} = (u_1 y_1^{(n)} + \dots + u_n y_n^{(n)}) + (u_1' y_1^{(n)} + \dots + u_n' y_n^{(n)})} \quad (9)$$

To satisfy the eq., Sub. Y and its derivatives in (1) from (3), (6), (8), (9). Then we group terms involving $y_1, \dots, y_n$ and their der. Then most terms because each $y_1, \dots, y_n$ is a sol. to the homogeneous eq. so $L[y] = 0$

The remaining terms

$$\boxed{u_1' y_1^{(n-1)} + \dots + u_n' y_n^{(n-1)} = g} \quad (10)$$

So we have n algebraic eq. for $u_1', \dots, u_n'$

$$\begin{cases} y_1 u_1' + \dots + y_n u_n' = 0 \\ y_1' u_1' + y_2' u_2' + \dots + y_n' u_n' = 0 \\ y_1'' u_1' + \dots + y_n'' u_n' = 0 \\ \vdots \\ y_1^{(n-1)} u_1' + \dots + y_n^{(n-1)} = g(t) \end{cases} \quad (11)$$

By solving (11), we find $u_1', \dots, u_n'$. Integrating we find $u_1, \dots, u_n$.

This pos. since $W(y_1, \dots, y_n) \neq 0$

Using Cramer's rule

$$\boxed{u_m'(t) = \frac{g(t) W_m(t)}{W(t)}, \quad m = 1, 2, \dots, n} \quad (12)$$

Where $W(t) = W(y_1, \dots, y_n)(t)$ and W_m is the det. obtained from W by replacing m-th column by the col. $(0, 0, 0, \dots, 1)$

Hence a particular sol. of (1) is given by

$$\boxed{Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)} \, ds}$$

Abel's identity

$$\boxed{W(t) = W(y_1, \dots, y_n)(t) = Ce}$$

Ex: Consider the func. given by $f(t) = \begin{cases} 2, & 0 \le t < 4 \\ 5, & 4 \le t < 7 \\ -1, & 7 \le t < 9 \\ 1, & 9 \le t \end{cases}$

Express $f(t)$ in terms of $u_c(t)$.

We start with $f_1(t) = 2$ which agrees with $f(t)$.

To produce the jump of 3 units of at $t=4$, we add $3u_4(t)$ to $f_1(t)$. So $f_2(t) = 2 + 3u_4(t)$.

The negative jump of 6 units at $t=7$, we add $-6u_7(t)$

$$f_3(t) = 2 + 3u_4(t) - 6u_7(t).$$

Finally we add $2u_9(t)$

$$f(t) = 2 + 3u_4(t) - 6u_7(t) + 2u_9(t).$$

The Laplace transform of u_c for $c \geq 0$ is given by.

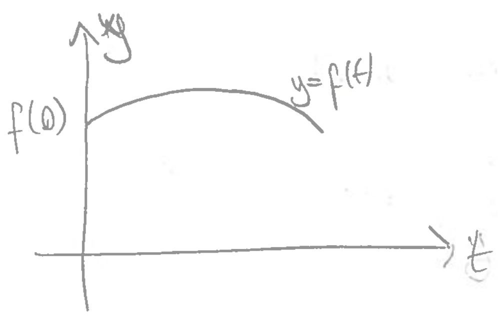
$$\boxed{\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st} dt = \frac{e^{-cs}}{s}, \quad s > 0.}$$

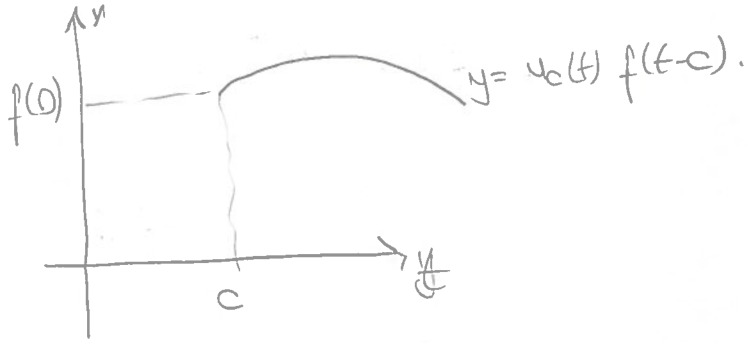
For a given func. f defined for $t \geq 0$, we often want to consider the func. g given by

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c. \end{cases}$$

which gives a translation of f a dist. c in the positive t direction.

In terms of $u_c(t)$, we can write $g(t) = u_c(t) \cdot f(t-c)$





7.4 Basic Theory of systems of first order linear eq.

The gen. theory of a system of n first order linear eq.

$$\begin{cases} x_1' = p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t) \\ \vdots \\ x_n' = p_{n1}(t)x_n + \cdots + p_{nn}(t)x_n + g_n(t) \end{cases} \quad (1)$$

closely parallels that of a single linear eq of n-th order. That is, we consider
We write system (1) in matrix notation
$x_1 = \phi(t), \ldots, x_n = \phi_n(t)$ to be components of a vector $\vec{x} = \phi(t)$.
$g_1(t), \ldots, g_n(t)$ are components of a vector $\vec{g}(t)$;
$p_{11}(t), \ldots, p_{nn}(t)$ are elements of an $n\times n$ matrix $P(t)$.

Then system (1) is equivalent to

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t) \quad (2)$$

Ex: $t^3y'' - t^2y' + (t-1)y = 0 \Rightarrow \vec{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$ [illegible] $\begin{pmatrix} y' \\ y'' \end{pmatrix}$

$y'' - \frac{1}{t}y' + \left(\frac{t-1}{t^3}\right)y = 0 \Rightarrow \vec{x}' = \underbrace{\begin{pmatrix} [illegible] & [illegible] \\ \frac{1-t}{t^3} & \frac{1}{t} \end{pmatrix}}_{P(t).}\vec{x} \Rightarrow \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ \frac{1-t}{t^3}y + \frac{1}{t}y' \end{pmatrix}$

A vector $\vec{x} = \phi(t)$ is said to be a sol. of (2). If its components satisfy (1).

We assume P and $\vec{g}$ are cont. on $\alpha < t < \beta$ which is sufficient to guarantee existence of sol.

First we consider the homogeneous eq., i.e $\vec{g}(t) = 0$;

$$\vec{x}' = P(t)\vec{x} \quad (3)$$

We use the notation

$$\vec{x}^{(1)}(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \ldots, \vec{x}^{(k)}(t) = \begin{pmatrix} x_{1k}(t) \\ x_{2k}(t) \\ \vdots \\ x_{nk}(t) \end{pmatrix}, \ldots \quad (4)$$

To specify sol. of system (3), note that ... $x_i^{(j)}(t)$ refers to the ith component of the j-th sol.

Thr 7.4.1: If the vector functions $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are sol. of system (3) then the linear comb. $c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$ is also a sol. for any c_1 and c_2.

(Principle of Superposition).

By repeated applications of 7.4.1 we can conclude that if $\vec{x}^{(1)}, \vec{x}^{(2)}, \ldots, \vec{x}^{(n)}$ are solutions of (3), then

$$\vec{x} = c_1\vec{x}^{(1)}(t) + \cdots + c_n\vec{x}^{(n)}(t)$$

is also a sol. for any constants $c_1, \ldots, c_n$. So ever finite linear comb of sol. of (3) is also a sol.

The question is whether all sol. can be found in this way. Let $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ be n sol. of (3). And consider the matrix $X(t)$ whose cols. are $\vec{x}^{(1)}(t), \ldots, \vec{x}^{(n)}(t)$;

$$X(t) = \begin{bmatrix} x_{11}(t), & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{bmatrix}$$

The cols. of $X(t)$ are linearly independent for a given value of t if and only if $\det X \neq 0$ for that t. This det called Wronskian of the n sol's. $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ is also denoted by $W[\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}]$;

$$W[\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}](t) = \det X(t)$$

The sol. $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are then linearly independent at a point if and only if their W is not zero there.

Thr 7.4.2: If the vector functions $\vec{x}^{(1)}, \ldots, \vec{x}^{(n)}$ are linearly independent sol. of (3) for each point in the interval $\alpha < t < \beta$, then each sol. $\vec{x} = \vec{\phi}(t)$ of system (3), can be expressed as a linear comb.

$$\vec{\phi}(t) = c_1\vec{x}^{(1)}(t) + \cdots + c_n\vec{x}^{(n)}(t) \qquad (6)$$

Note that according to Thr 7.4.1, all expressions of the form (6) are sol. of (3), while by Thr 7.4.2 all sol. of (3) can be written in the form (6). If constants $c_1, \dots, c_n$ are arbitrary, (6) is called the gen. sol.

Any set of sol. $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ of (3) that are linearly independent at each point in $\alpha < t < \beta$ is said to be a fundamental set of sol. for that interval.

Thr 7.4.3. If $\vec{x}^{(1)}, \dots, \vec{x}^{(n)}$ are sol. of (3) on $\alpha < t < \beta$, then in this interval $W[\vec{x}^{(1)}, \dots, \vec{x}^{(n)}]$ either is identically zero, or else never vanishes.

7.5 Homogeneous Linear systems with Constant Coefficients.

We'll consider

$$\vec{x}' = A\vec{x} \qquad (1)$$

where A is a constant $n \times n$ matrix. We'll assume that all elements of A are real numbers.

We'll seek sol. of the form

$$\vec{x} = \vec{\xi} e^{rt}$$

where the exponent r and the vector $\vec{\xi}$ are to be det.

Substituting into (1) we get

$$r\vec{\xi}\cancel{e^{rt}} = A\vec{\xi}\cancel{e^{rt}} \Rightarrow r\vec{\xi} = A\vec{\xi}$$

$$(A - rI)\vec{\xi} = \vec{0} \qquad (2)$$

where I is the identity matrix ($n \times n$). To solve (1), we must solve the system of algebraic eq. (2). The vector $\vec{x}$ is a sol. of (1) provided r is an eigenvalue and $\vec{\xi}$ is an associated eigenvector of the coefficent matrix A.

If the n eigenvalues are real and different, then associated with each eigenvalue r_i is a real eigenvector $\vec{\xi}^{(i)}$ and the n eigenvectors $\vec{\xi}^{(1)}, \dots, \vec{\xi}^{(n)}$ are linearly independent. The corresponding sol. of system (1)

$$\vec{x}^{(1)}(t) = \vec{\xi}^{(1)} e^{r_1 t}, \dots, \vec{x}^{(n)}(t) = \vec{\xi}^{(n)} e^{r_n t}$$

To show these form a fundamental set of sol., we evaluate their W

Ex Find the gen. sol. to the system:

$$\vec{x}^{(1)} = \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}}_{A} \vec{x}$$

We need to solve $(A - rI)\vec{\xi} = \vec{0}$

$$\Rightarrow \begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$$ we must have $\det(A - rI) = 0$.

$$\det(A - rI) = -r^3 + 3r + 2 = 0.$$

$$\Downarrow$$

$$\underbrace{r_1 = 2}_{\text{Simple eigenv.}}, \quad \underbrace{r_2 = -1, \ r_3 = -1}_{\text{Double eigenvalue.}}$$

To find the eigenvector $\vec{\xi}^{(1)}$ r_1;

sub. $r_1 = 2$.

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $r = -1$;

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \xi_1 + \xi_2 + \xi_3 = 0.$$

Two of ξ_1, ξ_2, ξ_3 can be chosen arbitrarily and the third is determined.

$$\xi_1 = c_1, \quad \xi_2 = c_2, \quad \xi_3 = -c_1 - c_2 \Rightarrow \vec{\xi}^{(2)} = \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

For ex. by choosing $c_1 = 1$, $c_2 = 0$ we obtain the eigenvector $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Any non zero multiple of c_2 but a second independent eigenvector found by making another choice of c_1 and c_2; say $c_1 = 0$, $c_2 = 1$.

$\vec{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ which is linearly independent of $\vec{\xi}^{(2)}$

So a fundamental set of sol. is the following

$$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}, \quad \vec{x}^{(2)}(t) = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t}, \quad \vec{x}^{(3)}(t) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

Gen. Sol. is $\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$.

Supposing the string is disturbed from its equilibrium with 0 velocity, we must have

$$u(x,0) = f(x) \qquad u_t(x,0) = 0 \ ; \quad 0 < x < l \qquad (5)$$

We use the method of sep. of var.

$$u(x,t) = X(x)\,T(t)$$

Taking the der. and sub. into (4) $\dfrac{X''}{X} = \dfrac{1}{\alpha^2}\dfrac{T''}{T} = -\lambda, \lambda > 0$

$$\begin{cases} X'' + \lambda X = 0 \\ T'' + \alpha^2 \lambda T = 0 \end{cases} \qquad (6)$$

By the boundary cond. we get $\boxed{X(0) = 0, \ X(l) = 0}$

From the second initial cond. gives $X(x)T'(0) = 0 \Rightarrow \boxed{T'(0) = 0}$

So, $X'' + \lambda X = 0$, $X(0) = 0$, $X(l) = 0$ gives.

$$\lambda_n = \frac{n^2\pi^2}{l^2}, \quad X_n(x) \sim \sin\left(\frac{n\pi x}{l}\right), \ n = 1, 2, \ldots$$

$$T'' + \alpha^2 \underset{\uparrow \ \frac{n^2\pi^2}{l^2}}{\lambda} T = 0 \Rightarrow T'' + \frac{n^2\pi^2\alpha^2}{l^2} T = 0$$

$$T(t) = k_1 \cos\left(\frac{n\pi\alpha t}{l}\right) + k_2 \sin\left(\frac{n\pi\alpha t}{l}\right)$$

So using $T'(0) = 0 \Rightarrow k_2 = 0$.

$$\boxed{T_n(t) \sim \cos\left(\frac{n\pi\alpha t}{l}\right)}$$

Thus, $u_n(x,t) = \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right), \ n = 1, 2, 3, \ldots$

To satisfy the remaining initial cond. $u(x,0) = f(x)$, we'll c

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi\alpha t}{l}\right) \ (\ldots$$

The cond. $u(x,0) = f(x)$ requires.

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = f(x).$$

So coefficients c_n must be coefficients of Fourier Sine ser.

$$c_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx. \qquad (8)$$

Case i) $\lambda > 0$ $\quad \lambda = \mu^2 > 0$

$$X'' + \mu^2 X = 0 \to r^2 + \mu^2 = 0 \to r_{1,2} = \mp i\mu$$

$X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$ the general solution

$X(0) = 0 \to c_1 = 0$

$\to X(x) = c_2 \sin(\mu x)$

$X'(x) = c_2 \mu \cos(\mu x)$

$X'(\pi) = 0 \to c_2 \cdot \mu \cdot \cos(\mu \pi) = 0$

Since $c_2 \neq 0$, $\mu \neq 0$ we get $\cos(\mu \pi) = 0$

$\mu = \left(\frac{2n-1}{2}\right)$ $(n = 1, 2, ..)$

$\cos\left[(2n-1/2)\frac{\pi}{2}\right] = 0$

$\lambda_n = \mu_n^2 = \left(\frac{2n-1}{2}\right)^2$ $\quad n = 1, 2 ..$ eigenvalues

$X_n(x) = \sin(\mu_n x) = \sin\left[\left(\frac{2n-1}{2}\right)x\right]$ eigenfunctions

Case ii) $\lambda = 0$

$X'' = 0 \to X(x) = c_1 + c_2 x$

$X(0) = 0 \to c_1 = 0$

$X(x) = c_2 . x$

$X'(x) = c_2 \to X'(\pi) = 0 \to c_2 = 0$

For $\lambda = 0$ we have the trivial solution $X(x) = 0$

$\lambda = \lambda_n = \mu_n^2 \to T'' + \mu_n^2 T = 0 \to r^2 + \mu_n^2 = 0$

$r_{1,2} = \mp i\mu_n \to T(t) = k_1 \cos(\mu_n t) + k_2 \sin(\mu_n t)$

$T'(t) = -k_1 \mu_n \sin(\mu_n t) + k_2 \mu_n \cos(\mu_n t)$

$T'(0) = 0 \to k_2 \mu_n = 0 \to k_2 = 0$ $\quad k_1$ arb.

Take $k_1 = 1$, $T_n(t) = \cos(\mu_n t) = \cos\left[\left(\frac{2n-1}{2}\right)t\right]$

$u_n(x,t) = T_n(t)\, X_n(x)$

$= \cos(\mu_n t) \sin(\mu_n x)$

$= \cos\left[\left(\frac{2n-1}{2}\right)t\right] . \sin\left[\left(\frac{2n-1}{2}\right)x\right]$ fundamental solutions

The series solution is:

$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \cos(\mu_n t) \sin(\mu_n x)$

$c_n = ?$

$u(x,0) = \sin\left(\frac{x}{2}\right) - 3\sin\left(\frac{3x}{2}\right)$

$\sum_{n=1}^{\infty} c_n \sin\left(\left[\frac{2n-1}{2}\right]x\right) = \sin\left(\frac{x}{2}\right) - 3\sin\left(\frac{3x}{2}\right)$

$\to c_1 = 1$, $c_2 = -3$, $c_n = 0$ when $n \neq 1, 2$

$u(x,t) = \cos\left(\frac{t}{2}\right)\sin\left(\frac{x}{2}\right) - 3\cos\left(\frac{3t}{2}\right)\sin\left(\frac{3x}{2}\right)$