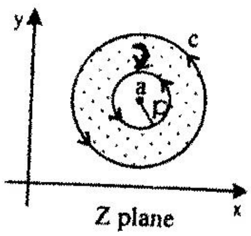
Given f(z) is analytic inside and on c. Now $\frac{f(z)}{z-a}$ is analytic inside and on c except at z =a

Draw a circle $c_1 : |z-a| = r$ with center at z = a and radius r units such that c_1 lies entirely inside c.



Now $\phi(z) = \frac{f(z)}{z-a}$ is analytic in the region enclosed between c and c_1

$$\therefore \int_c \phi(z)dz = \int_{c_1} \phi(z)dz$$

$$\text{(ie) } \therefore \int_c \frac{f(z)}{z-a}dz = \int_{c_1} \frac{f(z)}{z-a}dz$$

On c_1 any point z is given by $z = a + re^{i\theta}, 0 \le \theta \le 2\pi \therefore dz = ire^{i\theta}d\theta$

$$\therefore \int_c \frac{f(z)}{z-a}dz = \int_0^{2\pi} \frac{f(a+re^{i\theta})}{re^{i\theta}} ire^{i\theta}d\theta$$

$$= 2\pi i f(a)$$

$$\therefore f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

CAUCHY'S INTEGRAL FORMULA FOR DERIVATIVES OF AN ANALYTIC FUNCTION:

Statement:

If f(z) is analytic inside and on a simple closed curve c and z = a is any interior point of the region R enclosed by c, then

$$f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_c \frac{f(z)}{(z-a)^3}dz$$

In general,

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + \ldots + \int_{c_n} f(z)dz \qquad \ldots\ldots(1)$$

Now $z_1, z_2, z_3 \ldots z_n$ are the singular points of $f(z)$.

$\therefore$ **Res** $f(z)_{z=z_i}$ = **the coefficient of** $\frac{1}{z-z_i}$ **in the Laurent's series of**

$f(z)$ **about** $z = z_i$ **(by definition of residues)**

$$= b_1 = \frac{1}{2\pi i}\int_{c_1}\frac{f(z)}{(z-z_1)^{1-n}}dz$$

Since $\left(b_n = \frac{1}{2\pi i}\int_{c_1}\frac{f(z)}{(z-z_1)^{z-n}}dz \right)$

$$= \frac{1}{2\pi i}\int_{c_1}\frac{f(z)}{(z-z_1)^0}dz$$

$$= \frac{1}{2\pi i}\int_{c_1} f(z)dz$$

$$\Rightarrow \int_{c_1} f(z)dz = 2\pi i\left|\text{Res} f(z)\right|_{z=z_i} \qquad \ldots\ldots(2)$$

From (1) and (2)

$$\int_{c_1} f(z)dz = 2\pi\left|\text{Res} f(z)\right|_{z=z_1} + 2\pi\left|\text{Res} f(z)\right|_{z=z_2} + 2\pi\left|\text{Res} f(z)\right|_{z=z_n}$$

$$= 2\pi\left|\text{Res} f(z)\right|_{z=z_1} + \left|\text{Res} f(z)\right|_{z=z_2} + \ldots + \left|\text{Res} f(z)\right|_{z=z_n}$$

$= 2\pi i$ **{Sum of residues of** $f(z)$ **at** $z = z_1, z_2, z_3 \ldots z_n\}$

Example 1 Find the residue of f(z) $= \frac{z+2}{(z-2)(z+1)^2}$ **about each singularity.**

Solution: The poles of f (z) are given by

$$(z-2) = 0, z+1 = 0$$

$$\Rightarrow z = 2, z = -1$$

$\therefore$ **The poles of f (z) are z = 2 is a simple poles and z = -1 is a pole of order 2.**

$$\therefore \left[\text{Res} f(z)\right]_{z=2} = \lim_{z\to 2}(z-2)f(z)$$

The poles of f (z) are $5z^2 + 26iz - 5$

$$Z = \frac{-26 \pm \sqrt{(26i)^2 + 100}}{10}$$

$$= \frac{-26 \pm 24}{10} = \frac{-i}{5}, -5i$$

The pole z = $\frac{-i}{5}$ **lies inside the circle** $|z| = 1$ **and z = -5i lies outside the circle** $|z| = 1$

Now $\therefore [\operatorname{Res} f(z)]_{z=-i/5} = \lim_{z \to -i/5} \left(z + \frac{i}{5}\right) f(z)$

$$= \lim_{z \to -i/5} \left(z + \frac{i}{5}\right) \frac{1}{5\left(z + \frac{i}{5}\right)(z + 5i)}$$

$$= \frac{1}{5\left(\frac{-i}{5} + 5i\right)} = \frac{1}{24i}$$

$\therefore$ **By Cauchy's Residue theorem**

$$\int_c f(z)dz = 2\pi i \sum R$$

$$= 2\pi i \frac{1}{24i} = \frac{\pi}{12}$$

(1) becomes

$$I = \frac{2\pi}{12} = \frac{\pi}{6}$$

Example 2: Evaluate $\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta, a > b > 0$ **using contour integration.**

Solution: Let $I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta,$

$$= \int_0^{2\pi} \frac{1 - \cos 2\theta}{2a + 2b\cos\theta} d\theta$$

We can write $\cos 2\theta$ **= Real part of** $e^{2i\theta}, \because e^{2i\theta} = \cos 2\theta + i\sin 2\theta$

$$\therefore I = R.P \int_0^{2\pi} \frac{1 - e^{2i\theta}}{2a + 2b\cos\theta} d\theta$$

Put $z = e^{i\theta}$

(1) becomes

$$\mathbf{I} = R.P\frac{1}{i}\int_0^{2\pi} f(z)dz$$

$$= R.P2\pi\left(\frac{a-\sqrt{a^2-b^2}}{b^2}\right)$$

$$= \frac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$$

Type II Integration around semi-circular contour

Consider the integral

Improper integrals of the form $\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx$ **,where P(x) and Q(x)**

Are polynomials in x such that the degree of Q exceeds that of P atleast by two and Q(x) does not vanish for any x.

Example 1: Prove that $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx = \frac{5\pi}{12}$ **using contour integration.**

Solution: Let $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$

Consider $\int_C f(z)dz$ **where C is the closed contour consisting of** Γ**, semi- large circle of radius R and the real axis from –R to R.**

Then $\int_c f(z)dz = \int_\Gamma f(z)dz + \int_{-R}^{R} f(x)dx$ **………(1)**

Now $f(z) = \frac{z^2-z+2}{z^4+10z^2+9} \to 0$ **as** $z \to \infty$

$$\therefore \lim_{z\to\infty} zf(z) = 0$$

Hence from (1) $\int_{-\infty}^{\infty} f(x)dx = \int_C f(z)dz$

By using residue theorem, $\int_C f(z)dz = 2\pi i\sum R$

$$\therefore \int_{-\infty}^{\infty} f(x)dx = 2\pi i \operatorname{Re}s f(z)$$

The poles of f(z) are given by

$$z^4+10z^2+9=0$$

$$\left(z^2+9\right)\left(z^2+1\right)=0$$

$$z=\pm i, z=\pm 3i$$

The poles z = 3i, z = i lies in the upper half of the z – plane