### Vectors and vector spaces.

A vector space is simply a field which obeys the field axioms, with field multiplication being valid for scalars and a vector and field additions being valid for two vectors. Or in layman's terms 'normal vector algebra applies!'

The key thing about vector spaces is that you can't 'leave' the space just but doing multiplication and addition. If your vector space is  $\mathbb{R}^2$  (2D space - a sheet of paper) you can't move to  $\mathbb{R}^3$  by addition or multiplaction

Vector are elements of a vector spaces. (Helpful I know!) And can be written as a sum of the basis: e.g. they

look like  $\begin{pmatrix} a_2 \\ \dots \\ a_n \end{pmatrix}$  or  $(a_1 \quad a_2 \quad \dots \quad a_n)$  where each entry is the scalar (from a given field) on each basis. E.g.

 $(a_1 \quad a_2 \quad \dots \quad a_n) = e_1a_1 + e_2a_2 + \dots + e_na_n$  where  $e_x$  represents the xth baisis.

For this topic if you get a vector with n entries think of it as n dimensional space, e.g.  $\begin{pmatrix} a_1 & a_2 \end{pmatrix}$  as 2d space (points on a sheet of paper). This can be confusing for higher dimensions, but trying to work something out thinking about a 2d or 3d example can be very helpful

In proofs he talks about the 'vector spaces of functions' and a couple of other and betract vector spaces, which can be very confusing, because they don't seem to contain action fact the elements are vectors!

Exam tip: Although him asking about a weird and work and vector spare is unlikely, it could happen. If it does (for example if he rambles on about vsoro Staces of functions don't worry! Just treat them exactly like you would a normal vector. prev

#### Subspace.

A subspace is a vector space which is contained within another vector space. Think 2D plane in 3D space:



Exam tip: To prove anything is a subspace (or not) start by showing zero is in it, then show any scalar multiple or linear combination of vectors remain within the subspace. E.g. 'prove differentiable functions is a subspace of the space of real functions.' d/dx (0) = 0, so differentiable. X,Y differentiable d/dx (aX+bY) = a d/dx (X)+ b d/dx (Y) by rules of differentiation. Which then proves it since X and Y are differentiable.

# Category theory.

This had a brief appearance at the start!

This is generalising everything to broad ideas. We have already seen how all functions are maps and all maps can be represented as matrices which in turn link to vector spaces. This essentially is another step up of generalisation. It's all very boring and very unexaminable. It has not come up in any of the available past papers, and other than regurgitating facts there aren't any questions that he can really ask about this.

Two elements are orthonormal if they are both orthogonal and normalized.

An orthonormal basis is where all basis elements are orthogonal too all others, and are all normal.



Two subspaces U and V are orthogonal if  $\langle u | v \rangle = 0$  for all elements  $u \in$  $U, v \in V$ . This becomes more obvious thinking geometrically, for example these two planes are obviously at right angles to each other (orthogonal).

# **Gram-Schmidt Process**

You just need to know how to apply it, the theory behind it is rather useless to the exam, although Kahn academy has a really good video explaining it. Although they use different notation, so careful of that in the exam if you choose to use them.

The definition is in the notes or as an answers in the exam past papers. Just learn how to regurgitate it, and how to sub into it. The best worked examples are past exams 2012 and 2011.

Exam tip: This came up every year until the last two years, where it has the team of the set t is very lightly it could come up. It has always been 3 marks state then 4 marks do Pratice It until you can easily do G-S for any 3D or 4D space. Adjoints.

names for thes. I have tabled them to make it easier to read and learn. There are a million definition These things in call of are the same the tor spaces:

Linear operator  $\equiv$  linear transformation  $\equiv$  Linear map from the vector space to itself

#### Names related to adjoints in table form

$\phi^*$ is the adjoint to $\phi$	$\langle \phi^*(v) w \rangle = \langle v \phi(w) \rangle$
$\phi$ is self adjoint	$\phi^*=\phi$
$\phi$ skew adjoint	$\phi^* = -\phi$
$\phi$ orthogonal/unitary (orthogonal if real vector	$\phi^*=\phi^{-1}$ , and
space, unitary if complex)	$\langle \phi(v)   \phi(w) \rangle = \langle v   w \rangle$
O(V) or $U(V)$	The set of orthogonal linear operators is called $O(V)$
	The set of unitary linear operators is called $U(V)$

Rules of adjoints in table form, with  $\psi$ ,  $\phi$  linear operators, and  $\lambda$ ,  $\mu$  scalars:

Matrix form for $\mathbb{F}^n$ vector space	If $\phi_A$ is the map represented by matrix A, then
	$\phi_A^* = \phi_A^\dagger$
Adjoint of an adjoint	$(\phi^*)^* = \phi$
Composition rule	$(\psi\circ\phi)^*=\psi^*\circ\phi^*$
Conjugate linearity	$(\lambda\psi+\mu\phi)^*=ar\lambda\psi^*+ar\mu\phi^*$

Exam tip: Q3 almost always asks what 'self adjoint' is and what a second thing from the first table means, so learn them. The rules are less examinable although a proof could come up with them in it.