Complex numbers obey the fundamental rules of algebra. Thus, two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$. Just as real numbers have the fundamental operations of addition, subtraction, multiplication, and division, so too do complex numbers. These operations are defined:

Addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{1.1.1}$$

Subtraction

$$(a + bi) - (c + di) = (a - c) + (b - d)i \tag{1.1.2}$$

Multiplication

$$(a + bi)(c + di) = ac + bci + adi + i^2bd = (ac - bd) + (ad + bc)i \tag{1.1.3}$$

Division

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\frac{c-di}{c-di} = \frac{ac - adi + bci - bdi^2}{c^2 + d^2} = \frac{ac + bd + (bc - ad)i}{c^2 + d^2}. \tag{1.1.4}$$

The *absolute value* or *modulus* of a complex number $a + bi$, written $|a + bi|$, equals $\sqrt{a^2 + b^2}$. Additional properties include:

$$|z_1 z_2 z_3 \cdots z_n| = |z_1||z_2||z_3| \cdots |z_n| \tag{1.1.5}$$

$$|z_1/z_2| = |z_1|/|z_2| \quad \text{if} \quad z_2 \neq 0 \tag{1.1.6}$$

$$|z_1 + z_2 + z_3 + \cdots + z_n| \leq |z_1| + |z_2| + |z_3| + \cdots + |z_n| \tag{1.1.7}$$

and

$$|z_1 + z_2| \geq |z_1| - |z_2|. \tag{1.1.8}$$

The use of inequalities with complex variables has meaning only when they involve absolute values.

It is often useful to plot the complex number $x + iy$ as a point (x, y) in the xy plane, now called the *complex plane*. Figure 1.1.1 illustrates this representation.

This geometrical interpretation of a complex number suggests an alternative method of expressing a complex number: the polar form. From the polar representation of x and y,

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta), \tag{1.1.9}$$

where $r = \sqrt{x^2 + y^2}$ is the *modulus*, *amplitude*, or *absolute value* of z and θ is the *argument* or *phase*, we have that

$$z = x + iy = r[\cos(\theta) + i\sin(\theta)]. \tag{1.1.10}$$

• Example 1.1.1

Let us simplify the following complex number:

$$\frac{3-2i}{-1+i} = \frac{3-2i}{-1+i} \times \frac{-1-i}{-1-i} = \frac{-3-3i+2i+2i^2}{1+1} = \frac{-5-i}{2} = -\frac{5}{2} - \frac{i}{2}. \tag{1.1.17}$$

• Example 1.1.2

Let us reexpress the complex number $-\sqrt{6} - i\sqrt{2}$ in polar form. From (1.1.9) $r = \sqrt{6+2}$ and $\theta = \tan^{-1}(b/a) = \tan^{-1}(1/\sqrt{3}) = \pi/6$ or $7\pi/6$. Because $-\sqrt{6} - i\sqrt{2}$ lies in the third quadrant of the complex plane, $\theta = 7\pi/6$ and

$$-\sqrt{6} - i\sqrt{2} = 2\sqrt{2}e^{7\pi i/6}. \tag{1.1.18}$$

Note that (1.1.18) is not a unique representation because $\pm 2n\pi$ may be added to $7\pi/6$ and we still have the same complex number since

$$e^{i(\theta \pm 2n\pi)} = \cos(\theta \pm 2n\pi) + i\sin(\theta \pm 2n\pi) = \cos(\theta) + i\sin(\theta) = e^{i\theta}. \tag{1.1.19}$$

For uniqueness we will often choose $n = 0$ and define this choice as the *principal branch*. Other branches correspond to different values of n.

• Example 1.1.3

Find the curve described by the equation $|z - z_0| = a$.

From the definition of the absolute value,

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} = a \tag{1.1.20}$$

or

$$(x-x_0)^2 + (y-y_0)^2 = a^2. \tag{1.1.21}$$

Equation (1.1.21), and hence $|z - z_0| = a$, describes a circle of radius a with its center located at (x_0, y_0). Later on, we shall use equations such as this to describe curves in the complex plane.

• Example 1.1.4

As an example in manipulating complex numbers, let us show that

$$\left|\frac{a+bi}{b+ai}\right| = 1. \tag{1.1.22}$$

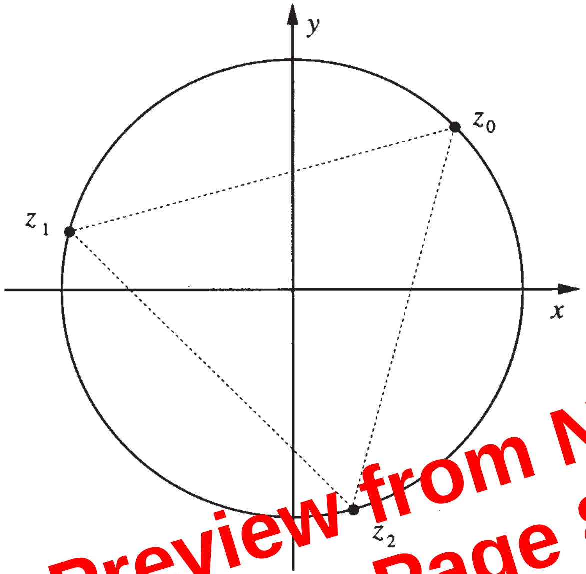

Figure 1.2.2: The zeros of $z^3 = -1 + i$.

or

$$z_0 = 2\exp\left(\frac{\pi i}{5}\right) = 2\left[\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right], \tag{1.2.9}$$

$$z_1 = 2\exp\left(\frac{3\pi i}{5}\right) = 2\left[\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right], \tag{1.2.10}$$

$$z_2 = 2\exp(\pi i) = -2, \tag{1.2.11}$$

$$z_3 = 2\exp\left(\frac{7\pi i}{5}\right) = 2\left[\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right] \tag{1.2.12}$$

and

$$z_4 = 2\exp\left(\frac{9\pi i}{5}\right) = 2\left[\cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)\right]. \tag{1.2.13}$$

Figure 1.2.1 shows the location of these roots in the complex plane.

• **Example 1.2.2**

Let us find the cube roots of $-1 + i$ and locate them graphically.

Any function that has continuous partial derivatives of second order and satisfies Laplace's equation (1.3.51) or (1.3.52) is called a *harmonic function*. Because both $u(x, y)$ and $v(x, y)$ satisfy Laplace's equation if $f(z) = u + iv$ is analytic, $u(x, y)$ and $v(x, y)$ are called *conjugate harmonic functions*.

• Example 1.3.8

Given that $u(x, y) = e^{-x}[x\sin(y) - y\cos(y)]$, let us show that u is harmonic and find a conjugate harmonic function $v(x, y)$ such that $f(z) = u + iv$ is analytic.

Because

$$\frac{\partial^2 u}{\partial x^2} = -2e^{-x}\sin(y) + xe^{-x}\sin(y) - ye^{-x}\cos(y) \tag{1.3.53}$$

and

$$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin(y) + 2e^{-x}\sin(y) + ye^{-x}\cos(y), \tag{1.3.54}$$

it follows that $u_{xx} + u_{yy} = 0$. Therefore, $u(x, y)$ is harmonic. From the Cauchy-Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin(y) - xe^{-x}\sin(y) + ye^{-x}\cos(y) \tag{1.3.55}$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^{-x}\cos(y) - xe^{-x}\cos(y) - ye^{-x}\sin(y). \tag{1.3.56}$$

Integrating (1.3.55) with respect to y,

$$v(x, y) = ye^{-x}\sin(y) + xe^{-x}\cos(y) + g(x). \tag{1.3.57}$$

Using (1.3.56),

$$\begin{aligned} v_x = -ye^{-x}\sin(y) - xe^{-x}\cos(y) + e^{-x}\cos(y) + g'(x) \\ = e^{-x}\cos(y) - xe^{-x}\cos(y) - ye^{-x}\sin(x). \end{aligned} \tag{1.3.58}$$

Therefore, $g'(x) = 0$ or $g(x) =$ constant. Consequently,

$$v(x, y) = e^{-x}[y\sin(y) + x\cos(y)] + \text{constant}. \tag{1.3.59}$$

Hence, for our real harmonic function $u(x, y)$, there are infinitely many harmonic conjugates $v(x, y)$ which differ from each other by an additive constant.

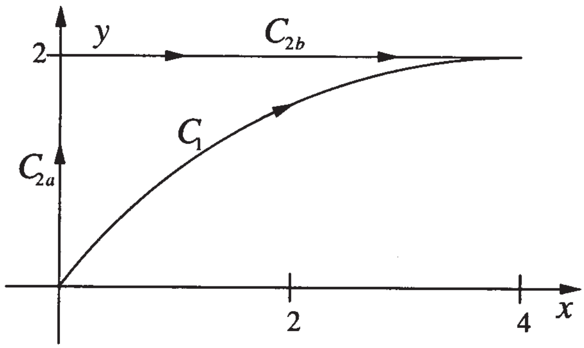

Figure 1.4.1: Contour used in Example 1.4.1.

axis from $z = 0$ to $z = 2i$ and then along a line parallel to the x-axis from $z = 2i$ to $z = 4 + 2i$. See Figure 1.4.1.

For the first case, the points $z = 0$ and $z = 4 + 2i$ on C_1 correspond to $t = 0$ and $t = 2$ respectively. Then the line integral equals

$$\int_{C_1} z^* \, dz = \int_0^2 (t^2 + it)^* \, d(t^2 + it) = \int_0^2 (2t^3 - it^2 + t) \, dt = 10 - \tfrac{8i}{3}. \tag{1.4.5}$$

The line integral for the second contour C_2 equals

$$\int_{C_2} z^* \, dz = \int_{C_{2a}} z^* \, dz + \int_{C_{2b}} z^* \, dz, \tag{1.4.6}$$

where C_{2a} denotes the integration from $z = 0$ to $z = 2i$ while C_{2b} denotes the integration from $z = 2i$ to $z = 4 + 2i$. For the first integral,

$$\int_{C_{2a}} z^* \, dz = \int_{C_{2a}} (x - iy)(dx + i\, dy) = \int_0^2 y \, dy = 2, \tag{1.4.7}$$

because $x = 0$ and $dx = 0$ along C_{2a}. On the other hand, along C_{2b}, $y = 2$ and $dy = 0$ so that

$$\int_{C_{2b}} z^* \, dz = \int_{C_{2b}} (x - iy)(dx + i\, dy) = \int_0^4 x \, dx + i \int_0^4 -2 \, dx = 8 - 8i. \tag{1.4.8}$$

Thus the value of entire C_2 contour integral equals the sum of the two parts or $10 - 8i$.

The point here is that integration along two different paths has given us different results even though we integrated from $z = 0$ to $z = 4 + 2i$ both times. This results foreshadows a general result that is extremely important. Because the integrand contains nonanalytic

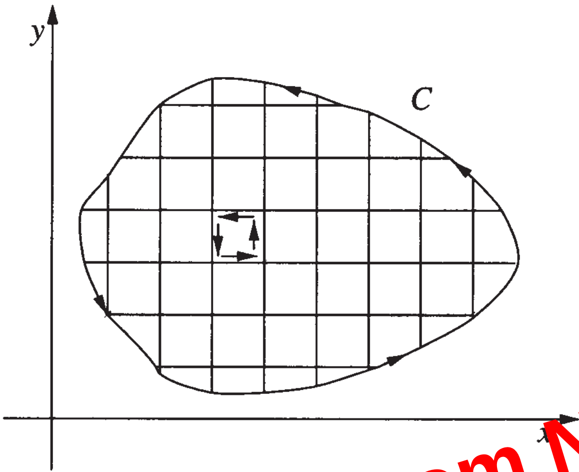

Figure 1.5.1: Diagram used in proving the Cauchy-Goursat theorem.

3. Evaluate $\int_C |z|\,dz$ along the right half of the circle $|z| = 1$ from $z = -i$ to $z = i$.

4. Evaluate $\int_C e^z\,dz$ along the line $y = x$ from $(-1, -1)$ to $(1, 1)$.

5. Evaluate $\int_C (z^*)^2\,dz$ along the line $y = x^2$ from $(0, 0)$ to $(1, 1)$.

6. Evaluate $\int_C z^{-1/2}\,dz$, where C is (a) the upper semicircle $|z| = 1$ and (b) the lower semicircle $|z| = 1$. If $z = re^{\theta i}$, restrict $-\pi < \theta < \pi$. Take both contours in the counterclockwise direction.

1.5 THE CAUCHY-GOURSAT THEOREM

In the previous section we showed how to evaluate line integrations by brute-force reduction to real-valued integrals. In general, this direct approach is quite difficult and we would like to apply some of the deeper properties of complex analysis to work smarter. In the remaining portions of this chapter we will introduce several theorems that will do just that.

If we scan over the examples worked in the previous section, we see considerable differences when the function was analytic inside and on the contour and when it was not. We may formalize this anecdotal evidence into the following theorem:

Cauchy-Goursat theorem[2]: *Let $f(z)$ be analytic in a domain D and*

[2] See Goursat, E., 1900: Sur la définition générale des fonctions analytiques, d'après Cauchy. *Trans. Am. Math. Soc.*, **1**, 14–16.

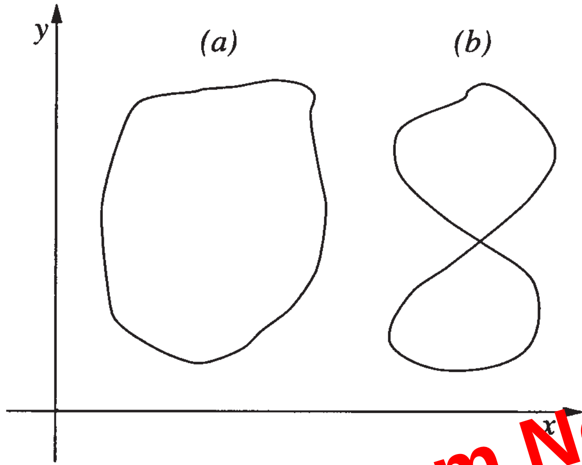

Figure 1.5.2: Examples of a (a) simply closed curve and (b) not simply closed curve.

let C be a simple Jordan curve[3] inside D so that $f(z)$ is analytic on and inside of C. Then $\oint_C f(z)\,dz = 0$.

Proof: Let C denote the contour around which we will integrate $w = f(z)$. We divide the region within C into a series of infinitesimal rectangles. See Figure 1.5.1. The integration around each rectangle equals the product of the average value of w on each side and its length,

$$\begin{aligned}&\left[w + \frac{\partial w}{\partial x}\frac{dx}{2}\right] dx + \left[w + \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial(iy)}\frac{d(iy)}{2}\right] d(iy)\\ &\quad + \left[w + \frac{\partial w}{\partial x}\frac{dx}{2} + \frac{\partial w}{\partial(iy)}d(iy)\right](-dx) + \left[w + \frac{\partial w}{\partial(iy)}\frac{d(iy)}{2}\right] d(-iy)\\ &\quad = \left(\frac{\partial w}{\partial x} - \frac{\partial w}{i\partial y}\right)(i\,dx\,dy) \end{aligned} \tag{1.5.1}$$

Substituting $w = u + iv$ into (1.5.1),

$$\frac{\partial w}{\partial x} - \frac{\partial w}{i\,\partial y} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \tag{1.5.2}$$

Because the function is analytic, the right side of (1.5.1) and (1.5.2) equals zero. Thus, the integration around each of these rectangles also equals zero.

[3] A Jordan curve is a simply closed curve. It looks like a closed loop that does not cross itself. See Figure 1.5.2.

not inside the contour, then the integrand would have been analytic inside and on the contour C. In this case, the answer would then be zero by the Cauchy-Goursat theorem.

Returning to the original problem, we construct the Laurent expansion for the integrand around the point $z = 1$ by noting that

$$\frac{z^2}{z+1} = \frac{[(z+1)-1]^2}{z+1} = \frac{1}{z+1} - 2 + (z+1). \qquad \textbf{(1.8.4)}$$

The singularity at $z = -1$ is a simple pole and by inspection the value of the residue equals 1. Therefore,

$$\oint_{|z|=2} \frac{z^2}{z+1}\,dz = 2\pi i. \qquad \textbf{(1.8.5)}$$

As it presently stands, it would appear that we must always construct a Laurent expansion for each singularity if we wish to use the residue theorem. This becomes increasingly difficult as the structure of the integrand becomes more complicated. In the following paragraphs we will show several techniques that avoid this problem in practice.

We begin by noting that many functions that we will encounter consist of the ratio of two *polynomials*, i.e., rational functions: $f(z) = g(z)/h(z)$. Generally, we can write $h(z)$ as $(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots$. Here we have assumed that we have divided out any common factors between $g(z)$ and $h(z)$ so that $g(z)$ does not vanish at $z_1, z_2, \ldots$. Clearly $z_1, z_2, \ldots,$ are singularities of $f(z)$. Further analysis shows that the nature of the singularities are a pole of order m_1 at $z = z_1$, a pole of order m_2 at $z = z_2$, and so forth.

Having found the nature and location of the singularity, we compute the residue as follows. Suppose we have a pole of order n. Then we know that its Laurent expansion is

$$f(z) = \frac{a_n}{(z-z_0)^n} + \frac{a_{n-1}}{(z-z_0)^{n-1}} + \cdots + b_0 + b_1(z-z_0) + \cdots \qquad \textbf{(1.8.6)}$$

Multiplying both sides of (1.8.6) by $(z-z_0)^n$,

$$\begin{aligned} F(z) &= (z-z_0)^n f(z) \\ &= a_n + a_{n-1}(z-z_0) + \cdots + b_0(z-z_0)^n + b_1(z-z_0)^{n+1} + \cdots \end{aligned} \qquad \textbf{(1.8.7)}$$

Because $F(z)$ is analytic at $z = z_0$, it has the Taylor expansion

$$F(z) = F(z_0) + F'(z_0)(z-z_0) + \cdots + \frac{F^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \cdots \qquad \textbf{(1.8.8)}$$