ST334 ACTUARIAL METHODS

version 2016/01

These notes are for ST334 Actuarial Methods. The course covers Actuarial CT1 and some related financial topics.

Actuarial CT1 is called 'Financial Mathematics' by the Institute of Actuaries. However, we reserve the term 'financial mathematics' for the study of stochastic models of derivatives and financial markets which is considerably more advanced mathematically than the material covered in this course.

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CHAPTER 1

Simple and Compound Interest

1 Simple interest

1.1 Background. The existence of a market where money can be borrowed and lent allows people to transfer consumption between today and tomorrow.

The amount of interest paid depends on several factors, including the risk of default and the amount of depreciation or appreciation in the value of the currency.

Interest on short-term financial instruments is usually *simple* rather than *compound*.

1.2 The simple interest problem over one time unit. This means there is *one* deposit (sometimes called the *principal*) and *one* repayment. The repayment is the sum of the principal and interest and is sometimes called the *future value*.

Suppose an investor deposits c_0 at time t_0 and receives a repayment c_1 at the later time $t_0 + 1$. Then the gain (or interest) on the investment c_0 is $c_1 - c_0$. The *interest rate*, i is given by

$$i = \frac{c_1 - c_0}{c_0}$$

Note that

$$c_1 = c_0(1+i) \quad \text{and} \quad c_0 = \frac{1}{1+i} c_1$$

The quantity c_0 is often called the *present value (at time t_0) of the amount c_1 at time $t_0 + 1$*. The last formula encapsulates the maxim "a pound today is worth more than a pound tomorrow".

> ***Example 1.2a.*** Suppose the amount of £100 is invested for one year at 8% per annum. Then the repayment in pounds is
>
> $$c_1 = (1 + 0.08) \times 100 = 108$$
>
> The present value in pounds of the amount £108 in one year's time is
>
> $$c_0 = \frac{1}{1+0.08} \times 108 = 100$$ ❑

Of course, the transaction can also be viewed from the perspective of the person who receives the money. A borrower borrows c_0 at time t_0 and repays c_1 at time $t_0 + 1$. The interest that the borrower must pay on the loan is $c_1 - c_0$.

1.3 The simple interest problem over several time units. Consider the same problem over n time units, where $n > 0$ is not necessarily integral and may be less than 1. The return to the investor will be

$$c_n = c_0 + nic_0 = c_0(1 + ni)$$

It follows that *present value (at time t_0) of the amount c_n at time $t_0 + n$* is

$$c_0 = \frac{1}{1+ni} c_n$$

1.4 Day and year conventions. Interest calculations are usually based on the exact number of days as a proportion of a year.
Domestic UK financial instruments usually assume there are 365 days in a year—even in a leap year. This convention is denoted ACT/365 which is short for *actual number of days divided by 365*.

2 Net present value and discounted cash flow

2.1 Net present value. The *net present value* of an investment is the sum of all incoming payments (which are regarded as positive) and all outgoings (which are regarded as negative) *after converting all payments into time 0 values*. The cash amounts must be discounted to allow for the fact that cash received tomorrow is worth less than cash received today—the reason for the term *discounted cash flow*.

Example 2.1a. *An investment of c_0 is made for one time unit at a simple interest rate of i per time unit. Find the net present value at time 0.*

Solution. The cash-flow sequence can be represented by the following table:

Time	0	1
Cash flow, $\boldsymbol{a}$	$-c_0$	$c_1 = c_0(1+i)$

The net present value at time 0 is

$$NPV(\boldsymbol{a}) = -c_0 + \frac{1}{1+i}c_1 = 0 \qquad \square$$

In general, suppose $\boldsymbol{c} = (c_0, c_1, \ldots, c_n)$ denotes the cash-flow sequence of payment c_0 at time 0 (the present time), and payment c_k after of k years for $k = 1, \ldots, n$. Suppose the current interest rate on investments is i per annum. Then the net present value is

$$NPV(\boldsymbol{c}) = c_0 + \frac{c_1}{1+i} + \frac{c_2}{(1+i)^2} + \cdots + \frac{c_n}{(1+i)^n} = \sum_{j=0}^{n} \frac{c_j}{(1+i)^j} \tag{2.1a}$$

Equation (2.1a) is sometimes called *the discounted cash flow* or (*DCF*) formula.
Clearly $NPV(\boldsymbol{c})$ decreases as i increases. Also, it decreases faster when later payments such as c_n are larger.

If the net present value of a project is positive, then the project is considered profitable; conversely, a negative *NPV* is evidence that a project should not proceed.

Example 2.1b. Consider the two cash flow sequences $\boldsymbol{a} = (12, 12, 12, 20, 24)$ and $\boldsymbol{b} = (20, 18, 14, 12, 12)$ at times $t = 0, \ldots, 4$. Find the net present values of the two cash-flows assuming an interest rate of (a) 3% and (b) 10%. Note that $\sum a_j = 80$ and $\sum b_j = 76$ and so $\boldsymbol{a}$ seems preferable.

Solution. (a) For 3%, we have $NPV(\boldsymbol{a}) = 12 + 12\nu + 12\nu^2 + 20\nu^3 + 24\nu^4 = 74.59$ where $\nu = 1/1.03$. Similarly $NPV(\boldsymbol{b}) = 72.3$[illegible] and so $\boldsymbol{a}$ is preferable to $\boldsymbol{b}$.
(b) For 10%, we have $NPV(\boldsymbol{a}) = 64.25$ and $NPV(\boldsymbol{b}) = 65.15$ and so $\boldsymbol{b}$ is preferable to $\boldsymbol{a}$.
If interest rates are high, **b** *is preferable because the cash is received earlier—and in principle this cash could be invested at this high interest rate.* $\square$

Example 2.1c. *Consider the following two investment projects:*
• *Project A. Purchase a 4 year lease on an office block for £1,000,000 with returns of £300,000 at time 1, £325,000 at time 2, £345,000 at time 3 and £360,000 at time 4.*
• *Project B. Invest in a 4-year bond which pays a nominal 12% interest p.a. payable half-yearly, currently priced at £95 per £100 and redeemable at par.*

Solution. For a £95 investment in a bond, the cash flows are as follows:

Time *in half-years*	0	1	2	3	4	5	6	7	8
Cash flow	−95	6	6	6	6	6	6	6	106

Let $\nu = 1/(1+r)$ denote the 6-month discount factor. Then

$$-95 + 6\nu + 6\nu^2 + 6\nu^3 + 6\nu^4 + 6\nu^5 + 6\nu^6 + 6\nu^7 + 106\nu^8 = 0$$

which leads to $101\nu + 100\nu^8 - 106\nu^9 = 95$ or $95(1+r)^9 - 101(1+r)^8 - 100r + 6 = 0$. Solving numerically gives $r = 0.06832$, or an effective annual rate of interest of $1 + i = (1+r)^2 = 1.06832^2 = 1.141308$ which is 14.13%.

Using this yield on the cash flows in project A we get the *NPV*:

$$-1{,}000{,}000 + \frac{300{,}000}{1+i} + \frac{325{,}000}{(1+i)^2} + \frac{345{,}000}{(1+i)^3} + \frac{360{,}000}{(1+i)^4} = 19{,}374.48$$

As *NPV* is positive, this suggests that the project A is preferable.
However, the above analysis has not allowed for the differential risks of the two projects. $\square$

2.3 Some standard examples.

Example 2.3a. *Replacing a piece of capital equipment.*
Suppose an old machine has current value w_0. For $j = 1, 2, \dots,$ suppose the machine has value w_j at the end of year j and has operating cost c_j for year j. Typically the w_j decrease with j due to depreciation, and the c_j increase with j.
Suppose a new machine will cost W_0 and will have value W_j at the end of year j of its operation. Also, its operating cost for year j of its operation is C_j. Suppose the interest rate is i per annum.

By considering the sequence of cash flows over the first 5 years, decide whether the machine be replaced immediately, or at the end of year 1, or at the end of year 2, etc.

Solution. Suppose the machine is replaced at time 0 (the beginning of year 1). Then the sequence of cash flows is:

$$\boldsymbol{a}_1 = (W_0 + C_1 - w_0, C_2, C_3, C_4, C_5, -W_5)$$

For replacement at start of year 2: $\boldsymbol{a}_2 = (c_1, W_0 + C_1 - w_1, C_2, C_3, C_4, -W_4)$
For replacement at start of year 3: $\boldsymbol{a}_3 = (c_1, c_2, W_0 + C_1 - w_2, C_2, C_3, -W_3)$
For replacement at start of year 4: $\boldsymbol{a}_4 = (c_1, c_2, c_3, W_0 + C_1 - w_3, C_2, -W_2)$

Clearly

$$NPV(\boldsymbol{a}_1) = W_0 + C_1 - w_0 + \frac{C_2}{1+i} + \frac{C_3}{(1+i)^2} + \frac{C_4}{(1+i)^3} + \frac{C_5}{(1+i)^4} - \frac{W_5}{(1+i)^5}$$

Similarly for $NPV(\boldsymbol{a}_2)$, etc. The best choice is the one with the smallest value for *NPV*. ❑

Example 2.3b. *Saving for a pension.*
Suppose someone plans to save the same amount £a at the beginning of every month for the next 240 months. This person then intends to withdraw £c at the beginning of every month for the subsequent 360 months. Assume the interest rate is i per annum compounded monthly. Find an expression for a in terms of i and c.

Solution. The monthly interest rate is $i/12$. Let $\alpha = 1 + i/12$.
At the end of 240 months, the capital accumulated is:

$$a\alpha^{240} + a\alpha^{239} + \cdots + a\alpha = a\alpha\frac{\alpha^{240} - 1}{\alpha - 1}$$

Using the same time point, the value of all the withdrawals is:

$$c + \frac{c}{\alpha} + \cdots + \frac{c}{\alpha^{359}} = c\frac{\alpha^{360} - 1}{\alpha^{359}(\alpha - 1)}$$

Setting these two quantities equal gives:

$$a = c\frac{\alpha^{360} - 1}{\alpha^{360}(\alpha^{240} - 1)}$$

Alternatively, calculations can be done in terms of the present value. At time 0, the value of the savings is:

$$a + \frac{a}{\alpha} + \cdots + \frac{a}{\alpha^{239}} = a\frac{\alpha^{240} - 1}{\alpha^{239}(\alpha - 1)}$$

At time 0, the value of the withdrawals is:

$$\frac{c}{\alpha^{240}} + \frac{c}{\alpha^{241}} + \cdots + \frac{c}{\alpha^{599}} = c\frac{\alpha^{360} - 1}{\alpha^{599}(\alpha - 1)}$$

Setting these quantities equal gives the same result as before. ❑

Example 2.3c. *Mortgage calculations. Suppose the capital borrowed is c_0 and the interest rate is i per annum compounded monthly. The capital is to be repaid by n equal payments of size x at the end of each month.*
(a) Express x in terms of n, c_0 and i.
(b) After the payment at the end of month j, what is the outstanding loan?
(c) How much of the loan is reduced by the payment made at the end of month j?

Solution. (a) Let $\alpha = 1 + i/12$. The present value of the n monthly payments is:

$$\frac{x}{\alpha} + \frac{x}{\alpha^2} + \cdots + \frac{x}{\alpha^n} = x\frac{\alpha^n - 1}{\alpha^n(\alpha - 1)}$$

Setting this equal to c_0 gives:

$$x = c_0\frac{\alpha^n(\alpha - 1)}{\alpha^n - 1}$$

(b) The value at time 0 of the amount repaid by the end of month j is the present value of the first j payments:

$$\frac{x}{\alpha} + \frac{x}{\alpha^2} + \cdots + \frac{x}{\alpha^j} = x\frac{\alpha^j - 1}{\alpha^j(\alpha - 1)} = \frac{c_0\alpha^{n-j}(\alpha^j - 1)}{\alpha^n - 1}$$

There is an interest payment of x for each of the n years plus a final payment of $c_n - x$. Hence the capital gain is $c_n - x - c_0$ and this capital gain can be approximated by a payment of $(c_n - x - c_0)/n$ every year. So the approximate rate of return is

$$i = \frac{x + (c_n - x - c_0)/n}{c_0}$$

Assuming there is a positive capital gain, we will have $x/c_0 < i < [x + (c_n - x - c_0)/n]/c_0$. This will usually be the best method for cash flows of this form.

Example 4.3a. (a) *Consider the following cash-flow sequence*

Time	0	1	2	3	4	5	6	7	8	9	10
Cash flow	−50	4	4	4	4	4	4	4	4	4	59

Use method (d) described above to find an initial approximation to the internal rate of return, r^*.

Solution. The return per year is $4 + (55 - 50)/10 = 4.5$. Hence $r^* \approx 4.5/50 = 0.09$. In fact, $r^* = 0.0866868$. ❑

Example 4.3b. (a) *Consider the cash-flow sequence $\boldsymbol{a} = (-87, 25, -40, 60, 60)$ at times 0, 1, 2, 3 and 4. Use the first 3 methods described above to find initial approximations to the internal rate of return, r^*.*
(b) Use the first 3 methods to find the yield on project B in example (2.1c) *on page 22.*

Solution. (a) The first method gives $r_0 = \sum c_j / \sum jc_j = 18/365 = 0.049$. The second method gives $r_0 = 18/293 = 0.061$. For the third method, we have $x = 105/4 = 26.25$. So we want r with $\boldsymbol{a}_{\overline{4}|,r} = 87/26.25 = 3.312$. From actuarial tables, $\boldsymbol{a}_{\overline{4}|,0.07} = 3.387$ and $\boldsymbol{a}_{\overline{4}|,0.08} = 3.312$; so take $r = 0.075$. The actual value of r^* is 0.056.
(b) The first method gives $r_0 = \sum c_j / \sum jc_j = 53/1016 = 0.052$. The second method gives $53/592 = 0.0895$. For the third method, we have $x = 148/8 = 18.5$. So we want r with $\boldsymbol{a}_{\overline{8}|,r} = 95/18.5 = 190/37 = 5.135$. From actuarial tables, $\boldsymbol{a}_{\overline{8}|,0.11} = 5.146$ and $\boldsymbol{a}_{\overline{8}|,0.115} = 5.0455$, so take $r = 0.11$.
The actual value is $r = 0.06832$. The third method gives a poor approximation because the mean of 18.5 is a poor summary of the sequence $(6, 6, 6, 6, 6, 6, 6, 106)$. ❑

4.4 Comparison with *NPV*. If the interest rate is i, then *NPV*(i) is the present value of the cash-flows of the project. The project is profitable if *NPV*(i) > 0.
Suppose r^*, the yield or *IRR* exists, and *NPV*(i) > 0 for $i < r^*$ and *NPV*(i) < 0 for $i > r^*$. Then the project is profitable if and only if $i < r^*$.

4.5 Problems with the use of the IRR. Consider the cash-flow sequence $\boldsymbol{a} = (c_0, c_1, \ldots, c_n)$. Let $x^* = 1/(1 + r^*)$. Then x^* is the solution for x of

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$$

This equation has n roots for x. Any root for x which satisfies $x > 0$ leads to a solution for r with $r > -1$. *There may be more than one root with $r > -1$.* Hence the internal rate of return cannot be defined.
Even if there is a unique root, it is possible that $g(r)$ is not monotone. It then follows that the property that *NPV* is positive for interest rates i on one side of r^* and negative for interest rates on the other side of r^* may not hold. So again it is not sensible to define the *IRR*.

Example 4.5a. *Consider the following sequence of cash-flows:*

Time	0	1	2	3
Cash flow	−32	128	−166	70

Find the internal rate of return, r^*.

Solution. The equation $c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$ gives $-32 + 128x - 166x^2 + 70x^3 = 0$. We can easily see that $x = 1$ is a solution. Hence we have $0 = (x - 1)(70x^2 - 96x + 32) = (x - 1)(7x - 4)(10x - 8)$. Hence $x = 1$, $4/7$, or $4/5$. Hence $r^* = 0$, 0.75, or 0.25. There is no unique value for the *IRR*. Also *NPV*(r) < 0 for $r \in (0, 0.25)$ and *NPV*(r) > 0 for $r \in (0.25, 0.75)$. ❑

Example 4.5b. *Find the IRR for the following cash-flow.*

Time	0	1	2
Cash flow	2	−4	3

Solution. We have $2 - 4x + 3x^2 = 0$ and this has no real roots. ❑

CHAPTER 3

Perpetuities, Annuities and Loan Schedules

1 Perpetuities

1.1 Perpetuities with constant payments. A perpetuity gives a fixed amount of income every year forever. Such securities are rare but do exist—for example, *consols*[4] issued by the UK government are undated.

Suppose the rate of interest per unit time is constant and equal to i. Let $\nu = 1/(1+i)$ denote the value at time 0 of the amount 1 at time 1. The value at time 0 of the following cash-flow sequence:

Time	0	1	2	3	...
Cash flow, c	0	1	1	1	...
	$\uparrow$ $a_{\overline{\infty}\rvert}$	$\uparrow$ $\ddot{a}_{\overline{\infty}\rvert}$			

is denoted $a_{\overline{\infty}\rvert}$ or $a_{\overline{\infty}\rvert,i}$ and is given by

$$a_{\overline{\infty}\rvert} = \nu + \nu^2 + \nu^3 + \cdots = \frac{\nu}{1-\nu} = \frac{1}{i} \tag{1.1a}$$

The value of this perpetuity at time 1 is denoted $\ddot{a}_{\overline{\infty}\rvert}$ or $\ddot{a}_{\overline{\infty}\rvert,i}$ and is given by

$$\ddot{a}_{\overline{\infty}\rvert} = (1+i)a_{\overline{\infty}\rvert} = 1 + \nu + \nu^2 + \nu^3 + \cdots = \frac{1}{1-\nu} = \frac{1}{d} = \frac{1+i}{i} \tag{1.1b}$$

where $d = 1 - \nu$ is the discount rate.

Example 1.1a. *Assume a University post costs £50,000 per year and that the interest rate is 4% per annum. How much must a benefactor give the University in order to establish a permanent post?*

Solution. Using the above formula gives

$$a_{\overline{\infty}\rvert} = \frac{50{,}000}{0.04} = 1{,}250{,}000 \qquad \square$$

1.2 In advance and in arrears. Suppose the rate of interest per unit time is constant and equal to i. Let $\nu = 1/(1+i)$ denote the value at time 0 of the amount 1 at time 1. Hence $\nu = 1 - d$ where d is the discount rate.

If A borrows £$(1-d)$ at time 0, then he has to repay £$(1-d)(1+i) = 1$ at time 1. This can be interpreted as follows:

- The principal £$(1-d)$ is returned at time 1 together with the interest £$i(1-d)$ = £d. The interest is paid ***in arrears***.

OR

- £1 is borrowed at time 0. The interest £d is paid ***in advance*** and the principal of £1 is returned at time 1. The interest £d paid at time 0 is the present value at time 0 of the quantity i at time 1. Hence $d = i/(1+i)$.

Thus $a_{\overline{\infty}\rvert}$ can be regarded as the value of a perpetuity when payments are made in arrears and $\ddot{a}_{\overline{\infty}\rvert}$ can be regarded as the value of a perpetuity when payments are made in advance.

[4] These were redeemed in 2015.

31. A financial regulator has brought in a new set of regulations and wishes to assess the cost of them. It intends to conduct an analysis of the costs and benefits of the new regulations in their first twenty years.

The costs are estimated as follows.

- The cost to companies who will need to devise new policy terms and computer systems is expected to be incurred at a rate of £50m in the first year increasing by 3% per annum over the twenty year period.
- The cost to financial advisers who will have to set up new computer systems and spend more time filling in paperwork is expected to be incurred at a rate of £60m in the first year, £19m in the second year, £18m in the third year, reducing by £1m every year until the last year, when the cost incurred will be at the rate of £1m.
- The cost to consumers who will have to spend more time filling in paperwork and talking to their financial advisers is expected to be incurred at a rate of £10m in the first year, increasing by 3% per annum over the twenty year period.

The benefits are estimated as follows.

- The benefits to consumers who are less likely to buy inappropriate policies is estimated to be received at a rate of £30m in the first year, £33m in the second year, £36m in the third year, rising by £3m per year until the end of twenty years.
- The benefits to companies who will spend less time dealing with complaints from customers is estimated to be received at a rate of £12m per annum for twenty years.

Calculate the net present value of the benefit or cost of the regulations in their first twenty years at a rate of interest of 4% per annum effective. Assume that all costs and benefits occur continuously throughout the year.

(Institute/Faculty of Actuaries Examinations, September 2006) [12]

32. A bank offers two repayment alternatives for a loan that is to be repaid over 10 years. The first requires the borrower to pay £1,200 per annum quarterly in advance and the second requires the borrower to make payments at an annual rate of £1,260 every second year in arrears.

Determine which terms would provide the best deal for the borrower at a rate of interest of 4% per annum effective.

(Institute/Faculty of Actuaries Examinations, September 2008) [5]

33. The force of interest, $\delta(t)$, is a function of time and at any time t (measured in years) is given by

$$\delta(t) = \begin{cases} 0.08 & \text{for } 0 \le t \le 4; \\ 0.12 - 0.01t & \text{for } 4 < t \le 9; \\ 0.05 & \text{for } t > 9. \end{cases}$$

(i) Determine the discount factor, $\nu(t)$, that applies at time t for all $t \ge 0$.

(ii) Calculate the present value at $t = 0$ of a payment stream, paid continuously from $t = 10$ to $t = 12$, under which the rate of payment at time t is $100e^{0.03t}$.

(iii) Calculate the present value of an annuity of £1,000 paid at the end of each year for the first 3 years.

(Institute/Faculty of Actuaries Examinations, April 2015) [5+4+3=12]

Questions on project analysis.

34. (i) In an annuity, the rate of payment per unit of time is continuously increasing so that at time t it is t. Show that the value at time 0 of such an annuity payable until time n, which is represented by $(\overline{Ia})_{\overline{n}|}$ is given by

$$(\overline{Ia})_{\overline{n}|} = \frac{\overline{a}_{\overline{n}|} - n\nu^n}{\delta}$$

where δ is the force of interest per unit of time and $\overline{a}_{\overline{n}|} = \int_0^n \exp(-\delta t)\,dt$.

(ii) A consortium is considering the purchase of a coal mine which has recently ceased production. The consortium forecasts that:

(1) the cost of reopening the mine will be £700,000, and this will be incurred continuously throughout the first twelve months

(2) after the first twelve months the revenue from sales of coal, less costs of sale and extraction, will grow continuously from zero to £2,500,000 at a constant rate of £250,000 per annum

(3) when the revenue from sales of coal, less costs of sale and extraction, reaches £2,500,000 it will then decline continuously at a constant rate of £125,000 per annum until it reaches £250,000

(4) when the revenue declines to £250,000 production will stop and the mine will have zero value

Additional costs are expected to be constant throughout at £200,000 p.a. excluding the first year. These are also incurred continuously.

What price should the consortium pay to earn an internal rate of return (IRR) of 20% p.a. effective?

(Institute/Faculty of Actuaries Examinations, September 1998) [14]

35. A property developer is constructing a block of offices. It is anticipated that the offices will take six months to build. The developer incurs costs of £40 million at the beginning of the project followed by £3 million at the end of each month for the following six months during the building period. It is expected that rental income from the offices will be £1 million per month, which will be received at the start of each month beginning with the seventh month. Maintenance and management costs paid be the developer are expected to be £2 million per annum payable monthly in arrear with the first payment at the end of the seventh month. The block of offices is expected to be sold 25 years after the start of the project for £60 million.

(i) Calculate the discounted payback period using an effective rate of interest of 10% per annum.

(ii) Without doing any further calculations, explain whether your answer to (i) would change if the effective rate of interest were less than 10% per annum.

(Institute/Faculty of Actuaries Examinations, April 2007) [7+3=10]

36. An investor borrows £120,000 at an effective interest rate of 7% per annum. The investor uses the money to purchase an annuity of £14,000 per annum payable half-yearly in arrears for 25 years. Once the loan is paid off, the investor can earn interest at an effective rate of 5% per annum on money invested from the annuity payments.

(i) Determine the discounted payback for this investment.

(ii) Determine the profit the investor will have made at the end of the term of the annuity.

(Institute/Faculty of Actuaries Examinations, April 2004) [5+7=12]

37. A piece of land is available for sale for £5,000,000. A property developer, who can lend and borrow money at a rate of 15% per annum, believes that she can build housing on the land and sell it for a profit. The total cost of development would be £7,000,000 which would be incurred continuously over the first two years after purchase of the land. The development would then be complete.

The developer has three possible strategies. She believes that she can sell off the completed housing:

- in three years time for £16,500,000;
- in four years' time for £18,000,000;
- in five years' time for £20,500,000.

The developer also believes that she can obtain a rental income from the housing between the time that the development is completed and the time of the sale,. The rental income is payable quarterly in advance and is expected to be £500,000 in the first year of payment. Thereafter, the rental income is expected to increase by £50,000 per annum at the beginning of each year that the income is paid.

(i) Determine the optimum strategy if this is based upon using net present value as the decision criterion.

(ii) Determine which strategy would be optimal if the discounted payback period were to be used as the decision criterion.

(iii) If the housing is sold in six years' time the developer believes that she can obtain an internal rate of return on the project of 17.5% per annum. Calculate the sale price that the developer believes that she can receive.

(iv) Suggest reasons why the developer may not achieve an internal rate of return of 17.5% per annum even if she sells the housing for the sale price calculated in (iii).

(Institute/Faculty of Actuaries Examinations, April 2006) [9+2+6+2=19]

38. (i) Explain what is meant by the following terms:

(a) equation of value;

(b) discounted payback period from an investment project.

(ii) An insurance company is considering setting up a branch in a country in which it has previously not operated. The company is aware that access to capital may become difficult in twelve years' time. It therefore has two decision criteria. The cash flows from the project must provide an internal rate of return greater than 9% per annum effective and the discounted payback period at a rate of interest of 7% per annum effective must be less than twelve years.

The following cash flows are generated in the development and operation of the branch.

Cash Outflows. Between the present time and the opening of the branch in three years' time, the insurance company will spend £1.5m per annum on research, development and the marketing of products. This outlay is assumed to be a constant continuous payment stream. The rent on the branch building will be £0.3m per annum paid quarterly in advance for twelve years starting in three years' time. Staff costs are assumed to be £1m in the first year, £1.05m in the second year, rising by 5% per annum each year thereafter. Staff costs are assumed to be incurred at the beginning of each year starting in three years' time and assumed to be incurred for twelve years.

Cash Inflows. The company expects the sale of products to produce a net income at a rate of £1m per annum for the first three years after the branch opens rising to £1.9m per annum in the next three years and to £2.5m for the following six years. This net income is assumed to be received continuously throughout each year. The company expects to be able to sell the branch operation 15 years from the present time for £8m.

Determine which, if any, of the decision criteria the project fulfils.

(Institute/Faculty of Actuaries Examinations, September 2005) [4+17=21]

39. An investor is considering investing in a capital project. The project requires a capital outlay of £500,000 at outset and further payments at the end of each of the first 5 years, the first payment being £100,000 and each successive payment increasing by £10,000. The project is expected to provide a continuous income at a rate of £80,000 in the

- Every bank will want to hold a proportion of its assets in a form in which it can quickly get at the money.
- Companies, local authorities, and other bodies may have a temporary surplus or shortage of money.
- The money market forms a bridge between the government and the private sector. If the government sells T-bills then money is taken out of the market; if an *eligible** bank is short of money it can get a loan from the government by selling a financial instrument to the Bank of England. These instruments include bills of exchange, gilt repos and Euro bills. The rate of interest at which the Bank is prepared to lend to the commercial banks is called the *Official Dealing Rate* or the *repo rate* and is fixed by the infamous *Monetary Policy Committee* which meets every month. Its decisions are widely reported in the press.

1.4 Interest rates in the money market. Commercial banks have a *bank base rate* which is determined by the Bank of England's repo rate. Deposit rates and borrowing rates for individuals are calculated from these bank base rates.

The *London Inter-Bank Offered Rate* or *LIBOR* is the rate for wholesale funds—it is the rate at which a bank will lend money to another bank of top credit worthiness. The *London Inter-Bank Bid Rate* or *LIBID* is the rate at which a bank bids for money from another bank. *LIMEAN* is the average of LIBOR and LIBID. Clearly, LIBOR > LIBID. LIBOR is a constantly changing measure of the cost of a large amount of money to a bank—it changes throughout the day!

The money market is linked to other markets through arbitrage mechanisms. For example, it is linked to the capital market by the ability to create long-term instruments from a series of short-term instruments.

2 Fixed interest government borrowings

2.1 Bonds. Governments (or government bodies) can raise money by issuing bonds. In most developed economies, government bonds form the largest and most liquid section of the bond market. They also constitute the most secure long-term investment available. In the UK, bonds guaranteed by the government are called *gilt-edged securities* or *gilts*. (Note that *Treasury bills* are for less than one year and are sold at a discount.)

In general, a borrower who issues a bond agrees to pay interest at a specified rate until a specified date, called the *maturity date* or *redemption date*, and at that time pays a fixed sum called the *redemption value*. The price of the bond at issue is called the *issue price*.

The interest rate on a bond is called the *coupon rate*. This rate is often quoted as a nominal rate convertible semi-annually and is applied to the *face* or *par value* of the bond. The face and redemption values are often the same.

Let

f = face or par value of the bond

r = the coupon rate (the effective coupon rate per interest period)

C = the redemption value of the bond

n = the number of interest periods until the redemption date

P = current price of the bond

i = the yield to maturity (this is the same as the internal rate of return)

The values of f, r, C and the redemption date are specified by the terms of the bond and are fixed throughout the lifetime of the bond.

The values of P and i vary throughout the lifetime of the bond. As the price of a bond rises, the yield falls. Also, the price of a bond (and hence its yield) depends on the prevailing interest rates in the market.

If $f = C$ then the bond is *redeemable at par*; if $f > C$ then the bond is *redeemable below par* or is *redeemable at a discount*; if $f < C$ then the bond is *redeemable above par* or is *redeemable at a premium*.

When a bond is sold between two coupon dates, the seller will be entitled to some *accrued interest*. This is explained in section 1.8 of chapter 5 on page 83.

* The list of eligible banks can be found on the web site of the Bank of England.

5.2 Forward contract.

A *forward contract* is a legally binding contract to buy/sell an agreed quantity of an asset at an agreed price at an agreed time in the future. The contract is usually tailor-made between two financial institutions or between a financial institution and a client. Thus forward contracts are *over-the-counter* or *OTC*. Such contracts are not normally traded on an exchange.

Settlement of a forward contract occurs entirely at maturity and then the asset is normally delivered by the seller to the buyer. Forward contracts are subject to credit risk—the risk of default.

5.3 Futures contract.

A *futures contract* is also a legally binding contract to buy/sell an agreed quantity of an asset at an agreed price at an agreed time in the future. Futures contracts can be traded on an exchange—this implies that futures contracts have standard sizes, delivery dates and, in the case of commodities, quality of the underlying asset. The underlying asset could be a financial asset (such as equities, bonds or currencies) or commodities (such as metals, wool, live cattle). A futures contract based on a financial asset is called a *financial future*.

Suppose the contract specifies that A will buy the asset S from B at the price K at the future time T. Then B is said to hold a *short forward position* and A is said to hold a *long forward position*. Therefore, a futures contract says that the short side must deliver to the long side the asset S for the *exchange delivery settlement price (EDSP)*. Thus the long side profits if the price rises and the short side profits if the price falls.

On maturity, physical delivery of the underlying asset is often not made—often the short side closes the contract by entering into a reverse contract with the long side and a *cash settlement* is made.

Each party to a futures contract must deposit a sum of money called the *margin* with the clearing house. Margins act as cushions against possible future adverse price movements. When the contract is first struck, the *initial margin* is deposited with the clearing house. Additional payments of *variation margin* are made daily to ensure the credit risk is controlled.

Note that futures contracts are guaranteed by the exchange on which they are traded. Many futures markets are more liquid than the spot market in the underlying asset. For example, there are many types of UK government bonds but only one UK bond futures contract. Futures markets enable a trader to speculate on price movements for only a small initial cash payment and they also enable a trader to take a *short position* if he believes a price is going to fall—he would sell a future contract.

A note on the differences between forward contracts and futures contracts.

- Forward contracts are tailor-made contracts between the two parties. Futures contracts are traded on an exchange—consequently, futures contracts are standardised (size of contract, delivery date, etc.)
- Settlement of forward contracts occurs at maturity. With futures contracts, there is the process of *marking to market* described in chapter 5.
- For most futures contracts, delivery of the underlying asset is not normally made. A cash flow is made to close out the position. With forward contracts, delivery is usually carried out.

I understand forward contracts started in Chicago in the 1840s. Farmers in the Midwest shipped their grain to Chicago for sale and distribution. Because all the grain was harvested at the same time, the price of grain dropped drastically at harvest time and then gradually increased as the stock was consumed. It made sense for farmers to agree to a forward contract in which they agreed to supply an agreed amount of grain at an agreed price at some time in the future.

Subsequently, hedgers and speculators wished to trade in these contracts. This led to the establishment of an exchange which laid down rules for contracts which could be traded and also checked the financial status of both parties to the contract—this led to futures contracts. The explosion in the use of futures contracts occurred with the introduction of financial futures. I believe that Chicago is still the second largest futures exchange in the world.
For further information see the two books by J.C. Hull and M. Brett (page 290 onwards).

5.4 Some special cases of futures contracts.

- **The bond future.** Government bond futures contracts are one of the most popular futures contracts.

A bond futures contract is an agreement to buy or sell a bond at a specified price at a specified time in the future. Some contracts are usually specified in terms of a notional bond and then there needs to be an agreed list of bonds which are eligible for fulfilling the contract. The short side will then choose the bond from the list which is cheapest to deliver. The price paid by the long side may have to be adjusted to allow for the fact

at the exercise price. The holder of a put option cannot lose more than the premium paid—this occurs if the value of the underlying asset is above the exercise price. In that situation, the holder of the put option will not exercise his right to sell the asset and so he will just lose the premium he has paid.
The profit/loss diagram for the writer of a put option is the mirror image in the y-axis of the previously described graph—this means that his maximum loss is the exercise price minus the premium.

5.9 Dangers of derivatives. Trading in futures can be more dangerous than trading in the underlying asset because of *gearing* or *leverage*. The initial margin will be small compared with the potential loss. Thus it is possible to lose large amounts (and gain large amounts) with a relatively small initial outlay.

For purchasers of options, the maximum loss is limited to the initial outlay.

6 Exercises

(exs4-1.tex)

1. A CD (Certificate of Deposit) is issued for £1,000,000 on 12 March for 90 days with a 5% coupon. Hence it matures on 10 June.

(a) What are the proceeds on maturity?

(b) On 4 April, what should the secondary market price be in order that the yield is then 4.5%?

(c) Assume the CD is purchased in the secondary market on 4 April for the price calculated in part (b). By 4 May, the yield has dropped to 4%. What is the rate of return for holding the CD from 4 April to 4 May?
(Assume ACT/365.)

2. (a) Distinguish between a future and an option.

(b) Explain why convertibles have "option-like" characteristics.
(Institute/Faculty of Actuaries Examinations, September 2006) [3]

3. Describe how cash flows are exchanged in an "interest rate swap".
(Institute/Faculty of Actuaries Examinations, September 2005) [2]

4. State the main differences between a preference share and an ordinary share.
(Institute/Faculty of Actuaries Examinations, April 2003) [3]

5. Describe the features and risk characteristics of a "Government Bill".
(Institute/Faculty of Actuaries Examinations, September 2002) [4]

6. (i) Define the characteristics of a government index-linked bond.

(ii) Explain why most index-linked securities carry some inflation risk, in practice.
(Institute/Faculty of Actuaries Examinations, September 2003) [2+2=4]

7. A company has entered into an interest rate swap. Under the terms of the swap, the company makes fixed annual payments equal to 6% of the principal of the swap. In return, the company receives annual interest payments on the principal based on the prevailing variable short-term interest rate which currently stands at 5.5% per annum.

(a) Describe briefly the risks faced by a counterparty to an interest rate swap.

(b) Explain which of the risks described in (a) are faced by the company.
(Institute/Faculty of Actuaries Examinations, April 2006) [4]

8. State the characteristics of an equity investment.
(Institute/Faculty of Actuaries Examinations, September 2007) [4]

9. Describe the characteristics of the following investments:

(a) Eurobonds

(b) Certificates of Deposit *(Institute/Faculty of Actuaries Examinations, April 2008)* [4]

10. Describe the characteristics of the cash flows that are paid and received in respect of

(i) an index-linked security

(ii) an equity *(Institute/Faculty of Actuaries Examinations, September 2013)* [2+3=5]

11. Describe the characteristics of commercial property (i.e. commercial real estate) as an investment.
(Institute/Faculty of Actuaries Examinations, September 2008) [5]

12. Explain why the running yield from property investments tends to be greater than that from equity investments.
(Institute/Faculty of Actuaries Examinations, April 2015) [3]

Here is the general result: suppose the coupon rate is nominal $r\%$ p.a. payable m times per year. Then the price P in order to achieve an effective yield of i per annum is given by

$$P = fr\boldsymbol{a}^{(m)}_{\overline{n}|,i} + \frac{C}{(1+i)^n} = \frac{fr}{m}\boldsymbol{a}_{\overline{mn}|,i'} + \frac{C}{(1+i')^{mn}} \tag{1.2a}$$

where $(1+i')^m = 1+i$ and $i' = \dfrac{i^{(m)}}{m}$ is the effective interest rate per period. If the price rises, then the yield falls, and conversely.

If $P = f = C$, then using the relation $\boldsymbol{a}^{(m)}_{\overline{n}|} = (1-\nu^n)/i^{(m)}$ and equation (1.2a) shows that $i^{(m)} = r$. This means that if the face value, redemption value and price are all the same, then the nominal yield convertible $m$thly equals the coupon rate.

If the income tax rate is t_1, then equation (1.2a) becomes:

$$P = fr(1-t_1)\boldsymbol{a}^{(m)}_{\overline{n}|,i} + \frac{C}{(1+i)^n} \tag{1.2b}$$

Example 1.2b. *Suppose a bond is issued for 15 years with a coupon rate of 6% per annum convertible 6-monthly. The bond is redeemable at 105%. (This means that $C = f \times 1.05$.)*
(a) Find the price per unit nominal[1] so that the effective yield at maturity is 7% per annum.
(b) Assume the income tax rate is 35%. Find the price per unit nominal so that the effective yield at maturity is 7% p.a.

Solution. For this example, we have $f = 100$, $C = 105$, $r = 0.06$, $i = 0.07$ and $n = 15$.
(a) The size of each coupon is 3. Hence

$$P = 6\boldsymbol{a}^{(2)}_{\overline{15}|,i} + \frac{105}{(1+i)^{15}} = 6\frac{i}{i^{(2)}}\boldsymbol{a}_{\overline{15}|,i} + \frac{105}{(1+i)^{15}} \approx 93.64$$

by using tables. Hence the bond should be issued at a discount.

(b)

$$P = 6 \times 0.65 \times \boldsymbol{a}^{(2)}_{\overline{15}|,i} + \frac{105}{(1+i)^{15}} = 3.9\frac{i}{i^{(2)}}\boldsymbol{a}_{\overline{15}|,i} + \frac{105}{1.07^{15}} \approx 74.19$$ ❑

1.3 Gross yield, redemption yield and flat yield.

The terms *gross interest yield, direct yield, flat yield, current yield* or *gross running yield* all refer to $100r/P_1$, which is the ratio of $100r$, the annual coupon per £100, to the current price P_1 per unit nominal[1]. This is the same as the annual coupon (fr) divided by the current price, P.
The terms *net interest yield* and *net running yield* refer to the same quantity but after allowing for tax: hence it is $100(1-t_1)r/P_1$, the ratio of the (after tax) annual coupon per £100 to the current price P_1 per unit nominal. This is the same as $(1-t_1)fr/P$, the annual coupon after tax divided by the current price, P.

Example 1.3a. *The current price of a bond with an annual coupon of 10% is £103 per £100 nominal. Then the gross interest yield (also known as the flat yield or running yield) is* $10/103 = 0.0971$ *or 9.71%.* ❑

The *interest yields* defined in the last paragraph are useful when a bond is undated and there are no redemption proceeds. If there are redemption proceeds, then we should also take these into account when calculating the yield.
The term *gross redemption yield* is the same as the yield to maturity defined in paragraph 1.1 and ignores taxation. The term *net redemption yield* or *net yield to redemption* or *net yield* refers to the yield after allowing for tax.

If the gross redemption yield on a bond is an effective 3% every six months, then the gross redemption yield is 6.09% per annum (because $1.03^2 - 1 = 0.0609$). Alternatively, it is described as a gross redemption yield of 6% convertible 6-monthly.[2]

1 In the UK, *per unit nominal* means per £100.

2 In Europe, the gross redemption yield is the same as the internal rate of return as defined in paragraph 1.1. However, in the USA and the UK, the gross redemption yield of a bond with 6-monthly coupons is defined to be twice the effective yield for 6 months—what we have defined as the gross redemption yield convertible 6-monthly. In actuarial questions, the distinction will be clarified by using phrases such as "a gross redemption yield of 6.09% per annum effective" or "a gross redemption yield of 6% per annum convertible 6-monthly". See *Basic Bond Analysis* by Joanna Place which can be downloaded from `http://www.bankofengland.co.uk/education/Pages/ccbs/handbooks`

First case: suppose $i^{(m)} < (1-t_1)g$, equivalently, suppose $P(n,i) > C$.
Then for a fixed price, a longer term implies a higher yield—see paragraph 1.6 above. So suppose $n_1 < n^*$ and $P(n_1, i) = P(n^*, i^*)$ for some i and i^*; then $i^* > i$.
This means that whatever price we pay for the bond, the yield if the bond is redeemed at any date n^* with $n^* > n_1$ must be greater than i, the yield if it is redeemed at the earliest time n_1.

If $i^{(m)} < (1-t_1)g$, the purchaser of the bond will receive no capital gain and should price the bond on the assumption that redemption will take place at the earliest possible time n_1. Then, no matter when the bond is actually redeemed, the yield that will be obtained will be at least the value of the yield calculated by assuming the bond is redeemed at the earliest possible time, n_1.

Second case: suppose $i^{(m)} > (1-t_1)g$, equivalently suppose $P(n,i) < C$.
Then for a fixed price, a shorter term implies a higher yield—see paragraph 1.6 above. So suppose $n^* < n_2$ and $P(n_2, i) = P(n^*, i^*)$ for some i and i^*; then $i^* > i$.

If $i^{(m)} > (1-t_1)g$, the purchaser of the bond will receive a capital gain and should price the bond on the assumption that redemption will take place at the latest possible time. Then, no matter when the bond is actually redeemed, the yield that will be obtained will be at least the value of the yield calculated by assuming the bond is redeemed at the latest possible time.

Third case: suppose $i^{(m)} = (1-t_1)g$, equivalently suppose $P(n,i) = C$ for all n.
If $i^{(m)} = (1-t_1)g$, the yield is same whatever the redemption date.

• *The minimum yield i if the price is P.* Let P and C denote the price and redemption price per £100 face value, respectively. By the above remarks, we have the following:

If $P < C$ then it is better for the investor that the bond is redeemed sooner rather than later. Suppose i_2 denotes the yield if the bond is redeemed at the latest possible time n_2; then the yield will be at least i_2 whatever the redemption date.

If $P > C$ then it is better for the investor that the bond is redeemed later rather than sooner. Suppose i_1 denotes the yield if the bond is redeemed at the earliest possible time n_1; then the yield will be at least i_1 whatever the redemption date.

If $P = C$ then the yield will be i where $i^{(m)} = (1-t_1)g$ whatever the redemption date.

1.8 Clean and dirty prices. Suppose investor A sells a bond to another investor, B, between two coupon dates. Then B will be the registered owner of the bond at the next coupon date and so receive the coupon. But A will feel entitled to the “accrued coupon” or “accrued interest”—this is the interest A earned by holding the bond from the previous coupon date to the date of the sale. He therefore sells the bond for the ***dirty price***—this is the *NPV* of future cash flows from the bond[6].
The ***clean price*** is defined to be equal to the dirty price minus the ***accrued interest***. So we have

$$\boxed{\begin{aligned} \textit{dirty price} &= \textit{NPV of future cash flows} \\ \textit{clean price} &= \textit{dirty price} - \textit{accrued interest} \end{aligned}}$$

The price quoted in the market is the clean price but the bond is sold for the dirty price.
The way that accrued interest is calculated depends on the bond market. For U.S. treasury bonds, the convention is ACT/ACT. Hence:

$$\text{accrued interest} = \text{value of next coupon} \times \frac{\text{days since last coupon}}{\text{days between coupons}}$$

For eurobonds, the convention is 30/360. This implies that the accrued interest on the 31st of the month is the same at the accrued interest on the 30th of the month. Also, the accrued interest for the 7 days from 22 February to 1 March would be 9/360 of the annual coupon. In the UK and Japan, the convention is ACT/365. Hence:

$$\text{accrued interest} = \text{annual coupon} \times \frac{\text{days since last coupon}}{365}$$

Example 1.8a. A bond has a semi-annual coupon with payment dates of 27 April and 27 October. Suppose the coupon rate is 10% per annum convertible semi-annually. Calculate the accrued interest per £100 if the bond is purchased on 4 July. Assume ACT/365.

6 Of course, this *NPV* depends on the current yield which in turn depends on the general market sentiment about the bond.

2 Exercises

(exs5-1.tex)

1. Consider a bond with face value f, redemption value C, current price P which has a coupon rate of $r\%$ payable m times per year for n years. Taxation is at the rate t_1 *and is paid at the end of each year*—hence the annual tax is $t_1 fr$. Find an equation for the yield to maturity.

2. Suppose bond A is redeemed after n_1 years and bond B is redeemed after n_2 years where $n_1 < n_2$. Suppose further that both bonds have the redemption value C, the same annual coupon fr, the same purchase price P and both have a coupon which is payable m times per year.
Let i_1 and i_2 denote the yields of bonds 1 and 2 respectively. Show the following:
(a) If $P = C$ then $i_1 = i_2$. (b) If $P < C$ then $i_1 > i_2$. (c) If $P > C$ then $i_1 < i_2$.

3. Suppose the first coupon on an undated bond with current price P and face value f is delayed for t_0 years. The coupon rate is $r\%$ nominal p.a. payable m times per year and the income tax rate is t_1.
Show that the effective annual yield, i, is given by

$$P = fr(1-t_1)\nu^{t_0}\ddot{a}^{(m)}_{\overline{\infty}|} = \frac{fr(1-t_1)\nu^{t_0}}{d^{(m)}} \qquad \text{where } \nu = \frac{1}{1+i}$$

4. A bond with an annual coupon of 8% per annum has just been issued with a gross redemption yield of 6% per annum effective. It is redeemable at par at the option of the borrower on any coupon payment date from the tenth anniversary of issue to the twentieth anniversary of issue. The gross redemption yield on bonds of all terms to maturity is 6% per annum effective.

Ten years after issue, the gross redemption yield on bonds of all terms to maturity is 10% per annum effective.
(a) Is the bond likely to be redeemed earlier or later than was assumed at issue?
(b) Is the gross redemption yield likely to be higher or lower than was assumed at issue?
(Institute/Faculty of Actuaries Examinations, September 1998 (adapted)) [3]

5. An investor purchases a bond, redeemable at par, which pays half-yearly coupons at a rate of 8% per annum. There are 8 days until the next coupon payment and the bond is ex-dividend. The bond has 7 years to maturity after the next coupon payment.
Calculate the purchase price to provide a yield to maturity of 6% per annum.
(Institute/Faculty of Actuaries Examinations, April, 2002) [4]

6. A fixed interest stock bears a coupon of 7% per annum payable half yearly on 1 April and 1 October. It is redeemable at par on any 1 April between 1 April 2004 and 1 April 2010 inclusive at the option of the borrower.
On 1 July 1991 an investor purchased £10 000 nominal of the stock at a price to give a net yield of 6% per annum effective after allowing for tax at 25% on the coupon payments.
On 1 April 1999 the investor sold the holding at a price which gave a net yield of 5% per annum effective to another purchaser who is also taxed at a rate of 25% on the coupon payments.
(i) Calculate the price at which the stock was bought by the original investor.
(ii) Calculate the price at which the stock was sold by the original investor.
(Institute/Faculty of Actuaries Examinations, April 2000) [4]

7. An investor purchases a bond on the issue date at a price of £96 per £100 nominal. Coupons at an annual rate of 4% are paid annually in arrears. The bond will be redeemed at par 20 years after the issue date.
Calculate the gross redemption yield from the bond.
(Institute/Faculty of Actuaries Examinations, September 2000) [4]

8. A new issue of a fixed interest security has a term to redemption of 20 years and is redeemable at 110%. The security pays a coupon of 9½% per annum payable half-yearly in arrears.

An investor who is liable to tax on all income at a rate of 23% and on all capital gains at a rate of 34% bought all the stock at the date of issue at a price which gave the investor a yield to maturity of 8% per annum effective. What price did the investor pay per £100 nominal of the stock?
(Institute/Faculty of Actuaries Examinations, April 1999) [5]

9. (i) Describe the risk characteristics of a government-issued, conventional, fixed-interest bond.
(ii) A particular government bond is structured as follows.
Annual coupons are paid in arrears of 8% of the nominal value of the bond. After 5 years, a capital repayment is made, equal to half of the nominal value of the bond. The capital is repaid at par. The repayment takes place immediately after the payment of the coupon due at the end of the 5th year. After the end of the 5th year, coupons are only paid on that part of the capital that has not been repaid. At the end of the 10th year, all the remaining capital is repaid.
Calculate the purchase price of the bond per £100 nominal, at issue, to provide a purchaser with an effective net rate of return of 6% per annum. The purchaser pays tax at a rate of 30% on coupon payments only.
(Institute/Faculty of Actuaries Examinations, September 2003) [2+5=7]

A tax-free investor purchased the bond on 1 March 2007, immediately after payment of the coupon then due, and sold the bond on 1 March 2012 immediately after payment of the coupon then due.

(iii) Calculate the gross annual rate of return achieved by the investor over this period.

(iv) Explain, without doing any further calculations, how your answer to part (iii) would change if the bond were due to be redeemed on 1 March 2035 (rather than 1 March 2025). You may assume that the gross redemption yield at both the date of purchase and the date of sale remains the same as in parts (i) and (ii) above.

(Institute/Faculty of Actuaries Examinations, April 2012) [3+1+2+3=9]

22. A fixed interest security pays coupons of 4% per annum, half-yearly in arrear and will be redeemed at par in exactly ten years.

(i) Calculate the price per £100 nominal to provide a gross redemption yield of 3% per annum convertible half yearly.

(ii) Calculate the price, 91 days later, to provide a net redemption yield of 3% per annum convertible half-yearly, if income tax is payable at 25%. *(Institute/Faculty of Actuaries Examinations, September 2013)* [2+2=4]

23. An investor is considering the purchase of two government bonds, issued by two countries A and B respectively, both denominated in euro.

Both bonds provide a capital repayment of €100 together with a final coupon payment of €6 in exactly one year. The investor believes that he will receive both payments from the bond issued by country A with certainty. He believes that there are four possible outcomes for the bond from country B, shown in the table below.

Outcome	*Probability*
No coupon or capital payment	0.1
Capital payment received, but no coupon payment received	0.2
50% of capital payment received, but no coupon payment received	0.3
Both coupon and capital payments received in full	0.4

The price of the bond issued by country A is €101.

(i) Calculate the price of the bond issued by country B to give the same expected return as that for the bond issued by country A.

(ii) Calculate the gross redemption yield from the bond issued by country B assuming that the price is as calculated in part (i).

(iii) Explain why the investor might require a higher expected return from the bond issued by country B than from the bond issued by country A. *(Institute/Faculty of Actuaries Examinations, September 2013)* [3+1+2=6]

3 Equity Calculations

3.1 Calculating the yield. Owners of equities receive a stream of dividend payments. The sizes of these dividend payments are uncertain and depend on the performance of the company. In the UK they are often paid six-monthly with an interim and a final dividend.

Consider the following special case of annual dividends (the results for six-monthly dividend payments are obtained in the same manner). Suppose the present price of a share just after the annual dividend has been paid is P. Suppose dividends are paid annually and d_k is the estimated gross dividend in k years time. Then the cash flow is displayed in the following table:

Time	0	1	2	3	...
Cash flow	$-P$	d_1	d_2	d_3	...

The yield i of the share given by:

$$P = \sum_{k=1}^{\infty} \frac{d_k}{(1+i)^k}$$

If dividends are assumed to grow by a constant proportion each year, then the cash flow is:

Time	0	1	2	3	...
Cash flow	$-P$	d_1	$d_1(1+g)$	$d_1(1+g)^2$	...

and so the yield i of the share given by:

$$P = \sum_{k=1}^{\infty} \frac{d_1(1+g)^{k-1}}{(1+i)^k} = \frac{d_1}{i-g}$$

Let $\nu = 1/(1+i)$ and $\alpha = fr/C$. Then

$$P = \sum_{k=1}^{n} \frac{fr}{(1+i)^k} + \frac{C}{(1+i)^n} = C\alpha \boldsymbol{a}_{\overline{n}|} + C\nu^n \tag{3.2c}$$

$$\begin{aligned}
\boldsymbol{d}_M(i) &= \frac{1}{P}\left[\sum_{k=1}^{n} \frac{kfr}{(1+i)^k} + \frac{Cn}{(1+i)^n}\right] = \frac{\alpha(\boldsymbol{Ia})_{\overline{n}|} + n\nu^n}{\alpha \boldsymbol{a}_{\overline{n}|} + \nu^n} \\
&= \frac{\alpha\nu(1-\nu^n - n(1-\nu)\nu^n) + (1-\nu)^2 n\nu^n}{\alpha\nu(1-\nu)(1-\nu^n) + (1-\nu)^2\nu^n} \\
&= \frac{1}{1-\nu} - \nu^n \frac{n\alpha\nu + 1 - n(1-\nu)}{\alpha\nu(1-\nu^n) + (1-\nu)\nu^n} && (3.2d) \\
&= 1 + \frac{1}{i} + \frac{n(i-\alpha) - (1+i)}{\alpha[(1+i)^n - 1] + i} && (3.2e)
\end{aligned}$$

after some algebra. Alternatively, use $P = fr\boldsymbol{a}_{\overline{n}|,i} + C\nu^n$. Hence

$$\frac{dP}{d\nu} = fr(1 + 2\nu + \cdots + n\nu^{n-1}) + nC\nu^{n-1} = \frac{fr(\boldsymbol{Ia})_{\overline{n}|} + nC\nu^n}{\nu}$$

and hence

$$-\frac{dP}{di} = -\frac{dP}{d\nu}\frac{d\nu}{di} = \nu^2\frac{dP}{d\nu} = \nu\left[fr(\boldsymbol{Ia})_{\overline{n}|} + nC\nu^n\right]$$

and so on.
(b) In this case, $C = P = f$; hence $\alpha = r$. Substituting these equalities in equation (3.2c) gives $\alpha = i$. Hence

$$\boldsymbol{d}_M(i) = \frac{1}{P}\left[\sum_{k=1}^{n} kfr\nu^k + Cn\nu^n\right] = i\sum_{k=1}^{n} k\nu^k + n\nu^n = \frac{1-\nu^n}{1-\nu}$$

Alternatively, just substitute $\alpha = i = (1-\nu)/\nu$ in equation (3.2d).
(c) The only payment is £100 at time n. Hence

$$\boldsymbol{d}_M(i) = \frac{100n\nu^n}{100\nu^n} = n$$ ❑

Example 3.2b. *Consider an n-year bond with face value f, redemption value C and coupon rate r payable half-yearly. Let i denote the effective annual yield and define $i^{(2)}$ in the usual way by $(1+i^{(2)}/2)^2 = 1+i$.*
(a) Find the Macaulay duration, $\boldsymbol{d}_M(i)$.
(b) Show that

$$\boldsymbol{d}_M(i) = -\left(1+\frac{i^{(2)}}{2}\right)\frac{1}{P}\frac{dP}{di^{(2)}}$$

Solution. Clearly,

$$P = \frac{fr}{2}\sum_{k=1}^{2n}\frac{1}{(1+i)^{k/2}} + \frac{C}{(1+i)^n}$$

Substituting into equation (3.2a) gives

$$\boldsymbol{d}_M(i) = \frac{1}{P}\left[\frac{fr}{2}\sum_{k=1}^{2n}\frac{k/2}{(1+i)^{k/2}} + \frac{Cn}{(1+i)^n}\right]$$

For part (b), note that

$$\frac{di}{di^{(2)}} = 1 + \frac{i^{(2)}}{2}$$

Using this result in equation (3.2b) gives

$$\boldsymbol{d}_M(i) = -(1+i)\frac{1}{P}\frac{dP}{di} = -\left(1+\frac{i^{(2)}}{2}\right)^2\frac{1}{P}\frac{dP}{di} = -\left(1+\frac{i^{(2)}}{2}\right)\frac{1}{P}\frac{dP}{di^{(2)}}$$ ❑

3.3 Effective duration or modified duration or volatility. Suppose i denotes the effective interest rate per annum. Then:

$$P = \sum_{k=1}^{n}\frac{c_{t_k}}{(1+i)^{t_k}}$$

The *effective duration*, $\boldsymbol{d}(i)$ or *volatility* or *modified duration* of the cash flow is defined to be

$$\boxed{\boldsymbol{d}(i) = -\frac{1}{P}\frac{dP}{di} = \frac{1}{P}\sum_{k=1}^{n}\frac{t_k c_{t_k}}{(1+i)^{t_k+1}}} \tag{3.3a}$$

where the last equality assumes that the values of the cash flows c_{t_j} do not depend on i.

Thus the effective duration is also a measure of the rate of change of the net present value with i which does not depend on the size of the net present value.

Note that

$$\boldsymbol{d}_M(i) = (1+i)\boldsymbol{d}(i)$$

Using the approximation $f(x+h) \approx f(x) + hf'(x)$ shows that if interest rates change from i to $i+\epsilon$ then

$$P(i+\epsilon) \approx \big(1 - \epsilon\boldsymbol{d}(i)\big)P(i)$$

and hence

$$\frac{P(i+\epsilon) - P(i)}{P(i)} \approx -\epsilon\boldsymbol{d}(i)$$

This leads to the following interpretation of the modified duration: *if the yield i decreases by ϵ then the price increases by the proportion $100\epsilon\boldsymbol{d}(i)$%.*

Example 3.3a. *Suppose the modified duration of a bond is equal to 8. Suppose further that the bond's yield increases by 10 basis points. (A basis point is 0.01%.) Find the approximate change in the price of the bond.*

Solution. The approximate change is $-\epsilon\boldsymbol{d}(i) = -0.001 \times 8 = -0.008$. It decreases by 0.8% approximately. ❑

3.4 Some results about duration.

- The Macaulay duration of a zero-coupon bond equals the time to maturity—by part (c) of example (3.2a).
- The duration increases as the yield decreases—see exercise 2 in section 4 below.
- The duration of a bond increases as the coupon rate decreases (other things being equal)—by equation (3.2e).

Duration is a measure of the riskiness of a bond; it is important for three reasons:

- it is a summary measure of the average payments which are due;
- it is a measure of the sensitivity of the price to interest rate changes;
- it is useful concept for immunising portfolios against interest rate changes.

Here are some standard examples:

Example 3.4a. *Find the Macaulay duration and the effective duration of a constant perpetuity:*

Time	0	1	2	3	...
*Cash flow, **c***	$-P(i)$	1	1	1	...

Solution. Now $P(i) = \nu + \nu^2 + \nu^3 + \cdots = \nu/(1-\nu)$. Hence:

$$\boldsymbol{d}_M(i) = \frac{1}{P}(\nu + 2\nu^2 + 3\nu^3 + \cdots) = \frac{1}{P}\frac{\nu}{(1-\nu)^2} = \frac{1}{1-\nu} = \frac{1+i}{i} \quad\text{and}\quad \boldsymbol{d}(i) = \frac{1}{i}$$ ❑

Example 3.4b. *Find the Macaulay duration of a constant annuity:*

Time	0	1	2	$\cdots$	$n-1$	n
*Cash Flow, **c***	$-P(i)$	1	1	$\cdots$	1	1

Solution. $P(i) = \nu + \nu^2 + \cdots + \nu^n$ and

$$\boldsymbol{d}_M(i) = \frac{\nu + 2\nu^2 + \cdots + n\nu^n}{\nu + \nu^2 + \cdots + \nu^n} = \frac{1}{1-\nu} - \frac{n\nu^n}{1-\nu^n} = \frac{1+i}{i} - \frac{n}{(1+i)^n - 1}$$

❑

Here are two memorable properties of the lognormal distribution:

• *The product of independent lognormals is lognormal.*
Suppose $Y_i \sim \text{lognormal}(\mu_i, \sigma_i^2)$ for $i = 1, \ldots, n$ and $Y_1, \ldots, Y_n$ are independent. Using the standard result about the distribution of the sum of independent normal random variables gives

$$\prod_{i=1}^n Y_i = Y_1 \times \cdots \times Y_n \sim \text{lognormal}\Big(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\Big)$$

In particular, if $Y_1, \ldots, Y_n$ are i.i.d. lognormal(μ, σ^2) then $\prod Y_i \sim \text{lognormal}(n\mu, n\sigma^2)$.

• *Mean and variance of a lognormal.*
If $Z \sim N(\mu, \sigma^2)$ then the moment generating function of Z is

$$\mathbf{E}[e^{tZ}] = e^{t\mu + \frac{1}{2}t^2\sigma^2} \quad \text{for all } t \in \mathbb{R}$$

If $Y \sim \text{lognormal}(\mu, \sigma^2)$ then $Y = e^Z$ where $Z \sim N(\mu, \sigma^2)$. Hence

$$\mathbf{E}[Y] = \mathbf{E}[e^Z] = e^{\mu + \frac{1}{2}\sigma^2} \quad \text{and} \quad \mathbf{E}[Y^2] = \mathbf{E}[e^{2Z}] = e^{2\mu + 2\sigma^2}$$

and so

$$\mathbf{var}[Y] = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

Example 5.3a. *Suppose £1 is invested at time 0 for n time units. Suppose further that the interest rate is i_j for the time period $(j-1, j)$ where $j = 1, \ldots, n$ and the $i_1, i_2, \ldots, i_n$ are independent and $1 + i_j \sim$ lognormal(μ_j, σ_j^2).*
(a) Let £S_n denote the accumulated amount at time n. Find the distribution of S_n.
(b) Let £V_n denote the present value of £1 due at the end of n years. Find the distribution of V_n.

Solution. (a) Now $S_n = (1+i_1)(1+i_2)\cdots(1+i_n)$ and $\ln S_n = \sum_{j=1}^n \ln(1+i_j)$.
But $\ln(1+i_j) \sim N(\mu_j, \sigma_j^2)$ and the $1+i_j$ are independent. Hence $\ln S_n \sim N(\sum_{j=1}^n \mu_j, \sum_{j=1}^n \sigma_j^2)$.
Hence

$$S_n \sim \text{lognormal}\Big(\sum_{j=1}^n \mu_j, \sum_{j=1}^n \sigma_j^2\Big)$$

(b) Now $V_n(1+i_1)(1+i_2)\cdots(1+i_n) = 1$ and hence $\ln V_n = -\sum_{j=1}^n \ln(1+i_j)$. Hence

$$V_n \sim \text{lognormal}\Big(-\sum_{j=1}^n \mu_j, \sum_{j=1}^n \sigma_j^2\Big)$$ ❑

Example 5.3b. *Suppose $Y : (\Omega, \mathscr{F}, \mathbf{P}) \to ((0,\infty), \mathscr{B}(0,\infty))$ has the lognormal distribution with parameters (μ, σ^2) and $\mathbf{E}[Y] = \alpha$ and $\mathbf{var}[Y] = \beta$. Express μ and σ^2 in terms of α and β.*
Solution. We know that $\alpha = e^{\mu + \frac{1}{2}\sigma^2}$ and $\beta = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$. Hence

$$\sigma^2 = \ln\left(1 + \frac{\beta}{\alpha^2}\right) \quad \text{and} \quad e^\mu = \frac{\alpha}{\sqrt{1 + \beta/\alpha^2}} = \frac{\alpha^2}{\sqrt{\beta + \alpha^2}} \quad \text{or} \quad \mu = \ln \frac{\alpha^2}{\sqrt{\beta + \alpha^2}}$$ ❑

6 Exercises

(exs6-3.tex)

1. $\ln(1 + i_t)$ is normally distributed with mean 0.06 and standard deviation 0.01. Find $\Pr[i_t \le 0.05]$.
(Institute/Faculty of Actuaries Examinations, September 1999) [3]

2. The rate of interest in any year has a mean of 6% and standard deviation of 1%. The yield in any year is independent of the yields in all previous years. Find the mean and standard deviation of the accumulated value at time 12 of an investment of £10 at time 0.
(Institute/Faculty of Actuaries Examinations, September 1997, adapted) [3]

3. An individual purchases £100,000 nominal of a bond on 1 January 2003 which is redeemable at 105 in 4 years' time and pays coupons of 4% per annum at the end of each year.
The investment manager wishes to invest the coupon payments on deposit until the bond is redeemed. It is assumed that the rate of interest at which the coupon payments can be invested is a random variable and the rate of interest in any one year is independent of that in any other year.

Deriving the necessary formulæ, calculate the mean value of the total accumulated investment on 31 December 2006 if the annual effective rate of interest has an expected value of 5½% in 2004, 6% in 2005 and 4½% in 2006.
(Institute/Faculty of Actuaries Examinations, April 2004) [5]

19. A company is adopting a particular investment strategy such that the expected annual effective rate of return from investments is 7% and the standard deviation of annual returns is 9%. Annual returns are independent and $(1 + i_t)$ is lognormally distributed where i_t is the return in the tth year. The company has received a premium of £1,000 and will pay the policyholder £1,400 after 10 years.

(i) Calculate the expected value and standard deviation of an investment of £1,000 over 10 years, deriving all the formulæ that you use.
(ii) Calculate the probability that the accumulation of the investment will be less than 50% of its expected value in 10 years' time.
(iii) The company has invested £1,200 to meet its liability in 10 years' time. Calculate the probability that it will have insufficient funds to meet its liability.

(Institute/Faculty of Actuaries Examinations, April, 2002) [9+8+3=20]

20. An investment fund provides annual rates of return, which are independent and identically distributed, with annual accumulation factors following the lognormal distribution with mean 1.04 and variance 0.02.

(i) An investor knows that she will have to make a payment of £5,000 in 5 years' time.
Calculate the amount of cash she should invest now in order that she has a 99% chance of having sufficient cash available in 5 years' time from the investment to meet the payment.
(ii) Comment on your answer to (i). *(Institute/Faculty of Actuaries Examinations, April 2004)* [9+3=12]

21. (i) In any year, the interest rate per annum effective on monies invested with a given bank has mean value j and standard deviation s and is independent of the interest rates in all previous years.
Let S_n be the accumulated amount after n years of a single investment of 1 at time $t = 0$.
(a) Show that $\mathbf{E}[S_n] = (1 + j)^n$.
(b) Show that $\mathbf{var}[S_n] = (1 + 2j + j^2 + s^2)^n - (1 + j)^{2n}$.

(ii) The interest rate per annum effective in (i), in any year, is equally likely to be i_1 or i_2 where $i_1 > i_2$. No other values are possible.
(a) Derive expressions for j and s^2 in terms of i_1 and i_2.
(b) The accumulated value at time $t = 25$ years of £1 million invested with the bank at time $t = 0$ has expected value £5.5 million and standard deviation £0.5 million. Calculate the values of i_1 and i_2.

(Institute/Faculty of Actuaries Examinations, April 2005) [5+8=13]

22. An actuarial student has created an interest rate model under which the annual effective rate of interest is assumed to be fixed over the whole of the next 10 years. The annual effective rate is assumed to be 2%, 4% and 7% with probabilities 0.25, 0.55 and 0.2 respectively.

(a) Calculate the expected accumulated value of an annuity of £800 per annum payable annually in advance over the next 10 years.
(b) Calculate the probability that the accumulated value will be greater than £10,000.

(Institute/Faculty of Actuaries Examinations, April 2006) [4]

23. The rate of interest is a random variable that is distributed with mean 0.07 and variance 0.016 in each of the next 10 years. The value taken by the rate of interest in any one year is independent of its value in any other year. Deriving all necessary formulæ, calculate:

(i) The expected accumulation at the end of 10 years, if one unit is invested at the beginning of 10 years.
(ii) The variance of the accumulation at the end of 10 years, if one unit is invested at the beginning of 10 years.
(iii) Explain how your answers in (i) and (ii) would differ if 1,000 units had been invested.

(Institute/Faculty of Actuaries Examinations, September 2006) [3+5+1=9]

24. £80,000 is invested in a bank account which pays interest at the end of each year. Interest is always reinvested in the account. The rate of interest is determined at the beginning of each year and remains unchanged until the beginning of the next year. The rate of interest applicable in any one year is independent of the rate applicable in any other year.

During the first year, the annual effective rate of interest will be one of 4%, 6% or 8% with equal probability.
During the second year, the annual effective rate of interest will be either 7% with probability 0.75 or 5% with probability 0.25.
During the third year, the annual effective rate of interest will be either 6% with probability 0.7 or 4% with probability 0.3.

(i) Derive the expected accumulated amount in the bank account at the end of 3 years.
(ii) Derive the variance of the accumulated amount in the bank account at the end of 3 years.
(iii) Calculate the probability that the accumulated amount in the bank account is more than £97,000 at the end of 3 years. *(Institute/Faculty of Actuaries Examinations, April 2007)* [5+8+3=16]

2 Forward Contracts

2.1 Historical background. The *spot price* for a commodity such as copper is the current price for immediate delivery. This will be determined by the current supply and demand.

Suppose a manufacturer requires a supply of copper. He cannot plan his production if he does not know how much he will have to pay in 6 months' time for the copper he then needs. Similarly, the copper supplier does not know how to plan his production if he does not know how much he will receive for the copper he produces in 6 months' time. Both supplier and consumer can reduce the uncertainty by agreeing a price today for the "6 months' copper".

2.2 Forward contracts. A *forward contract* is a legally binding contract to buy/sell an agreed quantity of an asset at an agreed price at an agreed time in the future. The contract is usually tailor-made between two financial institutions or between a financial institution and a client. Thus forward contracts are *over-the-counter* or *OTC*. Such contracts are not normally traded on an exchange.

Settlement of a forward contract occurs entirely at maturity and then the asset is normally delivered by the seller to the buyer.

2.3 Futures contracts. A *futures contract* is also a legally binding contract to buy/sell an agreed quantity of an asset at an agreed price at an agreed time in the future. Futures contracts can be traded on an exchange—this implies that futures contracts have standard sizes, delivery dates and, in the case of commodities, quality of the underlying asset. The underlying asset could be a financial asset (such as equities, bonds or currencies) or commodities (such as metals, wool, live cattle).

Suppose the contract specifies that A will buy the asset S from B at the price K at the future time T. Then B is said to hold a *short forward position* and A is said to hold a *long forward position*. Therefore, a futures contract says that the short side must deliver to the long side the asset S for the *exchange delivery settlement price (EDSP)*.

On maturity, physical delivery of the underlying asset is not normally made — the short side closes the contract with the long side by making a *cash settlement*.

Let S_T denote the actual price of the asset at time T.

If $K > S_T$ then the short side B makes the profit $K - S_T$.

If $K < S_T$ then the long side A makes the profit $S_T - K$.

With futures contracts, there is daily settlement—this process is called *mark to market*. At the time the contract is made, the long side is required to place a deposit called the *initial margin* in a *margin account*. At the end of each trading day, the balance in the margin account is calculated. If the amount falls below the *maintenance margin* then the long side must top-up the account; the long side is also allowed to withdraw any excess over the initial margin. Generally, interest is earned on the balance in the margin account. Similarly, the short side also has a margin account.

We assume in this chapter that there is no difference between the pricing of forward contracts and futures contracts.

> ***Example 2.3a.*** Suppose the date is May 1, 2000 and the current spot price of copper is £1,000 per tonne[1]. This is the current price of copper for immediate delivery. Suppose the current price of September copper is £1,100 per tonne. Suppose further that the standard futures contract for copper is for 25 tonnes. This means that the current price of a September copper futures contract is 25 × £1,100 = £27,500.
>
> Suppose that on May 1, A buys one September futures contract for copper from B. This means that A agrees on May 1 to pay £27,500 to B for 25 tonnes of copper on September 1 whilst B agrees to sell 25 tonnes of copper to A on September 1 for £27,500. ❑

[1] One tonne is 1,000 kilograms. For current information, see the web site of the London Metal Exchange: `www.lme.com`.

19. (i) If $0 \le t \le 9$ then $\int_0^t \delta(s)\,ds = 0.03t + 0.005t^2$. Hence $\int_0^9 \delta(s)\,ds = 0.27 + 0.405 = 0.675$. Hence for $t > 9$ we have $\int_0^t \delta(s)\,ds = 0.675 + 0.06(t-9) = 0.135 + 0.06t$.
Hence $\nu(t) = e^{-0.03t-0.005t^2}$ for $0 \le t \le 9$ and $\nu(t) = e^{-0.135-0.06t}$ for $t > 9$.
(ii)(a) We need $5{,}000\nu(15) = 5{,}000e^{-0.135-0.9} = 5{,}000e^{-1.035} = 1776.131905$ or £1,776.13.
(b) We need δ with $1{,}776.13e^{15\delta} = 5{,}000$. Hence $\delta = \ln(5{,}000/1{,}776.13)/15 = 0.06900007148965$ or 0.069.
(iii) Now $NPV = \int_{11}^{15} \rho(s)\nu(s)\,ds = \int_{11}^{15} 100e^{-0.02s}e^{-0.135-0.06s}\,ds = 100\int_{11}^{15} e^{-0.135-0.08s}\,ds = 100e^{-0.135}(e^{-0.88} - e^{-1.2})/0.08 = 124.055318$ or 124.055.

Answers to Exercises: Chapter 2 Section 6 on page 35

(exs2-2.tex)

1. The following function does more than the question requires. It also returns the interest rate and *NPV* just before the sign changes (if one exists).

```
"irr" <- function(vec.cashflow, r){
    n <- length(vec.cashflow)
    numr <- length(r)
    tmp <- outer(1/(1 + r), 0:(n - 1), FUN = "^")
    NPV <- c(tmp %*% vec.cashflow)
    signs <- sign(NPV)
    signchanges <- (1:numr)[(signs[ - numr] * signs[-1]) <= 0]
    if(length(signchanges) == 0)
        noroot <- T
    else {
        noroot <- F
        firstchange <- signchanges[1]
    }
    results <- cbind(interest = r, NPV = NPV)
    if(noroot) {
        list(results)
    }
    else list(table = results, interest = r[firstchange], NPV = NPV[firstchange])
}
```

2. Applying the bond yield to the cash flow of project A gives: $-50 + 8/1.07 + 8/1.07^2 + \cdots + 8/1.07^8 = -2.23$. This suggests that project A is not profitable.

3. Tke *IRR* is the solution for r of the equation

$$\sum_{j=0}^{m} \frac{c_j}{(1+i)^{t_j}} = \sum_{j=0}^{n} \frac{b_j}{(1+i)^{t'_j}}$$

Define the function $g : (-1,\infty) \to \mathbb{R}$ by

$$g(r) = \sum_{j=0}^{n} \frac{b_j}{(1+r)^{t'_j - t_m}} - \sum_{j=0}^{m} c_j(1+r)^{t_m - t_j}$$

Then g is continuous with $\lim_{r\to -1} g(r) = \infty$, $\lim_{r\to\infty} g(r) = -\infty$ and $g(r) \downarrow\downarrow$ as $r \uparrow\uparrow$ for $r \in (-1,\infty)$. Hence there exists a unque $r^* \in (-1,\infty)$ with $g(r^*) = 0$ and then

$$\sum_{j=0}^{m} \frac{c_j}{(1+r^*)^{t_j}} = \sum_{j=0}^{n} \frac{b_j}{(1+r^*)^{t'_j}}$$

4. Use units of £1,000. We have:

Time	0	8	9	10	11
Cash flow for project A (£ million)	-1,000	1,700	0	0	0
Cash flow for project B (£ million)	-1,000	1,000	321	229	245

(i) Let i' denote the required effective interest rate per annum and let $\nu = 1/(1+i')$. Then $1{,}700\nu^8 = 1{,}000\nu^8 + 321\nu^9 + 229\nu^{10} + 245\nu^{11}$ or $700(1+i)^3 - 321(1+i)^2 - 229(1+i) - 245 = 0$. So let $f(i) = 700(1+i)^3 - 321(1+i)^2 - 229(1+i) - 245$. Now the return on project A is 7%. So try $f(0.07) = -0.0128$ and $f(0.071) = 1.4774767$. Interpolation gives $i' = 0.07 + 0.001 \times 0.0128/(0.0128 + 1.4774767) = 0.7001$ or 7.00%.
(ii) The cash is received earlier with project A than with project B. Hence an interest rate higher than i' favours project A; an interest rate lower than i' favours project B.
In particular, if $i = 0.06$ then $NPV_A = -1{,}000 + 1{,}700/1.06^8 = 66.6010$ and $NPV_B = -1{,}000 + 1{,}000/1.06^8 + 321/1.06^9 + 229/1.06^{10} + 245/1.06^{11} = 74.3471$.

$$\text{Total } NPV \text{ of costs} = 0.2 + \int_2^{7.25} \nu^u\,du = 0.2 + \frac{\nu^{7.25} - \nu^2}{\ln \nu} = 3.613811$$

$$NPV \text{ of TV rights } = 0.3 \int_7^{7.25} \nu^u\,du = 0.3\frac{\nu^{7.25} - \nu^7}{\ln \nu} = 0.0380320$$

Now for the other revenue:

Year	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Time in years	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	11.5
Cash flow	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.8	0.6	0.4	0.2

This has *NPV*:

$$\begin{aligned}
NPV &= \frac{\nu^{0.5}}{10}\left[1 + 2\nu + 3\nu^2 + \cdots + 12\nu^{11} - \nu^8 - 4\nu^9 - 7\nu^{10} - 10\nu^{11}\right]\\
&= \frac{\nu^{0.5}}{10}\left[1 + 2\nu + 3\nu^2 + \cdots + 12\nu^{11} - \nu^8(1 + 4\nu + 7\nu^2 + 10\nu^3)\right]\\
&= \frac{\nu^{0.5}}{10}\left[\frac{1-\nu^{12}}{(1-\nu)^2} - \frac{12\nu^{12}}{1-\nu} - \nu^8\left(\frac{1-10\nu^4}{1-\nu} + 3\nu\frac{1-\nu^3}{(1-\nu)^2}\right)\right]\\
&= \frac{\nu^{0.5}}{10}\left[\frac{1 + 2\nu^{12} - 3\nu^9}{(1-\nu)^2} - \frac{2\nu^{12} + \nu^8}{1-\nu}\right] = 3.052965
\end{aligned}$$

Hence we need C with $C\nu^{11} = 3.613811 - 3.052965 - 0.038032$ leading to $C = 1.49165$ or £149.165 million.

13. Use units of £1,000. Let $\nu = 1/1.08$. For the discrete payments we have (in thousands) $NPV_O = 105 + 105\nu^{1/2} + 105\nu + 200\nu^{15}$. Let $\mathbf{N}_1$ denote the net present value of the continuous income for years 2, 3 and 4. Then

$$\begin{aligned}
\mathbf{N}_1 &= 20\int_1^2 \nu^t\,dt + 23\int_2^3 \nu^t\,dt + 26\int_3^4 \nu^t\,dt\\
&= \frac{20(\nu^2-\nu)}{\ln\nu} + \frac{23(\nu^3-\nu^2)}{\ln\nu} + \frac{26(\nu^4-\nu^3)}{\ln\nu} = \frac{\nu-1}{\ln\nu}\left[20\nu + 23\nu^2 + 26\nu^3\right]
\end{aligned}$$

For $k \in \{5, 6, \ldots, 30\}$, let $\mathbf{N}(k)$ denote the net present value of the continuous income for years 5, …, k. Let $\alpha = 1.03$. Then

$$\mathbf{N}(k) = \sum_{j=5}^k 29\alpha^{j-5}\int_{j-1}^j \nu^t\,dt = \frac{29(\nu-1)}{\ln\nu}\sum_{j=5}^k \alpha^{j-5}\nu^{j-1} = \frac{\nu-1}{\ln\nu}29\nu^4\frac{1-(\alpha\nu)^{k-4}}{1-\alpha\nu}$$

Let $NPV(k) = -NPV_O + \mathbf{N}_1 + \mathbf{N}(k)$. (i) We need $NPV(30) = -NPV_O + \mathbf{N}_1 + \mathbf{N}(30) = 4.30097626$. Answer is £4,300.98 (ii) Now $NPV(15) = -NPV_O + NPV_1 + NPV(15) = -129.57469246$ and even if we remove the £200,000 payment (which has net present value equal to £63,048.34) from NPV_O, this quantity is still negative. Also $NPV(2) < NPV(3) < \cdots < NPV(15)$ and so these are all negative. (iii) $NPV(29) = -1.97144191 < 0$. Hence the *DPP* occurs in the final year.

Answers to Exercises: Chapter 2 Section 8 on page 39

(exs2-3.tex)

1. We use equation (7.4a) which is:

$$(1+i)^t = \frac{f_{t_1-0}}{f_0}\frac{f_{t_2-0}}{f_{t_1}}\cdots\frac{f_{t_n-0}}{f_{t_{n-1}}}\frac{f_t}{f_{t_n}}$$

Now $t = 3$, $t_1 = 0.5$, etc. Hence

$$1.11^3 = \frac{460}{400}\times\frac{550}{510}\times\frac{500}{550}\times\frac{600}{540}\times\frac{650}{600}\times\frac{X}{710} = \frac{460}{400}\times\frac{500}{510}\times\frac{650}{540}\times\frac{X}{710}$$

and hence $X = 715.50062864$ or £715,500,629.

2. Using equation (7.4a) gives

$$(1+i)^3 = \frac{80}{120}\times\frac{200}{110}\times\frac{200}{210} = \frac{800}{693}$$

leading to $i = 0.049024$ or 4.90%.

3. The TWRR, i is given by

$$(1+i)^3 = \frac{212}{180}\frac{261}{237}\frac{230}{261}\frac{273}{248}\frac{295}{273}\frac{309}{311} = \frac{212}{180}\frac{230}{237}\frac{295}{248}\frac{309}{311} = 1.35086$$

Hence $i = 0.10544$ or 10.54%.

11. (i) $(2.9/2.3) \times (4.2/4.4) = 1.203557$ or 20.36%. (ii) We want i with $2.3(1+i) + 1.5(1+i)^{8/12} = 4.2$. So let $f(i) = 23(1+i) + 15(1+i)^{2/3} - 42$. Using the approximation $(1+i)^{2/3} \approx 1 + 2i/3$ gives $i \approx 4/33 = 0.12$. Then $f(0.12) = -0.062803$ and $f(0.13) = 0.263347$. Linear interpolation gives $i = 0.12 + 0.01 \times 0.062803/(0.263347 + 0.062803) = 0.121926$ or 12.2%. (iii) The fund performs well from January to May when the amount in the fund is (relatively) small and performs badly from May to December—when the amount in the fund is large. Hence the MWRR will be less than the TWRR, because the TWRR ignores the amount in the fund.

12. (a) Ignoring inflation we have

Time, t	1/4/2001	1/4/2002	1/4/2003	1/4/2006
Fund value at time t − 0	0 0	$1000 \times \frac{90}{80}$ $= 1125$	$1000\left(1+\frac{90}{80}\right)\frac{98}{90}$ $= 2313.89$	$1000\left[\frac{130}{66} + \frac{473}{360}\frac{160}{98}\right]$ $= 4114.82$
Net cash flow at time t	1000	1000	0	
Fund value at time t + 0	1000	$1000\left(1+\frac{90}{80}\right)$	$1000\left(1+\frac{90}{80}\right)\frac{98}{90}$	

Hence the real TWRR is given by

$$(1+i_1)^5 = \frac{98}{80} \times \frac{\frac{130}{66} + \frac{473}{360}\frac{160}{98}}{\left(1+\frac{90}{80}\right)\frac{98}{90}} \times \frac{1}{1.22} = \frac{1048}{561} \times \frac{1}{1.22} = 1.7856$$

This gives $i_1 = 0.12294104322$ or 12.294% per annum.
(b) The equation for the real MWRR is

$$1000 + \frac{1000\nu}{1.02} = 1000\left[\frac{130}{66} + \frac{473}{360}\frac{160}{98}\right]\frac{\nu^5}{1.22} = 4114.82\frac{\nu^5}{1.22}$$

or $19961\nu^5 - 5802.17647059\nu - 5918.22 = 0$. Substituting $\nu = 1/1.12294$ gives $LHS > 0$. Hence MWRR > TWRR. (It is 12.50%.)

13. Use units of £1 million.

Time, t	*1 January 2010*	*1 January 2011*	*1 July 2012*
Fund value at time t − 0	120	140	600
Net cash flow at time t	0	200	
Fund value at time t + 0	120	340	

(i) Let i denote the annual time weighted rate of return. Then

$$(1+i)^{5/2} = \frac{140}{120} \times \frac{600}{340} = \frac{35}{17}$$

Hence $i = 0.3348967$ or 33.49%. (ii) The rate of growth in the first year is 16.67% but 46% in the last 18 months. Hence the rate of growth is higher when the fund contains more money; hence the money weighted rate of return will be higher than the time weighted rate of return. (It is 38.87%.)

14. (i) MWRR affected by amounts and timings of cash flows which are not under the control of the fund manager. The equation for the MWRR may not have a unique or, indeed, any solution. *But*, TWRR requires knowledge of all the cash flows and the fund values at all the cash flow dates.
(ii) Denote answer by i. Then $(1+i)^2 = (45/41) \times (72/19)$ leading to $i = 0.17745183$ or 17.75%.

Answers to Exercises: Chapter 3 Section 3 on page 51

(exs3-1.tex)

1. (a) and (b) Use $s_{\overline{n}|} = \sum_{j=0}^{n-1}(1+i)^j$, $s_{\overline{n+1}|} = \sum_{j=0}^{n}(1+i)^j$ and $\ddot{s}_{\overline{n}|} = \sum_{j=1}^{n}(1+i)^j$.
(c) If $i = 0$, $\ddot{a}_{\overline{n}|} - a_{\overline{n-1}|} = n - (n-1) = 1$. If $i \neq 0$, $\ddot{a}_{\overline{n}|} - a_{\overline{n-1}|} = (1+i)(1-\nu^n)/i - (1-\nu^{n-1})/i = 1$.
(d) If $i = 0$, $ia_{\overline{n}|} + \nu^n = 0 + 1 = 1$. If $i \neq 0$, $ia_{\overline{n}|} + \nu^n = (1-\nu^n) + \nu^n = 1$.
(e) Use (d) and $ia_{\overline{n}|} = d\ddot{a}_{\overline{n}|}$. (f) If $i = 0$, $LHS = RHS = 1$. If $i \neq 0$, use equation 2.2a. (g) Use (f) and $is_{\overline{n}|} = d\ddot{s}_{\overline{n}|}$
Interpretations: (b) (Value of $n+1$ payments at time $n+1$) = (value of first n payments at time $n+1$) + 1(for payment at time $n+1$). (c) (value of n payments at time 1) = (value of $n-1$ payments at time 1)+1(payment at time 1). (d) Value at time 0 of loan of size 1, interest rate i with n annual interest payments. (e) As for (d), but interest payments in advance. (f) and (g) correspond to (d) and (e) but valued at time n.

2. Now $\delta = 0.06$; hence $1 + i = e^\delta$ and $i = 0.0618$. Either,

$$(\overline{Ia})_{\overline{10}|} = \frac{\overline{a}_{\overline{10}|} - 10\nu^{10}}{\delta} = \frac{ia_{\overline{10}|,i}/\delta - 10\nu^{10}}{\delta} = \frac{ia_{\overline{10}|,i}}{\delta^2} - \frac{10\nu^{10}}{\delta} = 33.86$$

or, from first principles:

$$(\overline{Ia})_{\overline{10}|} = \int_0^{10} t\nu^t\,dt = \left.\frac{t\nu^t}{\log\nu}\right|_0^{10} - \int_0^{10}\frac{\nu^t}{\log\nu}\,dt = \frac{10\nu^{10}}{\log\nu} - \left.\frac{\nu^t}{(\log\nu)^2}\right|_0^{10} = -\frac{10\nu^{10}}{\delta} + \frac{1-\nu^{10}}{\delta^2} = 33.86$$

3. The cash flow is:

Time	0	t_0	t_0+1/m	t_0+2/m	t_0+3/m	$\cdots$
Cash flow	$-P$	$(1-t_1)fr/m$	$(1-t_1)fr/m$	$(1-t_1)fr/m$	$(1-t_1)fr/m$	$\cdots$

$$P = \frac{(1-t_1)fr}{m}\left(\nu^{t_0}+\nu^{t_0+1/m}+\nu^{t_0+2/m}+\cdots\right) = \frac{(1-t_1)fr\nu^{t_0}}{m}\frac{1}{1-\nu^{1/m}} = (1-t_1)fr\nu^{t_0}\ddot{a}^{(m)}_{\overline{\infty}|}$$

Recall $(1-d^{(m)}/m)^m = 1-d = \nu$ and hence $d^{(m)} = m(1-\nu^{1/m})$. Hence an alternative expression is $P = (1-t_1)fr\nu^{t_0}/d^{(m)}$.

4. (a) Redemption is at the discretion of the bond issuer. A higher yield means that money is more expensive. This is good news for the purchaser of the bond (low price P and high coupons) but bad news for a bond issuer. The current yield on bonds is 10% p.a. and the coupons on the original bond are only 8% p.a. Hence, the bond issuer can make a profit by purchasing a new bond to pay the coupons on the existing bond rather than redeeming the existing bond. The bond issuer should retain the current bond—so the answer to part (a) is later.
(b) Use equation 1.5c. We assume no tax. Hence $P(n,i) = C + (g-i)Ca_{\overline{n}|,i} = C + (0.08-0.06)Ca_{\overline{n}|,i} = C\left[1+0.02a_{\overline{n}|,i}\right]$. Hence $n\nearrow$ implies $i\nearrow$. Hence later redemption implies means a higher yield.

5. The cash flow is as follows (where $t_1 = 8/365$):

Time	0	t_1	$t_1+{}^1/_2$	t_1+1	$t_1+{}^3/_2$	t_1+2	$\cdots$	$t_1+{}^{13}/_2$	t_1+7
Cash flow, $\boldsymbol{c}$	$-P$	0	4	4	4	4	$\cdots$	4	104

Because the bond is ex-dividend, the payment at time t_1 is zero. The purchase price at time t_1 should be $8a^{(2)}_{\overline{7}|}+100\nu^7$. Using tables gives

$$P = \frac{1}{1.06^{8/365}}\left[8a^{(2)}_{\overline{7}|}+100\nu^7\right] = \frac{8\times1.014782\times5.582381+66.5057}{1.06^{8/365}} = 111.682$$

Alternatively, if $x = 1/1.06^{0.5}$, we have

$$P = \frac{1}{1.06^{8/365}}4\left[x+x^2+\cdots+x^{13}+x^{14}+25x^{14}\right] = \frac{1}{1.06^{8/365}}4x\left[\frac{1-x^{14}}{1-x}+25x^{13}\right] = 111.682$$

6. Now each coupon after tax is $350\times0.75 = 262.50$. If the bond is redeemed at the final possible date, the cash flow is

Time	1/7/91	1/10/91	1/4/92	$\ldots$	1/10/2009	1/4/2010
Payment no.	0	1	2	$\ldots$	37	38
Cash flow	$-P$	262.50	262.50	$\ldots$	262.50	10,262.50

Now $i^{(2)} = 0.0591$ and $(1-t_1)g = 0.75\times0.07 = 0.0525$. As $i^{(2)} > (1-t_1)g$, assume bond will be redeemed at the latest possible date (and then yield will be at least 6% whatever the redemption date). Let $\nu = 1/1.06$. Hence

$$P = 262.50\sum_{k=0}^{37}\nu^{0.25+0.5k} + 10000\nu^{18.75}$$
$$= 262.50\nu^{0.25}\frac{1-\nu^{19}}{1-\nu^{1/2}} + 10000\nu^{18.75} = 9385.46$$

(ii) Now $i^{(2)} = 2\left(1.05^{1/2}-1\right) = 0.0494$ and $(1-t_1)g = 0.75\times0.07 = 0.0525$. As $i^{(2)} < (1-t_1)g$, the second purchaser should price the bond on the assumption it will be redeemed at the earliest possible date of 1/4/04 (and then yield will be at least 5% whatever the redemption date). Hence the cash flow for the second purchaser is as follows:

Time	1/4/99	1/10/99	1/4/00	$\ldots$	1/10/2003	1/4/2004
Payment no.	0	1	2	$\ldots$	9	10
Cash flow	$-P$	262.50	262.50	$\ldots$	262.50	10,262.50

Let $\nu = 1/1.05$. Hence

$$P = 262.50\sum_{k=1}^{10}\nu^{k/2} + 10000\nu^5 = 262.50\nu^{0.5}\frac{1-\nu^5}{1-\nu^{1/2}} + 10000\nu^5 = 10136.30$$

7.

Time (years)	0	1	2	$\ldots$	19	20
Cash flow	−96	4	4	$\ldots$	4	104

Hence $96 = 4a_{\overline{20}|} + 100\nu^{20}$ and so $24 = a_{\overline{20}|} + 25\nu^{20}$.
Using method (d) in paragraph 4.3 gives $i \approx \left[4+(100-96)/20\right]/96 = 0.044$. If $i = 0.04$, $RHS = 25.001$; if $i = 0.045$, $RHS = 23.374$. Using interpolation gives $(x-0.04)/(0.045-0.04) = (f(x)-f(0.04))/(f(0.045)-f(0.04))$ and hence $x = 0.04+0.005(24-25.001)/(23.374-25.001) = 0.043$. So answer is 4.3%.

8. Now $i = 0.08$ and so $i^{(2)} = 2[1.08^{1/2} - 1] = 0.078461$. Also $(1-t_1)g = 0.77 \times 95/110 = 0.665$. Hence $i^{(2)} > (1-t_1)g$ and so CGT is payable. Hence P is given by

$$P = 0.77 \times 9.5\boldsymbol{a}^{(2)}_{\overline{20}|,0.08} + 110\nu^{20} - 0.34(110 - P)\nu^{20}$$
$$= 0.77 \times 9.5\boldsymbol{a}^{(2)}_{\overline{20}|,0.08} + (72.6 + 0.34P)\nu^{20}$$

and so

$$P(1 - 0.34\nu^{20}) = 0.77 \times 9.5\boldsymbol{a}^{(2)}_{\overline{20}|,0.08} + 72.6\nu^{20}$$

which gives $P = 95.79$.

9. (i) Credit risk: most governments will not default. Low volatility risk. Income stream may be volatile relative to inflation. (ii) The cash flow is as follows:

Time (years)	0	1	2	3	4	5	6	7	8	9	10
Cash flow, before tax	$-P$	8	8	8	8	58	4	4	4	4	54

Let $\nu = 1/1.06$. Then $P = 8 \times 0.7\boldsymbol{a}_{\overline{5}|} + 50\nu^5 + 4 \times 0.7\boldsymbol{a}_{\overline{5}|}\nu^5 + 50\nu^{10} = (5.6 + 2.8\nu^5)\boldsymbol{a}_{\overline{5}|} + 50\nu^5(1 + \nu^5) = 97.6855$. Or, $P = 0.7 \times 4(\boldsymbol{a}_{\overline{5}|} + \boldsymbol{a}_{\overline{10}|}) + 50\nu^5(1 + \nu^5) = 97.6855$.

10. Now $i^{(4)} = 4[(1+i)^{1/4} - 1] = 4[1.04^{1/4} - 1] = 0.0394$. Also $(1-t_1)g = 0.8 \times 5/103 = 0.0388$. Hence $i^{(4)} > (1-t_1)g$ and so CGT is payable and the investor should assume latest possible redemption.
Using units of £1,000 we have $P = 5 \times 0.8\boldsymbol{a}^{(4)}_{\overline{20}|} + 103\nu^{20} - 0.25(103 - P)\nu^{20} = 4\boldsymbol{a}^{(4)}_{\overline{20}|} + 77.25\nu^{20} + 0.25P\nu^{20}$. Hence $(1 - 0.25\nu^{20})P = 4\boldsymbol{a}^{(4)}_{\overline{20}|} + 77.25\nu^{20}$ leading to $P = 102.07201$ or £102,072.01.

11. (i) Now $t_1 = 0.25$, $i^{(2)} = 2(1.05^{1/2} - 1) = 0.04939$, and $(1 - t_1)g = 0.75 \times 7/110 = 0.0477$. Hence $i^{(2)} > (1 - t_1)g$. Using the formula $P(n, i) = C + \left[(1 - t_1)g - i^{(m)}\right] C\boldsymbol{a}^{(m)}_{\overline{n}|,i}$ shows there is a capital gain.
(ii) Because $i^{(2)} > (1 - t_1)g$, it follows that the loan is least valuable to the investor if repayment is made by the borrower at the latest possible date.
(iii) Assuming the loan is redeemed at the latest possible date, $P = 0.75 \times 3.5[x + x^2 + \cdots + x^{30}] + [110 - 0.3(110 - P)]x^{30}$ where $x = 1/1.05^{1/2}$. Hence $P = 2.625x(1 - x^{30})/(1 - x) + [77 + 0.3P]x^{30}$. Hence $(1 - 0.3x^{30})P = 2.625x(1 - x^{30})/(1 - x) + 77x^{30}$ giving $P = 107.75380$ or £107,753.80.
Or, $P = (5.25\boldsymbol{a}^{(2)}_{\overline{15}|} + 77\nu^{15})/(1 - 0.3\nu^{15}) = 107.75383$ or £107,753.83.

12. (i) The term "gross redemption yield" means before tax. Let $\nu = 1/1.05$, then $P = 4\boldsymbol{a}^{(2)}_{\overline{15}|,0.05} + 100\nu^{15} = 4 \times 1.012348 \times 10.379658 + 100/1.05^{15} = 90.1330$.
(ii) (a) Now $i^{(2)} = 2[1.05^{1/2} - 1] = 0.049390$ and $(1 - t_1)g = 0.75 \times 4/100 = 0.03$. Hence $i^{(2)} > (1 - t_1)g$ and CGT is payable. Now let $\nu = 1/1.06$; then $P = 4 \times 0.75\boldsymbol{a}^{(2)}_{\overline{7}|,0.06} + 100\nu^7 - (100 - P) \times 0.4\nu^7$. Hence $(1 - 0.4\nu^7)P = 3\boldsymbol{a}^{(2)}_{\overline{7}|,0.06} + 60\nu^7$ leading to $P = 77.5203$.
(b) The cash flow was as follows:

Time (years)	0	0.5	1	1.5	2	2.5	3	3.5	$\cdots$	7.5	8
Cash flow before tax	-90.1330	2	2	2	2	2	2	2	$\cdots$	2	$2 + 77.5203$
Cash flow after tax	-90.1330	1.5	1.5	1.5	1.5	1.5	1.5	1.5	$\cdots$	1.5	$1.5 + 77.5203$

Let i denote the required effective rate of return per annum and let $x = 1/(1+i)^{1/2}$. Then $90.133 = 3\boldsymbol{a}_{\overline{8}|,i} + 77.5203\nu^8$ and we need to solve for i.
Using method (d) in paragraph 4.3 gives $i \approx \left[3 + (77.5203 - 90.133)/8\right]/90.133 = 0.016$.
Define $f(i) = 1.5x(1 - x^{16})/(1 - x) + 77.5203x^{16}$ where $x = 1/(1+i)^{1/2}$. Then $f(0.015) = 91.3573$ and $f(0.02) = 88.2486$. Using linear interpolation gives $(x - 0.015)/(90.133 - 91.3537) = (0.02 - 0.015)/(88.2486 - 91.3537)$ and hence $x = 0.015 + 0.005 \times 1.2207/3.1051 = 0.016966$. Hence the return is 1.7% to the nearest 0.1%. (More precisely, it is 1.694%.)

13. (i) Assuming £100 nominal is purchased, then price is

$$P_1 = 0.75 \times 10\boldsymbol{a}^{(2)}_{\overline{10}|,0.08} + \frac{110}{1.08^{10}} = 102.26$$

Hence he paid £102.26 per £100 nominal.
(ii) Now $(1 - t_1)g = 0.6 \times 10/110 = 6/110 = 0.054$ and $i^{(2)} = 2[(1 + i)^{1/2} - 1] = 2[1.06^{1/2} - 1] = 0.059$. Hence $i^{(2)} > (1 - t_1)g$ and so the investor is liable for CGT and the price is given by

$$P_2 = 0.6 \times 10\boldsymbol{a}^{(2)}_{\overline{2}|,0.06} + \frac{110 - 0.4(110 - P)}{1.06^2} = 6\boldsymbol{a}^{(2)}_{\overline{2}|,0.06} + \frac{66 + 0.4P}{1.06^2}$$

and hence $P_2 = 108.544$.
(iii) We need to find i with $P_1 = 7.5\boldsymbol{a}^{(2)}_{\overline{8}|,i} + P_2/(1 + i)^8$.
Using method (d) in paragraph 4.3 gives $i \approx \left[7.5 + (P_2 - P_1)/8\right]/P_1 = 0.081$.
Trying $i = 0.08$ gives $RHS = 102.588$. Trying $i = 0.085$ gives $RHS = 99.689$. Interpolating gives $i \approx 0.08 + 0.005(f(i) - f(0.08))/(f(0.085) - f(0.08)) = 0.08 + 0.005 \times (102.26 - 102.588)/(99.689 - 102.588) = 0.0806$ or 8.1% approximately.

9. Cash flow is as follows:

Time	0	1	2	3
Cash flow, unadjusted	−20	2	2	22
Inflation index, $Q(t)$	245.0	268.2	282.2	305.5
Cash flow, adjusted	−20	$\frac{2\times 245.0}{268.2}$	$\frac{2\times 245.0}{282.2}$	$\frac{22\times 245.0}{305.5}$

Hence

$$20 = \frac{2\times 245.0\nu}{268.2} + \frac{2\times 245.0\nu^2}{282.2} + \frac{22\times 245.0\nu^3}{305.5} = \frac{490\nu}{268.2} + \frac{490\nu^2}{282.2} + \frac{5390\nu^3}{305.5}$$

Inflation index goes from 245.0 to 305.5 in 3 years. Setting $(1+j)^3 = 305.5/245.0$ gives $j = 0.076$ as the average rate of inflation. General result $(1+i_R)(1+j) = 1+i_M$ where i_R is the real rate, j is the inflation rate and i_M is the money rate, leads to $i_R = 0.022$ as a first estimate of the real rate of return. (Or use approximation $\nu^k \approx 1-ki$.)

Trying $i = 0.022$ gives $\nu = 1/1.022$ and $RHS = 19.978$. Trying $i = 0.021$ gives $\nu = 1/1.022$ and $RHS = 20.032$. Interpolating gives $(x-0.021)/(0.022-0.021) = (f(x) - f(0.021)/(f(0.022) - f(0.021))$ and hence $x = 0.021 + 0.001(20-20.032)/(19.978-20.032) = 0.0216$ or 2.16%.

10. Effective real yield per annum,i_R, is given by $1+i_R = 1.0125^2$. Cash flow is:

Time	0	$10/12$	$22/12$	$34/12$	$\cdots$
Cash flow	0	5	5×1.03	5×1.03^2	$\cdots$
Inflation Index	1	$1.02^{10/12}$	$1.02^{22/12}$	$1.02^{34/12}$	$\cdots$

Hence, if $\beta = 1+i_R = 1.0125^2$, we have

$$x = \sum_{n=0}^{\infty} \frac{5\times 1.03^n}{1.02^{n+10/12}(1+i_R)^{n+10/12}} = \frac{5}{(1.02\beta)^{10/12}} \sum_{n=0}^{\infty} \frac{1.03^n}{1.02^n\beta^n} = 5\frac{(1.02\beta)^{1/6}}{1.02\beta - 1.03}$$

Hence $x = 321.68$ or £3.217.

11. Let $\nu_m = 1/(1+i_M)$ where i_M is the money rate of return. Hence

$$1 = \sum_{k=1}^{\infty} d(1+g)^{k-1}\nu^k = \frac{d\nu}{1-(1+g)\nu} = \frac{d}{i_M - g}$$

Hence $i_M = d+g$. Using the general result that $(1+i_R)(1+e) = 1+i_M$ where i_R is the real rate, e is the inflation rate and i_M is the money rate, we get $i_R = (d+g-e)/(1+e)$ as required.

12. Let $\alpha = 1.03$ and $\beta = 1.05$. The cash flow is as follows:

Time (in years)	0	0.75	1.75	2.75	3.75	$\cdots$
Cash flow, unadjusted	−250	10	10β	$10\beta^2$	$10\beta^3$	$\cdots$
Inflation factor	1	$\alpha^{0.75}$	$\alpha^{1.75}$	$\alpha^{2.75}$	$\alpha^{3.75}$	$\cdots$

Hence we require:

$$250 = \sum_{k=0}^{\infty} \frac{10\beta^k\nu^{0.75+k}}{\alpha^{0.75+k}} = \frac{10\nu^{0.75}}{\alpha^{0.75}} \sum_{k=0}^{\infty} \frac{\beta^k\nu^k}{\alpha^k} = \frac{10\nu^{0.75}}{1.03^{0.75}} \frac{1}{1-1.05\nu/1.03}$$

and this leads to

$$0 = 25\times 1.05\nu + 1.03^{0.25}\nu^{0.75} - 25\times 1.03 = 26.25\nu + 1.03^{0.25}\nu^{0.75} - 25.75$$

or, multiplying by $1+i$,

$$0 = 0.5 + 1.03^{0.25}(1+i)^{0.25} - 25.75i$$

We need an initial approximation.
Either, approximate $(1+i)^{0.25}$ by $1+0.25i$. This leads to $i \approx \left(0.5 + 1.03^{0.25}\right) / \left(25.75 - 0.25\times 1.03^{0.25}\right) = 0.059$. In fact, $(1+i)^{0.25} = 1 + 0.25i + \alpha$ where $\alpha > 0$; hence i is greater than the approximation 0.059.
Or, ignoring inflation and assuming the next dividend occurs in 12 months' time gives $250 = 10\nu + 10\beta\nu^2 + \cdots = 10\nu/(1-\beta\nu) = 10/(1+i_M-\beta) = 10/(i_M - 0.05)$. Hence $i_M \approx 0.05 + 1/25 = 0.09$. The approximation used implies that i_M is greater than 0.09.
Now use the general result that $(1+i_R)(1+e) = 1+i_M$ where i_R is the real rate, e is the inflation rate and i_M is the money rate. This gives $1+i_R = 1.09/1.03$ and hence $i_R \approx 0.058$. So i_R is slightly more than 0.058.

If $i = 0.058$ then $0.5 + 1.03^{0.25}(1+i)^{0.25} - 25.75i = 0.0282$.
If $i = 0.059$ then $0.5 + 1.03^{0.25}(1+i)^{0.25} - 25.75i = 0.0027$.
If $i = 0.0595$ then $0.5 + 1.03^{0.25}(1+i)^{0.25} - 25.75i = -0.010$.
Linear interpolation between 0.059 and 0.0595 gives $(x-0.059)/(0.0595-0.059) = (f(x) - f(0.059)/(f(0.0595) - f(0.059))$ and hence $x = 0.059 + 0.0005(0-0.0027)/(-0.010-0.0027) = 0.0591$ or 5.91%.

13. (i) Next expected dividend is d_1; rate of dividend growth is g; let $\nu = 1/(1+i)$. Then $P = \sum_{k=1}^{\infty} d_1(1+g)^{k-1}\nu^k = d_1\nu/(1-(1+g)\nu) = d_1/(i-g)$.
(ii) Analyst I has $g = 0$; hence $750 = 35/i$ and $i = 35/750 = 7/150 = 0.04667$ or 4.667%.
Analyst II has $g = 0.1$; hence $750 = 35/(i-0.1)$ leading to $i = 0.1 + 0.04667 = 0.14667$ or 14.667%.

14. See answer to exercise 12. Let $\alpha = 1.015$ and $\beta = 1.04$. The cash flow is as follows:

Time (in years)	0	0.25	1.25	2.25	3.25	...
Cash flow, unadjusted	-125	5	5β	$5\beta^2$	$5\beta^3$	...
Inflation factor	1	$\alpha^{0.25}$	$\alpha^{1.25}$	$\alpha^{2.25}$	$\alpha^{3.25}$	...

Hence we require:

$$125 = \sum_{k=0}^{\infty} \frac{5\beta^k \nu^{0.25+k}}{\alpha^{0.25+k}} = \frac{5\nu^{0.25}}{\alpha^{0.25}} \sum_{k=0}^{\infty} \frac{\beta^k \nu^k}{\alpha^k} = \frac{5\nu^{0.25}}{1.015^{0.25}} \frac{1}{1 - 1.04\nu/1.015}$$

and this leads to

$$0 = 26\nu + 1.015^{0.75}\nu^{0.25} - 25.375$$

or, multiplying by $1+i$,

$$0 = 0.625 + 1.015^{0.75}(1+i)^{0.75} - 25.375i$$

For an initial approximation, either of the methods in the answer to exercise 12 can be used. Using the first method means we approximate $(1+i)^{0.75}$ by $1 + 0.75i$. This leads to $0 = 0.625 + 1.015^{0.75}(1 + 0.75i) - 25.375i$ and hence $i = \left(0.625 + 1.015^{0.75}\right) / \left(25.375 - 0.75 \times 1.015^{0.75}\right) = 0.0664$. As in the answer to exercise 12, the value of i will be larger than this approximation.
If $i = 0.0664$, then $0.625 + 1.015^{0.75}(1+i)^{0.75} - 25.375i = 0.00128$. If $i = 0.067$, then $0.625 + 1.015^{0.75}(1+i)^{0.75} - 25.375i = -0.0135$. Linear interpolation between 0.0664 and 0.067 gives $(x - 0.0664)/(0.067 - 0.0664) = (f(x) - f(0.0664)/(f(0.067) - f(0.0664))$ and hence $x = 0.0664 + 0.0006(0 - 0.00128)/(-0.0135 - 0.00128) = 0.0665$ or 6.65%.

15. (i) The cash flow sequence is:

Time (in years)	0	4/12	10/12	16/12	...
Cash flow	$-P$	d_1	$d_1(1+g)^{1/2}$	$d_1(1+g)$	...

Hence

$$P = \sum_{k=0}^{\infty} \frac{d_1(1+g)^{k/2}}{(1+i)^{(4+6k)/12}} = \frac{d_1}{(1+i)^{4/12}} \sum_{k=0}^{\infty} \left(\frac{1+g}{1+i}\right)^{k/2} = \frac{d_1(1+i)^{1/6}}{(1+i)^{1/2} - (1+g)^{1/2}}$$

(ii) The cash flow sequence is

Time (in years)	0	2/12	8/12	14/12	20/12	...
Cash flow	$-£18$	d_1	$d_1(1+g)^{1/2}$	$d_1(1+g)$	$d_1(1+g)^{3/2}$	...

Hence

$$18 = \frac{d_1\nu^{2/12}}{1 - \nu^{1/2}(1+g)^{1/2}} = \frac{0.5(1+i)^{1/3}}{(1+i)^{1/2} - 1.04^{1/2}}$$

or

$$18(1+i)^{1/2} - 18 \times 1.04^{1/2} = 0.5(1+i)^{1/3}$$

Initial approximation: replace $(1+i)^{1/2}$ by $1 + i/2$ and $(1+i)^{1/3}$ by $1 + i/3$. This leads to $i = 0.096$, rounding down. As in the answer to exercise 12, the value of i will be larger than this approximation.
If $i = 0.096$, then $18(1+i)^{1/2} - 18 \times 1.04^{1/2} - 0.5(1+i)^{1/3} = -0.0278$.
If $i = 0.1$, then $18(1+i)^{1/2} - 18 \times 1.04^{1/2} - 0.5(1+i)^{1/3} = 0.0059$.
Hence answer lies between 9.6% and 10%, or 10% to nearest 1%.

16. (i) Are a share in the ownership of a company. Can be readily bought and sold. From point of view of investor: receive dividends and potential growth in share price. Size of dividends is not guaranteed. No fixed redemption time. Shareholders last in line to receive payment if company is liquidated. Shareholders can vote in AGM of company.

(ii) Denote the current price by P and the next dividend by d. Let $\alpha = 1.02$, $\beta = 1.03$ and $\nu = 1/(1+i)$ where $i = 0.05$. Then the cash flow is as follows:

Time (in years)	0	0.5	1.5	2.5	3.5	...
Cash flow	$-P$	d	$d\beta$	$d\beta^2$	$d\beta^3$	...
Inflation index	1	$\alpha^{0.5}$	$\alpha^{1.5}$	$\alpha^{2.5}$	$\alpha^{3.5}$	...

Hence

$$P = d\sum_{k=0}^{\infty} \frac{\beta^k \nu^{k+0.5}}{\alpha^{k+0.5}} = \frac{d\nu^{0.5}}{\alpha^{0.5}} \frac{1}{1 - \beta\nu/\alpha} = \frac{d\nu^{0.5}\alpha^{0.5}}{\alpha - \beta\nu} = \frac{d(1+i)^{0.5}\alpha^{0.5}}{\alpha(1+i) - \beta}$$

and hence

$$\frac{d}{P} = \frac{1.02 \times 1.05 - 1.03}{1.05^{0.5} \times 1.02^{0.5}} = 0.0396$$

31. Effective interest rate is i where $1+i = 1.02^2 = 1.0404$; hence $i = 0.0404$. Let $\nu = 1/(1+i)^{1/12} = 1/1.0404^{1/12}$. Let x be the required amount. Then we need $1.05^{20}x$ to equal the cost of the annuity which is

$$10{,}000\boldsymbol{a}^{(12)}_{\overline{25}|,i} = \frac{2500}{3}\nu\frac{1-\nu^{300}}{1-\nu} = \frac{2500}{3}\frac{1-1/1.0404^{25}}{1.0404^{1/12}-1} = 158422.3048754$$

or £158,422.30 and hence $x = 158{,}422.30/1.05^{20} = 59707.69872255$ or £59,707.70.
(ii) Now x grows to $1.05^{20}x$ in 20 years. Adjusting for inflation, x grows to $1.05^{20}x \times (143/340)$ in 20 years. If j denotes the annual effective real return, then $(1+j)^{20} = 1.05^{20}x \times (143/340)$ and hence $j = 0.0055002$ or 0.55%.
(iii) Capital gain is $x(1.05^{20}-1)$ and hence the tax is $0.25x(1.05^{20}-1)$ and the new payout is $x\left(3\times 1.05^{20}+1\right)/4$. Let j' denote the new real rate of return; then $(1+j')^{20} = (143/340)\times\left(3\times 1.05^{20}+1\right)/4$. Hence $j' = -0.002977411$ or -0.2977%. (iv) The tax is paid on the money capital gain. After deducting the tax, the remaining return is less than inflation which leads to a negative real return.

32. Adjusting for inflation, £95 grows to £100 × 220/222. Let i denote the required rate. Then $95(1+i)^{91/365} = 100\times 220/222$. Hence $i = \left((100\times 220)/(95\times 222)\right)^{365/91} - 1 = 0.18463896$ or 18.5%.

33. **Investment A.** After tax, the interest payments are £0.6 million per year. This is a net interest rate of 6%.
Investment B. After 10 years, we have 1.1^{10} million. The capital gains are $1.1^{10}-1$. Amount returned after tax is $1,1^{10} - 0.4(1 = .1^{10}-1) = 0.6\times 1.1^{10}+0.4$. So if i_B denotes the net rate of return, we have $(1+i_B)^{10} = 0.6\times 1.1^{10}+0.4$ which leads to $i_B = 0.06940531$ or 6.94%.
Investment C. Capital gains are $1.1^{10} - 1.04^{10}$. Hence, amount returned is $0.6\times 1.1^{10} + 0.4\times 1.04^{10}$. So if i_C denotes the net rate of return, we have $(1+i_C)^{10} = 0.6\times 1.1^{10} + 0.4\times 1.04^{10}$ which leads to $i_C = 0.079469$ or 7.95%.
(ii) Gross rate of return is 10% for all 3 investments. Clearly, net rate for Investment C is greater than that for Investment B because the capital gains are less and hence the tax is less. The net rate for Investment B is greater than that for Investment A because the tax is deferred until the end for Investment B and so greater compounding is achieved.

34. Let i be the effective money yield per annum and let $\nu = 1/(1+i)$; let $\alpha = 1.06$. Then we have $175 = 6\nu^{1/2} + 6\alpha\nu^{3/2} + 6\alpha^2\nu^{5/2} + \cdots = 6\nu^{1/2}\left[1 + \alpha\nu + (\alpha\nu)^2 + \cdots\right] = 6\nu^{1/2}/(1-\alpha\nu)$ which gives the equation $185.5\nu + 6\nu^{1/2} - 175 = 0$ or $175i - 6(1+i)^{1/2} - 10.5 = 0$. Approximating $(1+i)^{1/2}$ by $1 + {}^{i}\!/_{2}$ gives $i = 0.096$. Trying $i = 0.095$ gives LHS $= -0.1535349$ and trying $i = 0.096$ gives LHS $= 0.0185989$. Linear interpolation gives $i = 0.09589$ or 9.589%. Hence if i_R denotes the real rate, then $1 + i_R = 1.09589/1.04$ leading to $i_R = 0.0537$ or 5.37%.

Answers to Exercises: Chapter 6 Section 2 on page 105

(exs6-1.tex)

1. Let $f_{1,2}$ denote the required value. Then $1.08(1+f_{1,2})^2 = 1.095^3$. Hence $f_{1,2} = \sqrt{1.095^3/1.08} - 1 = 0.10258$ or 10.26%.

2. Let $f_{1,2}$ denote the answer. Then $1.045(1+f_{1,2})^2 = 1.055^3$. Hence $f_{1,2} = \sqrt{1.055^3/1.045} - 1 = 0.060036$ or 6.0036%.

3. For the par yield, we want the coupon rate, r which ensures that the bond price equals the face value. Hence we want:

Time	0	1	2	3
Cash Flow	-1	r	r	$1+r$

Hence

$$\begin{aligned}1 &= \frac{r}{1+y_1} + \frac{r}{(1+y_2)^2} + \frac{1+r}{(1+y_3)^2}\\ &= r\left(\frac{1}{1.06} + \frac{1}{1.06\times 1.065} + \frac{1}{1.06\times 1.065\times 1.07}\right) + \frac{1}{1.06\times 1.065\times 1.07}\end{aligned}$$

and hence

$$r = \frac{1.06\times 1.065\times 1.07 - 1}{1.065\times 1.07 + 1.07 + 1} = 0.06478$$

or 6.478%.

4. Let r denote the required answer. Then

$$1 = r\left(\frac{1}{1.06} + \frac{1}{1.065^2} + \frac{1}{1.06\times 1.066^2}\right) + \frac{1}{1.06\times 1.066^2}$$

and hence

$$r = \frac{1.06\times 1.066^2 - 1}{1.066^2 + 1.06\times 1.066^2/1.065^2 + 1} = 0.06395 \qquad \text{or } 6.395\%.$$

leading to

$$\begin{aligned} r &= \left[1 - \frac{1}{(1+y_3)^3}\right] \Big/ \left[\frac{1}{1+y_1} + \frac{1}{(1+y_2)^2} + \frac{1}{(1+y_3)^3}\right] \\ &= \left[1 - \frac{1}{(1.08-0.04e^{-0.3})^3}\right] \Big/ \left[\frac{1}{1.08-0.04e^{-0.1}} + \frac{1}{(1.08-0.04e^{-0.2})^2} + \frac{1}{(1.08-0.04e^{-0.3})^3}\right] \\ &= 0.050157 \end{aligned}$$

or 5.0157%.

19. (i) Now $(1+f_{2,2})^2 = (1+f_{2,1})(1+f_{3,1})$. This implies $1.05^2 = 1.045(1+f_{3,1})$ and hence $f_{3,1} = 1.05^2/1.045 - 1 = 0.0550239$ or 5.5024%. (ii) One year: $y_1 = 0.04$ or 4%. Two year: $y_2 = \sqrt{1.04 \times 1.0425} - 1 = 0.04124925$ or 4.125%. Three year: $y_3 = (1.04 \times 1.0425 \times 1.045)^{1/3} - 1 = 0.042498$ or 4.25%. Four year: $y_4 = (1.04 \times 1.0425 \times 1.05^2)^{1/4} - 1 = 0.0456155$ or 4.56%. (iii) Cash flow from bond is as follows:

Time	0	1	2	3	4
Cash flow	$-P$	3	3	3	103

Hence

$$P = \frac{3}{1.04} + \frac{3}{1.04 \times 1.0425} + \frac{3}{(1+y_3)^3} + \frac{103}{(1+y_4)^4} = 94.46813$$

Let i denote gross redemption yield. Hence

$$94.46813 = \frac{3}{1+i} + \frac{3}{(1+i)^2} + \frac{3}{(1+i)^3} + \frac{103}{(1+i)^4} = 3a_{\overline{4}|,i} + \frac{100}{(1+i)^4}$$

Let $f(i) = 3a_{\overline{4}|,i} + 100/(1+i)^4 - 94.46813$. Then $f(0.045) = 3 \times 3.587526 + 83.8561 - 94.46813 = 0.150548$ and $f(0.05) = 3 \times 3.545951 + 82.2702 - 94.46813 = -1.560077$. Interpolating gives $(i - 0.045)/(-0.150548) = 0.005/(-1.560077 - 0.150548)$ leading to $i = 0.04544$ or 4.544%. (iv) We have

$$\begin{aligned} &\frac{3}{1+i} + \frac{3}{(1+i)^2} + \frac{3}{(1+i)^3} + \frac{103}{(1+i)^4} \\ =&\frac{3}{1+f_{0,1}} + \frac{3}{(1+f_{0,1})(1+f_{1,1})} + \frac{3}{(1+f_{0,1})(1+f_{1,1})(1+f_{2,1})} + \frac{103}{(1+f_{0,1})(1+f_{1,1})(1+f_{2,1})(1+f_{3,1})} \end{aligned}$$

It follows that

$$\min\{f_{0,1}, f_{1,1}, f_{2,1}, f_{3,1}\} < i < \max\{f_{0,1}, f_{1,1}, f_{2,1}, f_{3,1}\}$$

The gross redemption yield, i, is a complicated average of the 1-year forward rates.

20. (i) The spot rates are $i_1 = a - b$ and $i_2 = a - 2b$. Also $f_{01} = 0.061$ and $f_{11} = 0.065$. Hence $a - b = 0.061$ and $(1+f_{01})(1+f_{11}) = (1+a-2b)^2$. Hence $b = 1.061 - \sqrt{1.061 \times 1.065} = 0.00199812$ and $a = 1.122 - \sqrt{1.061 \times 1.065} = 0.05900188$.
(ii) First we need i_4, the 4-year spot rate. Using the fact that the 4-year par yield is 0.07, we get

$$1 = 0.07\left(\frac{1}{1+a-b} + \frac{1}{(1+a-2b)^2} + \frac{1}{(1+a-3b)^3}\right) + \frac{1.07}{(1+i_4)^4}$$

Hence $i_4 = 0.07071303$. Then

$$P = 0.05\left(\frac{1}{1+a-b} + \frac{1}{(1+a-2b)^2} + \frac{1}{(1+a-3b)^3}\right) + \frac{1.08}{(1+i_4)^4} = 0.954501698798$$

or £0.9545 per £1 nominal.

21. (i) Let i denote the gross redemption yield and let $\nu = 1/(1+i)$. Then $103 = 6\nu + 6\nu^2 + 111\nu^3$. Spreading the capital gain of £2 equally over the 3 years, suggests $i \approx 6.7/103 = 0.065$. So try $i = 0.065$. Then $6\nu + 6\nu^2 + 111\nu^3 - 103 = -0.1849923$. Try $i = 0.06$; then $6\nu + 6\nu^2 + 111\nu^3 - 103 = 1.1980964$. Linear interpolation gives $i = 0.06 + 0.005 \times 1.1980964/(1.1980964 + 0.1849923) = 0.064331$ or 6.43%.
(ii) The 1 year spot rate, i_1 is given by $111 = (1+i_1)103$ and hence $i_1 = 0.0776699$ or 7.767%. The 2 year spot rate, i_2, is given by $103 = 6/(1+i_1) + 111/(1+i_2)^2$ and hence $i_2 = 111/\sqrt{103 \times 111 - 618} - 1 = 0.067357$ or 6.736%. The 3-year spot rate, i_3 is given by $103 = 6/(1+i_1) + 6/(1+i_2)^2 + 111/(1+i_3)^3$. Hence $i_3 = 0.0639414$ or 6.394%.
(iii) To answer to 3 decimal places, we need more than 3 decimal places for the spot rates. Now $f_{01} = i_1 = 0.07767$, $f_{02} = i_2 = 0.06736$ and $f_{03} = i_3 = 0.06394$.
Now $(1+i_1)(1+f_{12}) = (1+i_2)^2$ and hence $f_{12} = 0.05714286$ or 5.714%.
Now $(1+i_2)^2(1+f_{23}) = (1+i_3)^3$ and hence $f_{23} = 0.05714286$ or 5.714%.
Now $(1+i_1)(1+f_{13})^2 = (1+i_3)^3$ and hence $f_{13} = 0.05714286$ or or 5.714%.
In fact $f_{12} = f_{23} = f_{13}$.

22. (i) Let $y_n = 0.06 - 0.02e^{-0.1n}$. Then the price per £100 of the bond is $P = 3/(1+y_1) + 3/(1+y_2)^2 + 103/(1+y_3)^3 = 95.844762$. Let i denote the gross redemption yield. Then $P = 3/(1+i) + 3/(1+i)^2 + 103/(1+i)^3$. Now the capital gain is approximately £4.16 over 3 years. This suggests a yield of 3 + 1.4 = 4.4% approximately. So try $i = 0.044$. Let $f(i) = P - 3\nu - 3\nu^2 - 103\nu^3$ where $\nu = 1/(1+i)$. Then $f(0.044) = -0.299419$; $f(0.046) = 0.234823$. Using linear interpolation gives $i - 0.044/(-f(0.044)) = 0.002/(f(0.046) - f(0.044))$ and hence $i \approx 0.0451$ or 4.5%.
(ii) The 4-year par yield, r satisfies $1 = r/(1+y_1) + r/(1+y_2)^2 + r/(1+y_3)^3 + (1+r)/(1+y_4)^4$. Hence $r = 1 - 1/(1+y_4)^4)/(1/(1+y_1) + 1/(1+y_2)^2 + 1/(1+y_3)^3 + 1/(1+y_4)^4) = 0.046424$ or 4.64%.

Hence

$$\frac{d}{di}\boldsymbol{d}_M(i) = -\nu^2 \frac{\sum_{k=1}^n c_{t_k}\nu^{t_k} \sum_{k=1}^n t_k^2 c_{t_k}\nu^{t_k} - (\sum_{k=1}^n t_k c_{t_k}\nu^{t_k})^2}{(\sum_{k=1}^n c_{t_k}\nu^{t_k})^2}$$
$$= -\nu^2 \left[\frac{\sum_{k=1}^n t_k^2 c_{t_k}\nu^{t_k}}{\sum_{k=1}^n c_{t_k}\nu^{t_k}} - \left(\frac{\sum_{k=1}^n t_k c_{t_k}\nu_{t_k}}{\sum_{k=1}^n c_{t_k}\nu^{t_k}} \right)^2 \right]$$
$$= -\nu^2\sigma^2$$

and this quantity is clearly negative.

3. First the price:

$$P = 10\int_0^{20} \nu^t\,dt = 10 \left.\frac{\nu^t}{\ln\nu}\right|_0^{20} = \frac{10(\nu^{20}-1)}{\ln\nu}$$

Hence

$$\boldsymbol{d}_M(i) = \frac{10}{P}\int_0^{20} t\nu^t\,dt = \frac{10}{P}\left[\left.\frac{t\nu^t}{\ln\nu}\right|_0^{20} - \int_0^{20}\frac{\nu^t}{\ln\nu}\,dt \right] = \frac{10}{P}\left[\frac{20\nu^{20}}{\ln\nu} - \left.\frac{\nu^t}{(\ln\nu)^2}\right|_0^{20} \right]$$
$$= \frac{10}{P(\ln\nu)^2}\left[20\nu^{20}\ln\nu - \nu^{20} + 1\right] = \frac{1-\nu^{20}-20\nu^{20}\ln 1.04}{(1-\nu^{20})\ln 1.04} = 8.70586$$

or 8.706 years.

4. (a) We must have (1) $V_A(i_0) = V_L(i_0)$, the net present values are the same; (2) $\boldsymbol{d}_A(i_0) = \boldsymbol{d}_L(i_0)$, the effective durations are the same; and (3) $\boldsymbol{c}_A(i_0) > \boldsymbol{c}_L(i_0)$, the convexity of the assets is greater than the convexity of the liabilities.
(b) We have $\boldsymbol{d}_M(i) = (1+i)\boldsymbol{d}(i)$ where $\boldsymbol{d}_M$ is the duration and $\boldsymbol{d}$ is the volatility.
(c) Now $P(i) = \nu + \nu^2 + \cdots = \nu/(1-\nu) = 1/i$. Hence

$$\frac{dP(i)}{di} = -\frac{1}{i^2} \text{ and so } -\frac{1}{P}\frac{dP(i)}{di} = \frac{1}{i}$$

5. Let $\nu = 1/1.08$. We need to check

$$\sum_{k=1}^n a_{t_k}\nu^{t_k} = \sum_{k=1}^n l_{t_k}\nu^{t_k} \qquad \sum_{k=1}^n t_k a_{t_k}\nu^{t_k} = \sum_{k=1}^n t_k l_{t_k}\nu)^{t_k} \qquad \sum_{k=1}^n t_k^2 a_{t_k}\nu^{t_k} > \sum_{k=1}^n t_k^2 l_{t_k}\nu^{t_k}$$

First equality: $12.425\nu^{12} + 12.946\nu^{24} = 15\nu^{13} + 10\nu^{25}$. Both sides equal 6.9757.
Second equality: $12 \times 12.425\nu^{12} + 24 \times 12.946\nu^{24} = 13 \times 15\nu^{13} + 25 \times 10\nu^{25}$. Both sides equal 108.21.
Third relation: $lhs = 144 \times 12.425\nu^{12} + 576 \times 12.946\nu^{24} = 1886.46$ and $rhs = 169 \times 15\nu^{13} + 625 \times 10\nu^{25} = 1844.73$. hence we have the conditions for immunisation against small interest rate changes.

6. (i) For medium and long term investments. Unsecured. Bearer bonds. Usually one annual coupon. Issued in eurocurrency—currency owned by a non-resident of the country where the currency is legal tender.
(ii)(a) Now $97 = r\boldsymbol{a}_{\overline{20}|,0.05} + 100/1.05^{20}$. Hence $r = (97 - 100/1.05^{20})/\boldsymbol{a}_{\overline{20}|,0.05} = 4.75927$. (b) We have

$$\boldsymbol{d}_M(i) = \frac{1}{97}\left[r\sum_{k=1}^{20}\frac{k}{1.05^k} + \frac{100\times 20}{1.05^{20}} \right] = \frac{1}{97}\left[r(\boldsymbol{Ia})_{\overline{20}|,0.05} + \frac{100\times 20}{1.05^{20}} \right] = 13.2146691547$$

So the duration is 13.2147 years.

7. (i) Let $\nu = 1/1.07$.
First condition: present values are equal. For the assets we have $V_A = 7.404\nu^2 + 31.834\nu^{25} = 12.3323$.
For the liabilities we have $V_L = 10\nu^{10} + 20\nu^{15} = 12.3324$.
Second condition: the durations are the same.
For the assets we have

$$\sum \frac{t_k a_{t_k}}{(1+i)^{t_k}} = 2 \times 7.404\nu^2 + 25 \times 31.834\nu^{25} = 159.569$$

For the liabilities we have

$$\sum \frac{t_k \ell_{t_k}}{(1+i)^{t_k}} = 10 \times 10\nu^{10} + 15 \times 20\nu^{15} = 159.569$$

Hence the durations (which are these figures multiplied by ν/P) are equal.
(ii) Let $\nu_1 = 1/1.075$. Then profit is $7.404\nu_1^2 + 31.834\nu_1^{25} - 10\nu_1^{10} - 20\nu_1^{15} = 0.0158$.
(iii) The assets are clearly more spread out than the liabilities; hence all three of Redington's are satisfied and the company is immunised against small changes in the interest rate.

8. (i) Let $\nu = 1/1.07$. The discounted mean term is

$$\boldsymbol{d}_M(i) = \frac{1}{P}\sum \frac{t_k c_{t_k}}{(1+i)^{t_k}} = \frac{8 \times 87.5\nu^8 + 19 \times 157.5\nu^{19}}{87.5\nu^8 + 157.5\nu^{19}} = \frac{700 + 2992.5\nu^{11}}{87.5 + 157.5\nu^{11}} = 13.07061477$$

or 13.070615 years. The convexity is

$$\boldsymbol{c}(i) = \frac{1}{P}\sum \frac{t_k(t_k+1)c_{t_k}}{(1+i)^{t_k+2}} = \frac{8\times 9 \times 87.5\nu^{10} + 19 \times 20 \times 157.5\nu^{21}}{87.5\nu^8 + 157.5\nu^{19}} = \frac{6300\nu^2 + 59850\nu^{13}}{87.5 + 157.5\nu^{11}} = 186.8959854$$

20. (i)(a) and (b) in notes.
(ii) Present value of liabilities is:

$$P_L = 60\boldsymbol{a}_{\overline{9}|,0.07} + \frac{750}{1.07^{10}} = 772.176$$

Volatility of liabilities is:

$$V_L = \frac{60(\boldsymbol{Ia})_{\overline{9}|,0.07} + 10 \times 750\nu^{10}}{P_L}$$

Let y_A and y_B denote nominal amounts in bonds A and B respectively. Then net present value of assets is

$$y_A\nu^5 + y_B\nu^{20} = P_L$$

The volatility of the assets is

$$\frac{5y_A\nu^5 + 20y_B\nu^{20}}{P_L}$$

This leads to $y_B = 115.406$ or £115,406 and $y_A = 656.768$ or £656,768.
(iii) For Redington immunisation, have 3 conditions:
- value of assets = value of liabilities
- volatility of assets = volatility of liabilities
- convexity of assets $>$ convexity of liabilities

First two conditions are satisfied. Remains to check last condition, where convexity is given by:

$$\boldsymbol{c}(i) = \sum_{k=1}^{n} \frac{t_k(t_k+1)C_{t_k}}{(1+i)^{t_k+2}} \Bigg/ P$$

21. (i)(a) Present value of liabilities is $V_L = 100\boldsymbol{a}_{\overline{5}|,0.05} = 432.9477$.
(b) Let $\nu = 1/1.05$. Macaulay duration is $\boldsymbol{d}_M(i) = 100(\boldsymbol{Ia})_{\overline{5}|}/V_L = 100(\boldsymbol{a}_{\overline{5}|} - 5\nu^6)/(1-\nu)V_L = 2.90$ years.
(ii) Suppose purchase nominal y_A of 1-year bond and y_B of 5-year bond.
Then present value is $y_A\nu + y_B\nu^5 = V_L$. Macaulay duration is $(y_A\nu + 5y_B\nu^5)/V_L = \boldsymbol{d}_M(i)$. Hence $(y_A\nu + 5y_B\nu^5) = V_L\boldsymbol{d}_M(i) = 100(\boldsymbol{a}_{\overline{5}|} - 5\nu^6)/(1-\nu) = 1256.64$. Hence $4y_B\nu^5$ and so $y_B = 262.8158$. Hence $y_A = 238.3759$.
(iii) (a) Convexity of assets: $(2y_A\nu^3 + 30\nu_B\nu^7)/(y_A\nu + y_B\nu^5) = 13.89356$ or 13.89 years. (b) Assets more spread out than liabilities. Hence Redington's immunisation should have been achieved.
Check of convexity of liabilities: $100(2\nu + 6\nu^2 + 12\nu^3 + 20\nu^4 + 30\nu^5)/432.9477 = 12.08$. Hence convexity of assets is greater than convexity of liabilities.

22. For proof of first formula, see exercise 10 in section 3.3.
Liabilities (in 1,000s):

Time	0	1	2	3	$\cdots$	19	20
Cash flow	0	100	105	110	$\cdots$	190	195

(a) Present value $NPV = 95\boldsymbol{a}_{\overline{20}|,0.07} + 5(\boldsymbol{Ia})_{\overline{20}|,0.07} = 95\boldsymbol{a}_{\overline{20}|,0.07} + 5(\boldsymbol{a}_{\overline{20}|,0.07} - 20\nu^{21})/(1-\nu) = 1446.947$.
Duration: $\boldsymbol{d}_M(i) = \sum_{k=1}^{20}(95+5k)k\nu^k/NPV = (95(\boldsymbol{Ia})_{\overline{20}|,0.07} + 5\sum_{k=1}^{20} k^2\nu^k)/NPV = 13643.889/NPV = 9.429$ or 9.43 years.
(b) Suppose x_A and x_B denote nominal (or face values) of amounts on (A) and (B) respectively. Hence present value is $x_A\nu^{25} + 0.08x_B\boldsymbol{a}_{\overline{12}|,0.07} + x_B\nu^{12} = 1446.947$. Duration is $(25x_A\nu^{25} + 0.08x_B(\boldsymbol{Ia})_{\overline{12}|,0.07} + 12x_B\nu^{12})/1446.947 = 9.429$. We have two simultaneous equations for the two unknowns: x_A and x_B. Solving (take 25 $\times$ first equation minus second equation) gives $x_B = 1249.27$ and hence $x_A = 534.27$.

23. Cash flow:

Time	0	1/2	1	3/2	4/2	$\cdots$	39/2	40/2
Cash flow	0	3.75	3.75	3.75	3.75	$\cdots$	3.75	113.75

(i)(a) $P = 7.5\boldsymbol{a}^{(2)}_{\overline{20}|,0.1} + 110\nu^{20} = 7.5(i/i^{(2)})\boldsymbol{a}_{\overline{20}|,0.1} + 110\nu^{20} = 7.5 \times 1.024404 \times 8.513564 + 110/1.1^{20} = 81.76077$.
(b) Let $\boldsymbol{d}_M(i)$ denote the volatility. Hence

$$\boldsymbol{d}_M(i) \times P = \sum \frac{t_kC_{t_k}}{(1+i)^{t_k+1}} = \sum_{k=1}^{40} \frac{k}{2} 3.75\nu^{k/2+1} + 20 \times 110\nu^{21} = \frac{3.75\nu}{2}\sum_{k=1}^{40} k\nu^{k/2} + 2{,}200\nu^{21}$$

$$= \frac{3.75\nu}{2}\,\frac{2\boldsymbol{a}^{(2)}_{\overline{20}|} - 40\nu^{41/2}}{1-\nu^{1/2}} + 2{,}200\nu^{21}$$

and hence $\boldsymbol{d}_M(i) = 8.9104$.
(ii) Let x denote amount of each liability. Let t_1 denote time first liability is due. Then present value of liabilities is $x(\nu^{t_1} + \nu^{10}) = P = 81.76077$.
Let $\boldsymbol{d}^L_M$ denote volatility of the liabilities. Then $\boldsymbol{d}^L_M \times P = x(t_1\nu^{t_1+1} + 10\nu^{11})$.
Hence $(t_1\nu^{t_1+1} + 10\nu^{11})/(\nu^{t_1} + \nu^{10}) = 8.9104$ and substituting $t_1 = 9.61$ shows that this is the required value.
(b) Convexity:

$$\boldsymbol{c}(i) = \sum_{k=1}^{n} \frac{t_k(t_k+1)C_{t_k}}{(1+i)^{t_k+2}} \Bigg/ P = \frac{t_1(t_1+1)x\nu^{t_1+2} + 10 \times 11x\nu^{12}}{x\nu^{t_1} + x\nu^{10}} = 87.526$$

Payments which are more spread out will have greater convexity. In this case, liabilities are very close (9.61 and 10 years) and assets more spread out. Hence there should be immunisation.
(iii) Present value one year later (at time $t = 1$) is $7.5\boldsymbol{a}^{(2)}_{\overline{19}|,0.1} + 110\nu^{19} = 7.5 \times 1.024404 \times 8.36492 + 110/1.1^{19} = 82.254$.
Present value (times $t = 1$) of interest payments for next four years is $7.5\boldsymbol{a}^{(2)}_{\overline{4}|,0.1} = 24.354$.
Hence present value (time $t = 1$) of asset minus interest payments is 57.9. Hence forward price at time 5 (four years later) is $57.9 \times 1.1^4 = 84.77$.

24. (i) Let $\nu = 1/1.03$, $\alpha = 1.05$ and $\beta = \alpha\nu = 1.05/1.03$. Then

$$NPV = 100\left[\nu + \alpha\nu^2 + \alpha^2\nu^2 + \cdots + \alpha^{59}\nu^{60}\right] = \frac{100\nu(1-\beta^{60})}{1-\beta} = 10852.379704$$

(ii) Let x denote the nominal amount of the bond. Then for the assets

$$NPV = 0.04x\boldsymbol{a}_{\overline{20}|,0.03} + \frac{x}{1.03^{20}} = x\,[0.04 \times 14.877475 + 0.553676]$$

Hence $x = 9446.91494$ or £9,446.91494m.
(iii) Duration of liabilities:

$$\begin{aligned}\boldsymbol{d}_M(i) &= \frac{1}{NPV}\sum c_{t_k} t_k \nu^{t_k} = \frac{100}{NPV}\left[\nu + 2\alpha\nu^2 + 3\alpha^2\nu^3 + \cdots + 59\alpha^{58}\nu^{59} + 60\alpha^{59}\nu^{60}\right] \\ &= \frac{100\nu}{NPV}\left[1 + 2\beta + 3\beta^2 + \cdots + 59\beta^{58} + 60\beta^{59}\right] = \frac{100\nu}{NPV}\left[\frac{1-\beta^{60}}{(1-\beta)^2} - \frac{60\beta^{60}}{1-\beta}\right] = 36.143707\end{aligned}$$

or 36.1437 years.
(iv) Duration of the assets:

$$\begin{aligned}\frac{1}{NPV}\left[0.04x(1 + 2\nu + 3\nu^2 + \cdots + 20\nu^{20}) + 20x\nu^{20}\right] &= \frac{x}{NPV}\left[0.04\nu\left(\frac{1-\nu^{20}}{(1-\nu)^2} - \frac{20\nu^{20}}{1-\nu}\right) + 20\nu^{20}\right] \\ &= 14.572532\end{aligned}$$

or 14.57 years.
(v) For the liabilities, $\boldsymbol{d}(i) = \nu\boldsymbol{d}_M(i) = 35.090977$ and for the assets $\boldsymbol{d}(i) = 14.148089$. Hence if there is a change of 1.5% in the interest rate, then the liabilities change by $35 \times 1.5 = 52.5\%$ approximately whilst the assets change by $14.15 \times 1.5 = 21.2\%$ approximately. Thus the liabilities would increase by approximately $52.5 - 21.2 = 31.3\%$ of their original value more than the change in the value of the assets.

25. (i)(a) Now $P = 500\boldsymbol{a}_{\overline{20}|,0.08}$. Hence $\boldsymbol{d}_M(i) = (I\boldsymbol{a})_{\overline{20}|}/\boldsymbol{a}_{\overline{20}|} = (1.08/0.08) \times (9.818147 - 20/1.08^{21})/9.818147 = 8.03694757$ or 8.037 years. (i)(b) Let $\alpha = 1.08$ and $\nu = 1/1.08$. Then $P = 500\nu(1 + \alpha\nu + \cdots + \alpha^{19}\nu^{19}) = 500\nu \times 20$ and $\boldsymbol{d}_M(i) = 500\nu(1 + 2\alpha\nu + \cdots + 20\alpha^{19}\nu^{19})/P = (1 + 2 + 3 + \cdots + 20)/20 = 10.5$ or 10.5 years.
(ii) The duration is a measure of the mean term of the cash flows weighted by their present values. For (b), the later cash flows are higher than the corresponding cash flows in (a); hence the duration will be higher.

26. Let $\nu = 1/1.07$ and $\alpha = 1.05$.
First option.

$$\begin{aligned}\text{Outgoings} &= 0.25 + 0.1\nu + 0.2\nu^2 + \cdots + 0.9\nu^9 + \nu^{10} = 0.25 + \frac{1}{10}(\nu + 2\nu^2 + \cdots + 9\nu^9 + 10\nu^{10}) \\ &= 0.25 + \frac{1}{10}\left[\frac{\nu(1-\nu^{10})}{(1-\nu)^2} - \frac{10\nu^{11}}{1-\nu}\right]\end{aligned}$$

Incomings $= 0.5(\nu^8 + \nu^9 + \cdots + \nu^{27}) + 5\nu^{27} = 0.5\nu^8(1-\nu^{20})/(1-\nu) + 5\nu^{27}$. Then NPV = Incomings − Outgoings = 0.379448 or £379,448.
Second Option. $NPV = 0.21\nu[1 + \alpha\nu + \alpha^2\nu^2 + \cdots + \alpha^9\nu^9] + 5.64\nu^{10} - 4.2 = 0.21\nu(1 - \alpha^{10}\nu^{10})/(1 - \alpha\nu) + 5.64\nu^{10} - 4.2 = 0.472592$ or £472,592. Hence the second option has the larger NPV.
(ii) Recall the discounted mean term or Macaulay duration is mean term of the cash flows weighted by their present values: $\boldsymbol{d}_M(i) = \sum t_k c_{t_k}\nu^{t_k}/P$. Clearly the discounted mean term of the second option is less than 10 years. The discounted mean term of the first option, $\boldsymbol{d}_1$ is given by $\boldsymbol{d}_1 P = -0.1(\nu + 2^2\nu^2 + 3^2\nu^3 + \cdots + 10^2\nu^{10}) + 0.5(8\nu^8 + 9\nu^9 + \cdots + 27\nu^{27}) + 5 \times 27\nu^{27}$. The sum over the first 10 years (terms up to ν^{10}) is negative. Hence the discounted mean term must be larger than 10 years. (In fact, $\boldsymbol{d}_1 P = 49.404532$.)

27. Let $\nu = 1/1.08$. Use units of £1,000.
Present value of liabilities is $V_L = 400\nu^{10}$. Hence, amount held in cash is $40\nu^{10}$.
Let x denote nominal amount in the zero-coupon bond and let y denote the nominal amount in the fixed-interest stock. Hence the present value of the assets is $V_A = 40\nu^{10} + x\nu^{12} + 0.08y\boldsymbol{a}_{\overline{16}|,0.08} + 1.1y\nu^{16}$.
Second condition is $\sum_k t_k a_{t_k}\nu^{t_k} = \sum_k t_k \ell_{t_k}\nu^{t_k}$. The right hand side is $4{,}000\nu^{10}$ and the left hand side is $12x\nu^{12} + 0.08y(I\boldsymbol{a})_{\overline{16}|,0.08} + 1.1 \times 16y\nu^{16}$. This gives two equations for the two unknowns, x and y:

$$x\nu^{12} + 0.08y\boldsymbol{a}_{\overline{16}|,0.08} + 1.1y\nu^{16} = 360\nu^{10}$$

$$12x\nu^{12} + 0.08y(I\boldsymbol{a})_{\overline{16}|,0.08} + 17.6y\nu^{16} = 4{,}000\nu^{10}$$

(ii) The new expectation is $850 \times 1.035^5 = 1009.53336$ or £1,009,533.36. This is considerably higher. The new standard deviation is $850\sqrt{\mathbf{var}[S_5]} = 65.486418$ or £65,486.42 which is twice the previous amount. So there is greater probability of covering the liability of £1,000,000, but there is also a much higher probability of a large shortfall.

14. Let S_2 denote the amount after 2 years. Then $S_2 = 10{,}000(1 + I_1)(1 + I_2)$ where

$$I_1 = \begin{cases} 0.03 & \text{with probability } 1/3; \\ 0.04 & \text{with probability } 1/3; \\ 0.06 & \text{with probability } 1/3. \end{cases} \quad \text{and} \quad I_2 = \begin{cases} 0.05 & \text{with probability } 0.7; \\ 0.04 & \text{with probability } 0.3. \end{cases}$$

Using the independence of I_1 and I_2 gives $\mathbf{E}[S_2] = 10{,}000(1 + \mathbf{E}[I_1])(1 + \mathbf{E}[I_2]) = 10{,}000(1 + 0.13/3)(1.047) = 10{,}923.7$. (ii) $\mathbf{E}[S_2^2] = 10^8\mathbf{E}[(1+I_1)^2]\mathbf{E}[(1+I_2)^2] = 10^8\left[1 + 2 \times 0.13/3 + 0.0061/3\right]\left[1 + 2 \times 0.047 + 0.00223\right] = 10^8 \times 1.193465601$. Hence $\mathbf{var}[S_2] = 10^8(1.193465601 - 1.09237^2) = 19{,}338.41$

15. (i) $S_{10} = 1{,}000\Pi_{j=1}^{10}(1 + I_j)$. Now $\mathbf{E}[1 + I_1] = 1.06$. Hence $\mathbf{E}[S_{10}] = 1{,}000 \times 1.06^{10} = 1{,}790.85$.
(ii) Now $\mathbf{E}[(1 + I_1)^2] = 1 + 0.12 + 0.00392 = 1.12392$. Hence $\mathbf{E}[S_{10}^2] = 10^6 \times 1.12392^{10}$ and so $\mathbf{var}[S_{10}] = 10^6 \times [1.12392^{10} - 1.06^{20}] = 9{,}145.60$. Hence the standard deviation is 95.6326.
(iii) (a) Clearly $\mathbf{E}[I_1]$ is unchanged. Hence $\mathbf{E}[S_{10}]$ is unchanged. The variance of I_1 is smaller. Hence the standard deviation of S_{10} will become smaller. (b) Clearly both $\mathbf{E}[S_{10}]$ and the standard deviation of S_{10} will become larger.

16. The cash flow is as follows:

Time	0	1	2	3	...	9	10
Cash Flow (thousands)	150	20	20	20	...	20	20

Hence $A(10) = 150(1+i_1)\cdots(1+i_{10}) + 20(1+i_2)\cdots(1+i_{10}) + \cdots + 20$ where the $1+i_j$ are i.i.d. lognormal($\mu = 0.07, \sigma^2 = 0.006$). Let $\alpha = e^{\mu+\frac{1}{2}\sigma^2} = e^{0.073}$. Then $\mathbf{E}[A(10)] = 150\alpha^{10} + \sum_{j=9}^{0} 20\alpha^j = 150\alpha^{10} + 20(\alpha^{10} - 1)/(\alpha - 1) = 595.18471$ giving the answer £595,184.71.
(ii) Let x denote the required amount. We require $\mathbf{P}[x(1+i_1)\cdots(1+i_{10}) \geq 600] = 0.99$. Now $Z = (1+i_1)\cdots(1+i_{10}) \sim$ lognormal$(10\mu, 10\sigma^2)$ and hence $\ln Z \sim N(10\mu, 10\sigma^2)$. So we require $0.99 = \mathbf{P}[xZ \geq 600] = \mathbf{P}[\ln Z \geq \ln 600 - \ln x]$. Hence $0.99 = \mathbf{P}[(\ln Z - 10\mu)/\sqrt{10\sigma^2} \geq (\ln 600 - \ln x - 10\mu)/\sqrt{10\sigma^2}]$ where $(\ln Z - 10\mu)/\sqrt{10\sigma^2} \sim N(0, 1)$. Hence $(\ln 600 - \ln x - 10\mu)/\sqrt{10\sigma^2} = -2.326\,348$ or $\ln x = \ln 600 - 10\mu + 2.326\,348\sqrt{10\sigma^2}$ giving £526,771.15.

17. (i) $S_n = (1 + i_1)\cdots(1 + i_n)$ where $\mathbf{E}[1 + i_t] = 1 + j$, $\mathbf{var}[1 + i_t] = s^2$ and hence $\mathbf{E}[(1 + i_t)^2] = s^2 + (1 + j)^2$.
Hence $\mathbf{E}[S_n] = (1 + j)^n$ and $\mathbf{E}[S_n^2] = [s^2 + (1 + j)^2]^n$. Hence $\mathbf{var}[S_n] = [s^2 + (1 + j)^2]^n - (1 + j)^{2n}$.
(ii) Now $\ln(1 + i_t) \sim N(\mu = 0.08, \sigma = 0.04)$. Hence $\ln S_{16} \sim N(16\mu, 16\sigma^2)$.
Hence $\Pr[1000S_{16} > 4250] = \Pr[S_{16} > 4.25] = \Pr[\ln S_{16} > \ln 4.25] = \Pr[(\ln S_{16} - 16\mu)/(4\sigma) > (\ln 4.25 - 16\mu)/(4\sigma)] = 1 - \Phi((\ln 4.25 - 16\mu)/(4\sigma)) = 1 - \Phi((\ln 4.25 - 1.28)/0.16) = 1 - \Phi(1.0432436) = 0.1485$ by using interpolation on the tables.

18. (i) $S_n = (1 + i_1)\cdots(1 + i_n)$ where $i_1, \ldots, i_n$ are i.i.d with mean j and variance s^2. Hence $\mathbf{E}[S_n] = (1 + j)^n$ and $\mathbf{E}[S_n^2] = \mathbf{E}[(1 + i_1)^2]^n = [\mathbf{var}(1 + i_1) + \mathbf{E}(1 + i_1)^2]^n = [s^2 + (1 + j)^2]^n$. Hence $\mathbf{var}(S_n) = [s^2 + (1 + j)^2]^n - (1 + j)^{2n}$.
(ii) Suppose $j = 0.06$ and $s = 0.01$. Hence $s^2 = 0.0001$ and $Z = \ln(1 + i) \sim N(\mu, \sigma^2)$.
(a) $\mathbf{E}[1 + i] = e^{\mu+\frac{1}{2}\sigma^2} = 1.06$ and $s^2 = 0.0001 = \mathbf{var}[1 + i] = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$. Hence $e^{\sigma^2} = 0.0001/1.06^2 + 1$ and so $\sigma^2 = 0.00008996$ and $e^\mu = 1.059928$ and $\mu = 0.05822$.
(b) $S_{12} = (1 + i_1)\cdots(1 + i_{12})$. Hence $\ln S_{12} \sim N(12\mu, 12\sigma^2) = N(0.69869, 0.0010679)$.
(c) $\Pr[S_{12} > 2] = \Pr[\ln S_{12} > \ln 2] = \Pr[(\ln S_{12} - 0.69869)/\sigma > (\ln 2 - 0.69869)/0.032679] = 1 - \Phi(-0.1696) = \Phi(0.1696) = 0.5636 + 0.96 \times 0.0039 = 0.567$

19. Let $W_{10} = (1 + i_1)\cdots(1 + i_{10})$ where $i_1, i_2, \ldots, i_{10}$ are i.i.d. with expectation 0.07 and standard deviation 0.09. Now $\mathbf{E}[(1 + i_1)^2] = 1 + 2 \times 0.07 + 0.09^2 + 0.07^2 = 1.153$.
(i) Hence we are interested in $S_{10} = 1000W_{10}$. Now $\mathbf{E}[S_{10}] = 1000\mathbf{E}[W_{10}] = 1000 \times 1.07^{10} = 1967.15$. Using $\mathbf{E}[W_{10}^2] = 1.153^{10}$ gives the standard deviation of S_{10} to be $1000\sqrt{\mathbf{var}[W_{10}]} = 1000\sqrt{1.153^{10} - 1.07^{20}} = 531.66$.
(ii) We need the distribution of S_{10}. Now $\ln(1 + i_1)$ has a lognormal distribution. Define μ and σ by $\ln(1 + i_1) \sim N(\mu, \sigma^2)$. Hence $e^{\mu+\frac{1}{2}\sigma^2} = \mathbf{E}[1 + i_1] = 1.07$ and $e^{2\mu+2\sigma^2} = \mathbf{E}[(1 + i_1)^2] = 1.153$. Hence $e^{\sigma^2} = 1.153/1.07^2$ and so $\sigma^2 = 0.00705$. Hence $e^\mu = 1.07^2/\sqrt{1.153}$ and so $\mu = 0.064134$. It follows that $\ln(W_{10}) \sim N(10\mu, 10\sigma^2)$.
We want $\mathbf{P}[S_{10} < 0.5\mathbf{E}[S_{10}]] = \mathbf{P}[W_{10} < 0.5\mathbf{E}[W_{10}]] = \mathbf{P}[\ln(W_{10}) < \ln(0.5 \times 1.07^{10})] = \Phi[(\ln(0.5 \times 1.07^{10}) - 0.64134)/\sqrt{0.0705}] = \Phi(-2.4777974) = 0.00661$.
(iii) $\mathbf{P}[1200W_{10} < 1400] = \mathbf{P}[W_{10} < 7/6] = \mathbf{P}[\ln(W_{10}) < \ln(7/6)] = \Phi[(\ln(7/6) - 0.64134)/\sqrt{0.0705}] = \Phi[-1.835] = 0.0333$.

20. Suppose invest £x at time 0. Then value at time 5 is $A(5) = xS_5$ where $S_5 = R_1R_2R_3R_4R_5$ and $R_1, R_2, \ldots, R_5$ are i.i.d. lognormal with expectation 1.04 and variance 0.02. Using the method of example 5.3b gives $\ln R_k \sim N(\mu, \sigma^2)$ where $\sigma^2 = \ln(1 + 0.02/1.04^2) = 0.0183222$ and $\mu = \ln(1.04^2/\sqrt{0.02 + 1.04^2}) = 0.0300596$. Hence $\ln S_5 \sim N(5\mu, 5\sigma^2)$ or $\ln S_5 \sim N(0.150298, 0.091611)$.
(i) We want x with $0.01 = \mathbf{P}[xS_5 < 5{,}000] = \mathbf{P}[\ln S_5 < \ln 5{,}000 - \ln x] = \mathbf{P}[(\ln S_5 - 0.150298)/\sqrt{0.091611} < (\ln 5{,}000 - \ln x - 0.150298)/\sqrt{0.091611}]$. Hence $(\ln 5{,}000 - \ln x - 5\mu)/\sqrt{5\sigma^2} = \Phi^{-1}(0.01)$ and hence $x =$

16. We have, in pence,

$$1000 = \frac{50}{1.05^{1/12}} + \frac{50}{1.05^{7/12}} + \frac{K}{1.05^{11/12}}$$

Hence $K = 1.05^{4/12}(1000 \times 1.05^{7/12} - 50 \times 1.05^{6/12} - 50) = 942.84489$ or 9.43.

17. (i) Compare (A): buy asset at price B and (B): buy forward at price K and deposit $Ke^{-\delta T}$ in risk free asset. At time T, both lead to same value—ownership of the asset. Hence, by no arbitrage, $Ke^{-\delta T} = B$.
(ii) Let K denote the price of the forward contract. Then

$$200 - \frac{10}{1.02} = \frac{K}{1.02^2}$$

Hence $K = 200 \times 1.02^2 - 10 \times 1.02 = 197.88$ or £197.88.

18. *Note: this is a poorly phrased question!* (i) See page 129. (ii) Let $t = 0$ denote the time four years ago and let K_0 denote the price of the initial forward contract at that time. The price of the security was £7.20 at $t = 0$ with dividends of £1.20 at $t = 5, 6, 7, 8$ and 9. So we are comparing the following 2 cash flows:

Time	0	1	2	3	4	5	6	7	8	9
$\mathbf{c}_1$	-7.20	0	0	0	0	1.20	1.20	1.20	1.20	1.20
$\mathbf{c}_2$	$-K_0$									

Hence $K_0 = 7.20 - 1.2\boldsymbol{a}_{\overline{5}|,0.025}/1.025^4$. Let K_1 denote the price of a forward contract at $t = 4$. Then $K_1 = 10.45 - 1.2\boldsymbol{a}_{\overline{5}|,0.025}$. Hence the value of the original contract at $t = 4$ is $K_1 - 1.025^4 K_0 = 10.45 - 7.20 \times 1.025^4 = 2.502547$ or £2.50.
(iii) Assume 1 unit of the security grows to 1.03^t units of the security over the interval $(0, t)$. Then the price of the original forward contract at $t = 0$ is $7.2/1.03^9$ which has value $7.2 \times 1.025^4/1.03^9$ at $t = 4$. The price of a new forward contract at $t = 4$ is $10.45/1.03^5$. Hence the value of the original forward contract at $t = 4$ is $10.45/1.03^5 - 7.2 \times 1.025^4/1.03^9 = 2.923201$ or £2.92.

19. Let $\nu = 1/1.04^2$.

Date	1/9/12	1/12/12	1/3/13	1/6/13	1/7/13
Cash flow:	-10	1	1	1	K

So $10 = \nu^{1/4} + \nu^{1/2} + \nu^{3/4} + K\nu^{10/12}$. Hence $K = 7.595645867212$ or £7.5956.
(ii) The forward price is the price quoted today (1/9/12) for buying 1 share on 1/7/13. Whether or not the share price increases or decreases between 1/9/12 and 1/7/13 does not affect the forward price and indeed this information is unknown when the cost of the forward contract is determined. Owning a forward contract on 1/9/12 is like owning the share on 1/9/12 except that no dividends before 1/7/13 are received and there is no need to pay the seller of the share until 1/7/13.

20. (i)(a) A futures contract is a legally binding contract to buy or sell a specified quantity of an asset at a specified price at a specified time in the future. An option gives the right but not the obligation to buy or sell a specified quantity of an asset at a specified price at a specified time in the future.
(b) A call option gives the owner the right but not the obligation to buy a specified asset for a specified price at some specified time in the future. A put option gives the owner the right but not the obligation to sell a specified asset for a specified price at some specified time in the future.
(ii) We are comparing the following 2 cash flows:

Time	1/4/2013	30/9/2013	31/3/2014
$\mathbf{c}_1$	-10.50	1.10	1.10
$\mathbf{c}_2$			$-K$

Assuming no arbitrage, we must have $NPV(\mathbf{c}_1) = NPV(\mathbf{c}_2)$. Hence

$$-10.50 + \frac{1.1}{1.0225} + \frac{1.1}{1.025^2} = \frac{-K}{1.025^2}$$

Hence $K = 8.801306$ or £8.80.

21. Let K denote the forward price in pence. Then

$$180 = \frac{10}{1.04^{1/2}} + \frac{K}{1.04^{3/4}}$$

Hence $K = 1.04^{1/4}\left[180 \times 1.04^{1/2} - 10\right] = 175.274906066$ or 175.275p.

22. (i) See page 129. (ii) See section 2.6 on page 134. Equating time 0 values gives

$$Ke^{-0.09\times 0.75} = 6e^{-0.035\times 0.75} \quad \text{and hence} \quad K = 6e^{(0.09-0.035)\times 0.75} = 6.2526756 \quad \text{or £6.25268.}$$

(iii) Suppose that at time 0, the investor borrows the amount £$6e^{-0.035\times 0.75}$ for 9 months and uses this to buy the stock; he reinvests the dividends in purchasing further units of the stock. Hence at time 0, the investor has $e^{-0.035\times 0.75}$ units of the stock which grow to 1 unit after 9 months. Suppose further that at time 0, he enters into a forward contract to sell one unit of the stock at time 9 months for the amount £6.30.
After 9 months, the investor will owe £$6e^{-0.035\times 0.75} \times e^{0.09\times 0.75}$ = £$6e^{(0.09-0.035)\times 0.75}$ = £6.25268. Hence he makes a profit of £6.30 − £6.25268.