16.2.1 Linearity

The Laplace transform of a linear combination of two (or more) functions is equal to the linear combination of the respective Laplace transforms. Mathematically,

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathcal{L}\{f(t))\} + \beta\mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s). \tag{16.6}$$

This linear property easily follows from the linearity property for integrals.

Example 16.8

$$\mathcal{L}\{2t - 3\sin t\} = 2\mathcal{L}\{t\} - 3\mathcal{L}\{\sin t\} = \frac{2}{s^2} - \frac{3}{s^2+1}$$

using Table 16.1, entries #2 and #4 (with $b = 1$). ∎

16.2.2 Derivative formula

The derivative formula relates the Laplace transform of the derivative $f'(t)$ of a function $f(t)$ to the Laplace transform $F(s)$ of the function $f(t)$ itself, namely

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0). \tag{16.7}$$

The derivative formula (16.7) is recorded in Table 16.1 as entry #15.

Example 16.9

$$\mathcal{L}\{1\} = s\,\mathcal{L}\{t\} - 0$$

is easily verified using Table 16.1, entries #1 and #2 ($f(t) = t$). ∎

Example 16.10 For $f(t) = \sin(bt)$ we have $f'(t) = b\cos(bt)$ and $f(0) = 0$, so that

$$\mathcal{L}\{b\cos(bt)\} = s\,\mathcal{L}\{\sin(bt)\} - 0 = s\,\frac{b}{s^2+b^2}$$

using Table 16.1, entry #4. Using the linear property (16.6) we then obtain

$$b\mathcal{L}\{\cos(bt)\} = s\,\frac{b}{s^2+b^2},$$

i.e.,

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2},$$

which is recorded as entry #5 in Table 16.1. ∎

Example 16.11 For $f(t) = e^{at}$ we have $f'(t) = ae^{at}$ and $f(0) = 1$, so that

$$\mathcal{L}\{ae^{at}\} = s\,\mathcal{L}\{e^{at}\} - 1. \tag{16.8}$$

Using the linear property (16.6) and Table 16.1, entries #2 we verify that

$$a\,\frac{1}{s-a} = s\,\frac{1}{s-a} - 1$$

16.3 Solution of linear IPVs

Examples 16.11 and 16.12 show that the Laplace transform of certain functions can easily be obtained by using the linearity property (16.6) and the derivative formulas (16.7), (16.10). Here we formalize this idea by showing that the Laplace transform of the solution of the second-order constant coefficients linear IVP

$$\begin{cases} au'' + bu' + cu = g(t), \\ u(0) = \alpha, \\ u'(0) = \beta \qquad \text{(if } a \neq 0\text{)} \end{cases}$$

can be obtained this way.

16.3.1 Obtaining the Laplace transform of the solution

Apply the Laplace transform to both sides of the ODE and use linearity (#14) and the derivative formulas (#15, #16):

$$\begin{array}{ccccccc} au'' & + & bu' & + & cu & = & g(t) \\ & & & & & \downarrow \mathcal{L} & \\ a\,\boxed{\mathcal{L}\{u''\}} & + & b\,\boxed{\mathcal{L}\{u'\}} & + & c\,\boxed{\mathcal{L}\{u\}} & = & \boxed{\mathcal{L}\{g(t)\}} \end{array}$$

$$\mathcal{L}\{u''\}: \quad \boxed{\begin{aligned} &= s^2\mathcal{L}\{u\} - su(0) - u'(0) \\ &= s^2U(s) - \alpha s - \beta \end{aligned}}$$

$$\mathcal{L}\{u'\}: \quad \boxed{\begin{aligned} &= s\mathcal{L}\{u\} - u(0) \\ &= sU(s) - \alpha \end{aligned}}$$

$$\mathcal{L}\{u\}: \quad \boxed{= U(s)} \qquad \mathcal{L}\{g(t)\}: \quad \boxed{= G(s)}$$

Note how the ICs are used right away in the determination of $U(s)$. Collecting terms yields

$$(as^2 + bs + c)U(s) - a\alpha s - (a\beta + b\alpha) = G(s),$$

i.e.,

$$U(s) = \begin{array}{c} \dfrac{G(s)}{as^2 + bs + c} \\ \downarrow \\ U(s) \text{ with trivial ICs} \\ \alpha = 0,\ \beta = 0 \\ \text{(start at rest)} \end{array} + \begin{array}{c} \dfrac{a\alpha s + a\beta + b\alpha}{as^2 + bs + c} \\ \downarrow \\ U(s) \text{ with no forcing} \\ g(t) = 0,\ G(s) = 0 \\ \text{(HODE)} \end{array}. \tag{16.12}$$

The quantity $\dfrac{1}{as^2 + bs + c}$ is called a *transfer function.*

Example 16.13 Find $U(s)$ if $u(t)$ is the solution of the IVP

$$\begin{cases} u' = 2u, \\ u(0) = 1. \end{cases}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 2 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 10 \end{array}\right] \xrightarrow{r_1 \to \frac{1}{2}r_1} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 10 \end{array}\right]$$

$$\xrightarrow{\substack{r_2 \to r_2 - 3r_1 \\ r_3 \to r_3 - r_1}} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 2 & -\frac{1}{2} & -2 & 0 \\ 0 & 3 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 & 1 & 10 \end{array}\right] \xrightarrow{r_2 \to \frac{1}{2}r_2} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{1}{4} & -1 & 0 \\ 0 & 3 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & 1 & 1 & 10 \end{array}\right]$$

$$\xrightarrow{\substack{r_3 \to r_3 - 3r_2 \\ r_4 \to r_4 - r_2}} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{1}{4} & -1 & 0 \\ 0 & 0 & \frac{5}{4} & 4 & 0 \\ 0 & 0 & \frac{5}{4} & 2 & 10 \end{array}\right] \xrightarrow{r_3 \to \frac{4}{5}r_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{1}{4} & -1 & 0 \\ 0 & 0 & 1 & \frac{16}{5} & 0 \\ 0 & 0 & \frac{5}{4} & 2 & 10 \end{array}\right]$$

$$\xrightarrow{r_4 \to r_4 - \frac{5}{4}r_3} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{1}{4} & -1 & 0 \\ 0 & 0 & 1 & \frac{16}{5} & 0 \\ 0 & 0 & 0 & -2 & 10 \end{array}\right] \xrightarrow{r_4 \to -\frac{1}{2}r_4} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 1 & -\frac{1}{4} & -1 & 0 \\ 0 & 0 & 1 & \frac{16}{5} & 0 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]$$

$$\xrightarrow{\substack{r_1 \to r_1 - r_4 \\ r_2 \to r_2 + r_4 \\ r_3 \to r_3 - \frac{16}{5}r_4}} \left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{2} & 0 & 5 \\ 0 & 1 & -\frac{1}{4} & 0 & -5 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right] \xrightarrow{\substack{r_1 \to r_1 - \frac{1}{2}r_3 \\ r_2 \to r_2 + \frac{1}{4}r_3}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]$$

Therefore

$$a = -3, \quad b = -1, \quad c = 16, \quad d = -5,$$

(ii') The alternate approach of picking suitable values of s in (16.18) can right away deliver some of the coefficients:

$$\begin{aligned} s = -1: &\quad 10 = -2d \quad \Rightarrow \quad d = -5, \\ s = -\tfrac{1}{2}: &\quad 10 = \tfrac{5}{8}c \quad \Rightarrow \quad c = 16. \end{aligned}$$

Obtaining a and b (which correspond to a quadratic denominator $s^2 + 1$) would require substituting $s = i$ (so that $s^2 + 1 = 0$) and lead to calculations with complex numbers. Alternately we can substitute other interesting values:

$$\begin{aligned} s = 0: &\quad 10 = b + c + d && \Rightarrow \quad b = -1, \\ s = 1: &\quad 10 = 6(a + b) + 4c + 6d && \Rightarrow \quad a = -3. \end{aligned}$$

4. Inverse transform each term in the PFD of $F(s)$ using Table 16.1:

$$\begin{aligned} F(s) &= \frac{-3s - 1}{s^2 + 1} + \frac{16}{2s + 1} + \frac{-5}{s + 1} \\ &= -3\boxed{\frac{s}{s^2 + 1}} - \boxed{\frac{1}{s^2 + 1}} + 8\boxed{\frac{1}{s + \frac{1}{2}}} - 5\boxed{\frac{1}{s + 1}} \end{aligned}$$

$$\begin{array}{cccc} \downarrow \mathcal{L}^{-1} & \downarrow \mathcal{L}^{-1} & \downarrow \mathcal{L}^{-1} & \downarrow \mathcal{L}^{-1} \\ \cos t & \sin t & e^{-\frac{1}{2}t} & e^{-t} \\ (\#5) & (\#4) & (\#3) & (\#3) \end{array}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = -3\cos t - \sin t + 8e^{-\frac{1}{2}t} - 5e^{-t}.$$

here using a PFD of $U(s)$. From Example ... we have

$$U(s) = \frac{1}{s+1} + \frac{-s+1}{s^2+1} = \begin{array}{ccccc} \dfrac{1}{s+1} & + & \dfrac{-s}{s^2+1} & + & \dfrac{1}{s^2+1} \\ \downarrow \mathcal{L}^{-1} & & \downarrow \mathcal{L}^{-1} & & \downarrow \mathcal{L}^{-1} \\ e^{-t} & & -\cos(t) & & \sin(t) \\ (\#3) & & (\#5) & & (\#4) \end{array}$$

$$\Rightarrow u(t) = e^{-t} - \cos t + \sin t.$$

- [**mistake**] The Laplace transform of $f'(t)$ is NOT $F'(s)$ (but is $sF(s) - f(0)$).
- [**tip**] The solution of an IVP obtained using the Laplace transform can be checked using the IC(s). Remember that the ICs are used early in the solution process. Many errors can occur in the calculations until the final answer. Thus check whether your solution satisfies the ICs! (at least the first one, $u(0)$. For example the solution of the IVP (16.14) given by (16.17) satisfies

$$\begin{aligned} u(0) &= -3 - 0 + 8 - 4 = 1, \quad \checkmark \\ u'(0) &= 0 - 1 - 4 + 4 = -1. \quad \checkmark \end{aligned}$$

- [**tip**] The Laplace inversion process of a rational function $F(s)$ can be carried out without explicit knowledge of the coefficients of the PFD, provided it is set-up correctly. In Example 16.18 the set-up (16.17) yields

$$\begin{array}{rccccccc} F(s) = & \multicolumn{3}{c}{\dfrac{as+b}{s^2+1}} & + & \dfrac{c}{2s+1} & + & \dfrac{d}{s+1} \\ = & a\boxed{\dfrac{s}{s^2+1}} & + & b\boxed{\dfrac{1}{s^2+1}} & + & \dfrac{c}{2}\boxed{\dfrac{1}{s+\frac{1}{2}}} & + & d\boxed{\dfrac{1}{s+1}} \\ & \downarrow \mathcal{L}^{-1} & & \downarrow \mathcal{L}^{-1} & & \downarrow \mathcal{L}^{-1} & & \downarrow \mathcal{L}^{-1} \\ & \cos t & & \sin t & & e^{-\frac{1}{2}t} & & e^{-t} \\ & (\#5) & & (\#4) & & (\#3) & & (\#3) \end{array}$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\{F(s)\} = a\cos t + b\sin t + \frac{c}{2}e^{-\frac{1}{2}t} + de^{-t}.$$

So you can still proceed if you get stuck in the system solution for a, b, c, d.

- [**tip**] Large values of s correspond to small values of t. This remark is sometimes useful to check whether a Laplace transform or inverse transform makes sense. The diagram

$$\begin{array}{ccc} e^{at} & \xrightarrow{\mathcal{L}} & \dfrac{1}{s-a} \\ \approx \text{ for } t \simeq 0 \Big\downarrow & & \Big\downarrow \approx \text{ for } s \text{ large} \\ 1 & \xrightarrow{\mathcal{L}} & \dfrac{1}{s} \end{array}$$

illustrates this. If $s = i\omega$, large s means large ω, i.e., high frequency. A function which varies with a high frequency must be observed on a small time scale.