Engineering Mathematics

Since $3 = \frac{9}{3}$, then $3\frac{2}{3} = \frac{9}{3} + \frac{2}{3} = \frac{11}{3}$

Similarly, $2\frac{1}{6} = \frac{12}{6} + \frac{1}{6} = \frac{13}{6}$

Thus $3\frac{2}{3} - 2\frac{1}{6} = \frac{11}{3} - \frac{13}{6} = \frac{22}{6} - \frac{13}{6} = \frac{9}{6} = \mathbf{1\frac{1}{2}}$ as obtained previously.

Problem 3. Determine the value of $4\frac{5}{8} - 3\frac{1}{4} + 1\frac{2}{5}$

$$\begin{aligned}4\frac{5}{8} - 3\frac{1}{4} + 1\frac{2}{5} &= (4 - 3 + 1) + \left(\frac{5}{8} - \frac{1}{4} + \frac{2}{5}\right)\\ &= 2 + \frac{5\times5 - 10\times1 + 8\times2}{40}\\ &= 2 + \frac{25 - 10 + 16}{40}\\ &= 2 + \frac{31}{40} = \mathbf{2\frac{31}{40}}\end{aligned}$$

Problem 4. Find the value of $\frac{3}{7} \times \frac{14}{15}$

Dividing numerator and denominator by 3 gives:

$$\frac{{}^{1}\cancel{3}}{7} \times \frac{14}{\cancel{15}_{5}} = \frac{1}{7} \times \frac{14}{5} = \frac{1\times14}{7\times5}$$

Dividing numerator and denominator by 7 gives:

$$\frac{1\times\cancel{14}^{2}}{{}_{1}\cancel{7}\times5} = \frac{1\times2}{1\times5} = \mathbf{\frac{2}{5}}$$

This process of dividing both the numerator and denominator of a fraction by the same factor(s) is called **cancelling**.

Problem 5. Evaluate $1\frac{3}{5} \times 2\frac{1}{3} \times 3\frac{3}{7}$

Mixed numbers **must** be expressed as improper fractions before multiplication can be performed. Thus,

$$\begin{aligned}&1\frac{3}{5} \times 2\frac{1}{3} \times 3\frac{3}{7}\\ &= \left(\frac{5}{5} + \frac{3}{5}\right) \times \left(\frac{6}{3} + \frac{1}{3}\right) \times \left(\frac{21}{7} + \frac{3}{7}\right)\\ &= \frac{8}{5} \times \frac{{}^{1}\cancel{7}}{{}_{1}\cancel{3}} \times \frac{\cancel{24}^{8}}{\cancel{7}_{1}} = \frac{8\times1\times8}{5\times1\times1}\\ &= \frac{64}{5} = \mathbf{12\frac{4}{5}}\end{aligned}$$

Problem 6. Simplify $\frac{3}{7} \div \frac{12}{21}$

$$\frac{3}{7} \div \frac{12}{21} = \frac{\frac{3}{7}}{\frac{12}{21}}$$

Multiplying both numerator and denominator by the reciprocal of the denominator gives:

$$\frac{\frac{3}{7}}{\frac{12}{21}} = \frac{\frac{{}^{1}\cancel{3}}{{}_{1}\cancel{7}} \times \frac{\cancel{21}^{3}}{\cancel{12}_{4}}}{\frac{{}^{1}\cancel{12}}{{}_{1}\cancel{21}} \times \frac{\cancel{21}^{1}}{\cancel{12}_{1}}} = \frac{\frac{3}{4}}{1} = \mathbf{\frac{3}{4}}$$

This method can be remembered by the rule: invert the second fraction and change the operation from division to multiplication. Thus:

$\frac{3}{7} \div \frac{12}{21} = \frac{{}^{1}\cancel{3}}{{}_{1}\cancel{7}} \times \frac{\cancel{21}^{3}}{\cancel{12}_{4}} = \mathbf{\frac{3}{4}}$ as obtained previously.

Problem 7. Find the value of $5\frac{3}{5} \div 7\frac{1}{3}$

The mixed numbers must be expressed as improper fractions. Thus,

$$5\frac{3}{5} \div 7\frac{1}{3} = \frac{28}{5} \div \frac{22}{3} = \frac{{}^{14}\cancel{28}}{5} \times \frac{3}{\cancel{22}_{11}} = \mathbf{\frac{42}{55}}$$

Problem 8. Simplify

$\frac{1}{3} - \left(\frac{2}{5} + \frac{1}{4}\right) \div \left(\frac{3}{8} \times \frac{1}{3}\right)$

The order of precedence of operations for problems containing fractions is the same as that for integers, i.e. remembered by **BODMAS** (**B**rackets, **O**f, **D**ivision, **M**ultiplication, **A**ddition and **S**ubtraction). Thus,

$$\frac{1}{3} - \left(\frac{2}{5} + \frac{1}{4}\right) \div \left(\frac{3}{8} \times \frac{1}{3}\right)$$

The total number of parts is $3+7+11$, that is, 21. Hence 21 parts correspond to 273 cm

$$1 \text{ part corresponds to } \frac{273}{21} = 13 \text{ cm}$$

$$3 \text{ parts correspond to } 3 \times 13 = 39 \text{ cm}$$

$$7 \text{ parts correspond to } 7 \times 13 = 91 \text{ cm}$$

$$11 \text{ parts correspond to } 11 \times 13 = 143 \text{ cm}$$

i.e. **the lengths of the three pieces are 39 cm, 91 cm and 143 cm**.

(Check: $39+91+143=273$)

Problem 11. A gear wheel having 80 teeth is in mesh with a 25 tooth gear. What is the gear ratio?

$$\text{Gear ratio} = 80{:}25 = \frac{80}{25} = \frac{16}{5} = 3.2$$

i.e. gear ratio = **16 : 5** or **3.2 : 1**

Problem 12. An alloy is made up of metals A and B in the ratio 2.5 : 1 by mass. How much of A has to be added to 6 kg of B to make the alloy?

Ratio A : B: :2.5 : 1 (i.e. A is to B as 2.5 is to 1) or $\frac{A}{B} = \frac{2.5}{1} = 2.5$

When B = 6 kg, $\frac{A}{6} = 2.5$ from which,

$A = 6 \times 2.5 = \mathbf{15\,kg}$

Problem 13. If 3 people can complete a task in 4 hours, how long will it take 5 people to complete the same task, assuming the rate of work remains constant

The more the number of people, the more quickly the task is done, hence inverse proportion exists.

3 people complete the task in 4 hours,

1 person takes three times as long, i.e.

$4 \times 3 = 12$ hours,

5 people can do it in one fifth of the time that one person takes, that is $\frac{12}{5}$ hours or **2 hours 24 minutes**.

Now try the following exercise

Exercise 5 Further problems on ratio and proportion

1. Divide 621 cm in the ratio of 3 to 7 to 13. [81 cm to 189 cm to 351 cm]
2. When mixing a quantity of paints, dyes of four different colours are used in the ratio of 7:3:19:5. If the mass of the first dye used is $3\frac{1}{2}$ g, determine the total mass of the dyes used. [17 g]
3. Determine how much copper and how much zinc is needed to make a 99 kg brass ingot if they have to be in the proportions copper : zinc: :8 : 3 by mass. [72 kg : 27 kg]
4. It takes 21 hours for 12 men to resurface a stretch of road. Find how many men it takes to resurface a similar stretch of road in 50 hours 24 minutes, assuming the work rate remains constant. [5]
5. It takes 3 hours 15 minutes to fly from city A to city B at a constant speed. Find how long the journey takes if
   (a) the speed is $1\frac{1}{2}$ times that of the original speed and
   (b) if the speed is three-quarters of the original speed.
   [(a) 2 h 10 min (b) 4 h 20 min]

1.3 Decimals

The decimal system of numbers is based on the **digits** 0 to 9. A number such as 53.17 is called a **decimal fraction**, a decimal point separating the integer part, i.e. 53, from the fractional part, i.e. 0.17

2

Indices and standard form

2.1 Indices

The lowest factors of 2000 are $2\times2\times2\times2\times5\times5\times5$. These factors are written as $2^4 \times 5^3$, where 2 and 5 are called **bases** and the numbers 4 and 3 are called **indices**.

When an index is an integer it is called a **power**. Thus, 2^4 is called 'two to the power of four', and has a base of 2 and an index of 4. Similarly, 5^3 is called 'five to the power of 3' and has a base of 5 and an index of 3.

Special names may be used when the indices are 2 and 3, these being called 'squared' and 'cubed', respectively. Thus 7^2 is called 'seven squared' and 9^3 is called 'nine cubed'. When no index is shown, the power is 1, i.e. 2 means 2^1.

Reciprocal

The **reciprocal** of a number is when the index is -1 and its value is given by 1, divided by the base. Thus the reciprocal of 2 is 2^{-1} and its value is $\frac{1}{2}$ or 0.5. Similarly, the reciprocal of 5 is 5^{-1} which means $\frac{1}{5}$ or 0.2.

Square root

The **square root** of a number is when the index is $\frac{1}{2}$, and the square root of 2 is written as $2^{1/2}$ or $\sqrt{2}$. The value of a square root is the value of the base which when multiplied by itself gives the number. Since $3\times3 = 9$, then $\sqrt{9} = 3$. However, $(-3)\times(-3) = 9$, so $\sqrt{9} = -3$. There are always two answers when finding the square root of a number and this is shown by putting both a $+$ and a $-$ sign in front of the answer to a square root problem. Thus $\sqrt{9} = \pm3$ and $4^{1/2} = \sqrt{4} = \pm2$, and so on.

Laws of indices

When simplifying calculations involving indices, certain basic rules or laws can be applied, called the **laws of indices**. These are given below.

(i) When multiplying two or more numbers having the same base, the indices are added. Thus

$$3^2 \times 3^4 = 3^{2+4} = 3^6$$

(ii) When a number is divided by a number having the same base, the indices are subtracted. Thus

$$\frac{3^5}{3^2} = 3^{5-2} = 3^3$$

(iii) When a number which is raised to a power is raised to a further power, the indices are multiplied. Thus

$$(3^5)^2 = 3^{5\times2} = 3^{10}$$

(iv) When a number has an index of 0, its value is 1. Thus $3^0 = 1$

(v) A number raised to a negative power is the reciprocal of that number raised to a positive power. Thus $3^{-4} = \dfrac{1}{3^4}$ Similarly, $\dfrac{1}{2^{-3}} = 2^3$

(vi) When a number is raised to a fractional power the denominator of the fraction is the root of the number and the numerator is the power.

Thus $\quad 8^{2/3} = \sqrt[3]{8^2} = (2)^2 = 4$

and $\quad 25^{1/2} = \sqrt[2]{25^1} = \sqrt{25^1} = \pm5$

(Note that $\sqrt{\ } \equiv \sqrt[2]{\ }$)

2.2 Worked problems on indices

> Problem 1. Evaluate: (a) $5^2 \times 5^3$, (b) $3^2 \times 3^4 \times 3$ and (c) $2 \times 2^2 \times 2^5$

From law (i):

(a) $5^2\times5^3 = 5^{(2+3)} = 5^5 = 5\times5\times5\times5\times5 = \mathbf{3125}$

Table 3.1

Octal digit	Natural binary number
0	000
1	001
2	010
3	011
4	100
5	101
6	110
7	111

The '0' on the extreme left does not signify anything, thus $26.35_8 = 10\ 110.011\ 101_2$

Conversion of decimal to binary via octal is demonstrated in the following worked problems.

Problem 7. Convert 3714_{10} to a binary number, via octal

Dividing repeatedly by 8, and noting the remainder gives:

		Remainder
8	3714	
8	464	2
8	58	0
8	7	2
	0	7

7 2 0 2

From Table 3.1, $7202_8 = 111\ 010\ 000\ 010_2$

i.e. $\mathbf{3714_{10} = 111\ 010\ 000\ 010_2}$

Problem 8. Convert 0.59375_{10} to a binary number, via octal

Multiplying repeatedly by 8, and noting the integer values, gives:

$0.59375 \times 8 = 4.75$

$0.75 \times 8 = 6.00$

.4 6

Thus $0.59375_{10} = 0.46_8$

From Table 3.1, $0.46_8 = 0.100\ 110_2$

i.e. $\mathbf{0.59375_{10} = 0.100\ 11_2}$

Problem 9. Convert 5613.90625_{10} to a binary number, via octal

The integer part is repeatedly divided by 8, noting the remainder, giving:

		Remainder
8	5613	
8	701	5
8	87	5
8	10	7
8	1	2
	0	1

1 2 7 5 5

This octal number is converted to a binary number, (see Table 3.1)

$12755_8 = 001\ 010\ 111\ 101\ 101_2$

i.e. $5613_{10} = 1\ 010\ 111\ 101\ 101_2$

The fractional part is repeatedly multiplied by 8, and noting the integer part, giving:

$0.90625 \times 8 = 7.25$

$0.25 \times 8 = 2.00$

.7 2

This octal fraction is converted to a binary number, (see Table 3.1)

$0.72_8 = 0.111\ 010_2$

i.e. $0.90625_{10} = 0.111\ 01_2$

Thus, $\mathbf{5613.90625_{10} = 1\,010\,111\,101\,101.111\,01_2}$

Problem 10. Convert $11\ 110\ 011.100\ 01_2$ to a decimal number via octal

Grouping the binary number in three's from the binary point gives: $011\ 110\ 011.100\ 010_2$

Using Table 3.1 to convert this binary number to an octal number gives: 363.42_8 and

$$363.42_8 = 3 \times 8^2 + 6 \times 8^1 + 3 \times 8^0 + 4 \times 8^{-1} + 2 \times 8^{-2}$$

$$= 192 + 48 + 3 + 0.5 + 0.03125$$

$$= \mathbf{243.53125_{10}}$$

Now try the following exercise

Exercise 11 Further problems on conversion between decimal and binary numbers via octal

In Problems 1 to 3, convert the decimal numbers given to binary numbers, via octal.

1. (a) 343 (b) 572 (c) 1265

$\left[\begin{array}{ll}\text{(a) } 101010111_2 & \text{(b) } 1000111100_2 \\ \text{(c) } 10011110001_2 & \end{array}\right]$

2. (a) 0.46875 (b) 0.6875 (c) 0.71875

$\left[\begin{array}{ll}\text{(a) } 0.01111_2 & \text{(b) } 0.1011_2 \\ \text{(c) } 0.10111_2 & \end{array}\right]$

3. (a) 247.09375 (b) 514.4375 (c) 1716.78125

$\left[\begin{array}{l}\text{(a) } 11110111.00011_2 \\ \text{(b) } 1000000010.0111_2 \\ \text{(c) } 11010110100.11001_2\end{array}\right]$

4. Convert the following binary numbers to decimal numbers via octal:

(a) 111.011 1 (b) 101 001.01
(c) 1 110 011 011 010.001 1

$\left[\begin{array}{ll}\text{(a) } 7.4375_{10} & \text{(b) } 41.25_{10} \\ \text{(c) } 7386.1875_{10} & \end{array}\right]$

3.5 Hexadecimal numbers

The complexity of computers requires higher order numbering systems such as octal (base 8) and hexadecimal (base 16), which are merely extensions of the binary system. A **hexadecimal numbering system** has a radix of 16 and uses the following 16 distinct digits:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E and F

'A' corresponds to 10 in the denary system, B to 11, C to 12, and so on.

To convert from hexadecimal to decimal:

For example

$$1A_{16} = 1 \times 16^1 + A \times 16^0$$
$$= 1 \times 16^1 + 10 \times 1 = 16 + 10 = 26$$

i.e. $\mathbf{1A_{16} = 26_{10}}$

Similarly,

$$\mathbf{2E_{16}} = 2 \times 16^1 + E \times 16^0$$
$$= 2 \times 16^1 + 14 \times 16^0 = 32 + 14 = \mathbf{46_{10}}$$

and $\mathbf{1BF_{16}} = 1 \times 16^2 + B \times 16^1 + F \times 16^0$
$$= 1 \times 16^2 + 11 \times 16^1 + 15 \times 16^0$$
$$= 256 + 176 + 15 = \mathbf{447_{10}}$$

Table 3.2 compares decimal, binary, octal and hexadecimal numbers and shows, for example, that

$$23_{10} = 10111_2 = 27_8 = 17_{16}$$

Problem 11. Convert the following hexadecimal numbers into their decimal equivalents: (a) $7A_{16}$ (b) $3F_{16}$

(a) $7A_{16} = 7 \times 16^1 + A \times 16^0 = 7 \times 16 + 10 \times 1$
$= 112 + 10 = 122$

Thus $\mathbf{7A_{16} = 122_{10}}$

(b) $3F_{16} = 3 \times 16^1 + F \times 16^0 = 3 \times 16 + 15 \times 1$
$= 48 + 15 = 63$

Thus, $\mathbf{3F_{16} = 63_{10}}$

Problem 12. Convert the following hexadecimal numbers into their decimal equivalents: (a) $C9_{16}$ (b) BD_{16}

(a) $C9_{16} = C \times 16^1 + 9 \times 16^0 = 12 \times 16 + 9 \times 1$
$= 192 + 9 = 201$

Thus $\mathbf{C9_{16} = 201_{10}}$

(b) $BD_{16} = B \times 16^1 + D \times 16^0 = 11 \times 16 + 13 \times 1$
$= 176 + 13 = 189$

Thus $\mathbf{BD_{16} = 189_{10}}$

Problem 13. Convert $1A4E_{16}$ into a denary number

Table 3.2

Decimal	Binary	Octal	Hexadecimal
0	0000	0	0
1	0001	1	1
2	0010	2	2
3	0011	3	3
4	0100	4	4
5	0101	5	5
6	0110	6	6
7	0111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F
16	10000	20	10
17	10001	21	11
18	10010	22	12
19	10011	23	13
20	10100	24	14
21	10101	25	15
22	10110	26	16
23	10111	27	17
24	11000	30	18
25	11001	31	19
26	11010	32	1A
27	11011	33	1B
28	11100	34	1C
29	11101	35	1D
30	11110	36	1E
31	11111	37	1F
32	100000	40	20

$1A4E_{16}$

$$= 1 \times 16^3 + A \times 16^2 + 4 \times 16^1 + E \times 16^0$$
$$= 1 \times 16^3 + 10 \times 16^2 + 4 \times 16^1 + 14 \times 16^0$$
$$= 1 \times 4096 + 10 \times 256 + 4 \times 16 + 14 \times 1$$
$$= 4096 + 2560 + 64 + 14 = 6734$$

Thus, $\mathbf{1A4E_{16} = 6734_{10}}$

To convert from decimal to hexadecimal:

This is achieved by repeatedly dividing by 16 and noting the remainder at each stage, as shown below for 26_{10}

16 | 26 Remainder
16 | 1 $10 \equiv A_{16}$
0 $1 \equiv 1_{16}$

most significant bit → 1 A ← least significant bit

Hence $\mathbf{26_{10} = 1A_{16}}$

Similarly, for 447_{10}

16 | 447 Remainder
16 | 27 $15 \equiv F_{16}$
16 | 1 $11 \equiv B_{16}$
0 $1 \equiv 1_{16}$

1 B F

Thus $\mathbf{447_{10} = 1BF_{16}}$

Problem 14. Convert the following decimal numbers into their hexadecimal equivalents: (a) 37_{10} (b) 108_{10}

(a)
16 | 37 Remainder
16 | 2 $5 = 5_{16}$
0 $2 = 2_{16}$

most significant bit → 2 5 ← least significant bit

Hence $\mathbf{37_{10} = 25_{16}}$

(b)
16 | 108 Remainder
16 | 6 $12 = C_{16}$
0 $6 = 6_{16}$

6 C

Hence $\mathbf{108_{10} = 6C_{16}}$

Problem 15. Convert the following decimal numbers into their hexadecimal equivalents: (a) 162_{10} (b) 239_{10}

(a)
16 | 162 Remainder
16 | 10 $2 = 2_{16}$
0 $10 = A_{16}$

A 2

Hence $\mathbf{162_{10} = A2_{16}}$

(b)
16 | 239 Remainder
16 | 14 $15 = F_{16}$
0 $14 = E_{16}$

E F

Hence $\mathbf{239_{10} = EF_{16}}$

To convert from binary to hexadecimal:

The binary bits are arranged in groups of four, starting from right to left, and a hexadecimal symbol

Assignment 1

This assignment covers the material contained in Chapters 1 to 4. *The marks for each question are shown in brackets at the end of each question.*

1. Simplify (a) $2\frac{2}{3} \div 3\frac{1}{3}$

   (b) $\dfrac{1}{\left(\frac{4}{7} \times 2\frac{1}{4}\right)} \div \left(\frac{1}{3} + \frac{1}{5}\right) + 2\frac{7}{24}$ (9)

2. A piece of steel, 1.69 m long, is cut into three pieces in the ratio 2 to 5 to 6. Determine, in centimetres, the lengths of the three pieces. (4)

3. Evaluate $\dfrac{576.29}{19.3}$

   (a) correct to 4 significant figures

   (b) correct to 1 decimal place (2)

4. Determine, correct to 1 decimal places, 57% of 17.64 g (2)

5. Express 54.7 mm as a percentage of 1.15 m, correct to 3 significant figures. (3)

6. Evaluate the following:

   (a) $\dfrac{2^3 \times 2 \times 2^2}{2^4}$ (b) $\dfrac{(2^3 \times 16)^2}{(8 \times 2)^3}$

   (c) $\left(\dfrac{1}{4^2}\right)^{-1}$ (d) $(27)^{-\frac{1}{3}}$

   (e) $\dfrac{\left(\frac{3}{2}\right)^{-2} - \frac{2}{9}}{\left(\frac{2}{3}\right)^2}$ (14)

7. Express the following in standard form:

   (a) 1623 (b) 0.076 (c) $145\frac{2}{5}$ (3)

8. Determine the value of the following, giving the answer in standard form:

   (a) $5.9 \times 10^2 + 7.31 \times 10^2$

   (b) $2.75 \times 10^{-2} - 2.65 \times 10^{-3}$ (4)

9. Convert the following binary numbers to decimal form:

   (a) 1101 (b) 101101.0101 (5)

10. Convert the following decimal number to binary form:

    (a) 27 (b) 44.1875 (6)

11. Convert the following decimal numbers to binary, via octal:

    (a) 479 (b) 185.2890625 (6)

12. Convert (a) $5F_{16}$ into its decimal equivalent (b) 132_{10} into its hexadecimal equivalent (c) 110101011_2 into its hexadecimal equivalent (6)

13. Evaluate the following, each correct to 4 significant figures:

    (a) 61.22^2 (b) $\dfrac{1}{0.0419}$ (c) $\sqrt{0.0527}$ (3)

14. Evaluate the following, each correct to 2 decimal places:

    (a) $\left(\dfrac{36.2^2 \times 0.561}{27.8 \times 12.83}\right)^3$

    (b) $\sqrt{\dfrac{14.69^2}{\sqrt{17.42} \times 37.98}}$ (7)

15. If 1.6 km = 1 mile, determine the speed of 45 miles/hour in kilometres per hour. (3)

16. Evaluate B, correct to 3 significant figures, when $W = 7.20$, $v = 10.0$ and $g = 9.81$, given that $B = \dfrac{Wv^2}{2g}$. (3)

Let $\dfrac{3x-1}{(x-1)(x-2)} \equiv \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)}$

$$\equiv \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

Equating numerators gives:

$$3x-1 \equiv A(x-2)+B(x-1)$$

Let $x = 1$. Then $2 = -A$

i.e. $\mathbf{A = -2}$

Let $x = 2$. Then $\mathbf{5 = B}$

Hence $\dfrac{3x-1}{(x-1)(x-2)} \equiv \dfrac{-2}{(x-1)} + \dfrac{5}{(x-2)}$

Thus $\boldsymbol{\dfrac{x^2+1}{x^2-3x+2} \equiv 1 - \dfrac{2}{(x-1)} + \dfrac{5}{(x-2)}}$

Problem 4. Express $\dfrac{x^3-2x^2-4x-4}{x^2+x-2}$ in partial fractions

The numerator is of higher degree than the denominator. Thus dividing out gives:

$$\begin{array}{r l}
 & x-3 \\
x^2+x-2 \,\big) & x^3-2x^2-4x-4 \\
 & x^3+x^2-2x \\
\hline
 & -3x^2-2x-4 \\
 & -3x^2-3x+6 \\
\hline
 & x-10
\end{array}$$

Thus

$$\frac{x^3-2x^2-4x-4}{x^2+x-2} \equiv x-3+\frac{x-10}{x^2+x-2}$$

$$\equiv x-3+\frac{x-10}{(x+2)(x-1)}$$

Let $\dfrac{x-10}{(x+2)(x-1)} \equiv \dfrac{A}{(x+2)} + \dfrac{B}{(x-1)}$

$$\equiv \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

Equating the numerators gives:

$$x-10 \equiv A(x-1)+B(x+2)$$

Let $x = -2$. Then $-12 = -3A$

i.e. $\mathbf{A = 4}$

Let $x = 1$. Then $-9 = 3B$

i.e. $\mathbf{B = -3}$

Hence $\dfrac{x-10}{(x+2)(x-1)} \equiv \dfrac{4}{(x+2)} - \dfrac{3}{(x-1)}$

Thus $\boldsymbol{\dfrac{x^3-2x^2-4x-4}{x^2+x-2}}$

$$\boldsymbol{\equiv x-3+\frac{4}{(x+2)}-\frac{3}{(x-1)}}$$

Now try the following exercise

Exercise 25 Further problems on partial fractions with linear factors

Resolve the following into partial fractions:

1. $\dfrac{12}{x^2-9}$ $\quad\left[\dfrac{2}{(x-3)} - \dfrac{2}{(x+3)}\right]$

2. $\dfrac{4(x-4)}{x^2-2x-3}$ $\quad\left[\dfrac{5}{(x+1)} - \dfrac{1}{(x-3)}\right]$

3. $\dfrac{x^2-3x+6}{x(x-2)(x-1)}$ $\quad\left[\dfrac{3}{x} + \dfrac{2}{(x-2)} - \dfrac{4}{(x-1)}\right]$

4. $\dfrac{3(2x^2-8x-1)}{(x+4)(x+1)(2x-1)}$ $\quad\left[\dfrac{7}{(x+4)} - \dfrac{3}{(x+1)} - \dfrac{2}{(2x-1)}\right]$

5. $\dfrac{x^2+9x+8}{x^2+x-6}$ $\quad\left[1 + \dfrac{2}{(x+3)} + \dfrac{6}{(x-2)}\right]$

6. $\dfrac{x^2-x-14}{x^2-2x-3}$ $\quad\left[1 - \dfrac{2}{(x-3)} + \dfrac{3}{(x+1)}\right]$

7. $\dfrac{3x^3-2x^2-16x+20}{(x-2)(x+2)}$ $\quad\left[3x-2+\dfrac{1}{(x-2)} - \dfrac{5}{(x+2)}\right]$

Assignment 2

This assignment covers the material contained in Chapters 5 to 8. *The marks for each question are shown in brackets at the end of each question.*

1. Evaluate: $3xy^2z^3 - 2yz$ when $x = \frac{4}{3}$, $y = 2$ and $z = \frac{1}{2}$ (3)

2. Simplify the following:

   (a) $\dfrac{8a^2b\sqrt{c^3}}{(2a)^2\sqrt{b}\sqrt{c}}$

   (b) $3x + 4 \div 2x + 5 \times 2 - 4x$ (6)

3. Remove the brackets in the following expressions and simplify:

   (a) $(2x - y)^2$

   (b) $4ab - [3\{2(4a - b) + b(2 - a)\}]$ (5)

4. Factorise: $3x^2y + 9xy^2 + 6xy^3$ (3)

5. If x is inversely proportional to y and $x = 12$ when $y = 0.4$, determine

   (a) the value of x when y is 3, and

   (b) the value of y when $x = 2$. (4)

6. Factorise $x^3 + 4x^2 + x - 6$ using the factor theorem. Hence solve the equation $x^3 + 4x^2 + x - 6 = 0$ (6)

7. Use the remainder theorem to find the remainder when $2x^3 + x^2 - 7x - 6$ is divided by

   (a) $(x - 2)$ (b) $(x + 1)$

   Hence factorise the cubic expression. (7)

8. Simplify $\dfrac{6x^2 + 7x - 5}{2x - 1}$ by dividing out. (5)

9. Resolve the following into partial fractions:

   (a) $\dfrac{x - 11}{x^2 - x - 2}$ (b) $\dfrac{3 - x}{(x^2 + 3)(x + 3)}$

   (c) $\dfrac{x^3 - 6x + 9}{x^2 + x - 2}$ (24)

10. Solve the following equations:

    (a) $3t - 2 = 5t + 4$

    (b) $4(k - 1) - 2(3k + 2) + 14 = 0$

    (c) $\dfrac{a}{2} - \dfrac{2a}{5} = 1$ (d) $\sqrt{\dfrac{s + 1}{s - 1}} = 2$ (13)

11. A rectangular football pitch has its length equal to twice its width and a perimeter of 360 m. Find its length and width. (4)

Substituting $y = -2$ into either equation (1) or equation (2) will give $x = 3$ as in method (a). The solution $\boldsymbol{x = 3, y = -2}$ is the only pair of values that satisfies both of the original equations.

Problem 2. Solve, by a substitution method, the simultaneous equations:

$$3x - 2y = 12 \qquad (1)$$
$$x + 3y = -7 \qquad (2)$$

From equation (2), $x = -7 - 3y$

Substituting for x in equation (1) gives:

$$3(-7 - 3y) - 2y = 12$$

i.e. $$-21 - 9y - 2y = 12$$

$$-11y = 12 + 21 = 33$$

Hence $$y = \frac{33}{-11} = -3$$

Substituting $y = -3$ in equation (2) gives:

$$x + 3(-3) = -7$$

i.e. $$x - 9 = -7$$

Hence $$x = -7 + 9 = 2$$

Thus $\boldsymbol{x = 2, y = -3}$ is the solution of the simultaneous equations.
(Such solutions should always be checked by substituting values into each of the original two equations.)

Problem 3. Use an elimination method to solve the simultaneous equations:

$$3x + 4y = 5 \qquad (1)$$
$$2x - 5y = -12 \qquad (2)$$

If equation (1) is multiplied throughout by 2 and equation (2) by 3, then the coefficient of x will be the same in the newly formed equations. Thus

2 × equation (1) gives: $\quad 6x + 8y = 10 \qquad (3)$

3 × equation (2) gives: $\quad 6x - 15y = -36 \qquad (4)$

Equation (3) − equation (4) gives:

$$0 + 23y = 46$$

i.e. $$y = \frac{46}{23} = 2$$

(Note $+8y - -15y = 8y + 15y = 23y$ and $10 - (-36) = 10 + 36 = 46$. Alternatively, 'change the signs of the bottom line and add'.)

Substituting $y = 2$ in equation (1) gives:

$$3x + 4(2) = 5$$

from which $$3x = 5 - 8 = -3$$

and $$x = -1$$

Checking in equation (2), left-hand side $= 2(-1) - 5(2) = -2 - 10 = -12 =$ right-hand side.

Hence $\boldsymbol{x = -1}$ **and** $\boldsymbol{y = 2}$ is the solution of the simultaneous equations.

The elimination method is the most common method of solving simultaneous equations.

Problem 4. Solve:

$$7x - 2y = 26 \qquad (1)$$
$$6x + 5y = 29 \qquad (2)$$

When equation (1) is multiplied by 5 and equation (2) by 2 the coefficients of y in each equation are numerically the same, i.e. 10, but are of opposite sign.

5 × equation (1) gives: $\quad 35x - 10y = 130 \qquad (3)$

2 × equation (2) gives: $\quad 12x + 10y = 58 \qquad (4)$

Adding equation (3) and (4) gives: $\quad 47x + 0 = 188$

Hence $x = \dfrac{188}{47} = 4$

[Note that when the signs of common coefficients are **different** the two equations are **added**, and when the signs of common coefficients are the **same** the two equations are **subtracted** (as in Problems 1 and 3).]

Substituting $x = 4$ in equation (1) gives:

$$7(4) - 2y = 26$$
$$28 - 2y = 26$$
$$28 - 26 = 2y$$
$$2 = 2y$$

Hence $$y = 1$$

Checking, by substituting $x = 4$ and $y = 1$ in equation (2), gives:

$$\text{LHS} = 6(4) + 5(1) = 24 + 5 = 29 = \text{RHS}$$

Thus the solution is $x = 4$, $y = 1$, since these values maintain the equality when substituted in both equations.

Now try the following exercise

Exercise 32 Further problems on simultaneous equations

Solve the following simultaneous equations and verify the results.

1. $a + b = 7$
   $a - b = 3$ $\quad [a = 5, \quad b = 2]$
2. $2x + 5y = 7$
   $x + 3y = 4$ $\quad [x = 1, \quad y = 1]$
3. $3s + 2t = 12$
   $4s - t = 5$ $\quad [s = 2, \quad t = 3]$
4. $3x - 2y = 13$
   $2x + 5y = -4$ $\quad [x = 3, \quad y = -2]$
5. $5x = 2y$
   $3x + 7y = 41$ $\quad [x = 2, \quad y = 5]$
6. $5c = 1 - 3d$
   $2d + c + 4 = 0$ $\quad [c = 2, \quad d = -3]$

9.3 Further worked problems on simultaneous equations

Problem 5. Solve

$$3p = 2q \qquad (1)$$
$$4p + q + 11 = 0 \qquad (2)$$

Rearranging gives:

$$3p - 2q = 0 \qquad (3)$$
$$4p + q = -11 \qquad (4)$$

Multiplying equation (4) by 2 gives:

$$8p + 2q = -22 \qquad (5)$$

Adding equations (3) and (5) gives:

$$11p + 0 = -22$$
$$p = \frac{-22}{11} = -2$$

Substituting $p = -2$ into equation (1) gives:

$$3(-2) = 2q$$
$$-6 = 2q$$
$$q = \frac{-6}{2} = -3$$

Checking, by substituting $p = -2$ and $q = -3$ into equation (2) gives:

$$\text{LHS} = 4(-2) + (-3) + 11 = -8 - 3 + 11 = 0 = \text{RHS}$$

Hence the solution is $p = -2$, $q = -3$

Problem 6. Solve

$$\frac{x}{8} + \frac{5}{2} = y \qquad (1)$$
$$13 - \frac{y}{3} = 3x \qquad (2)$$

Whenever fractions are involved in simultaneous equations it is usual to firstly remove them. Thus, multiplying equation (1) by 8 gives:

$$8\left(\frac{x}{8}\right) + 8\left(\frac{5}{2}\right) = 8y$$

i.e. $\quad x + 20 = 8y \qquad (3)$

Multiplying equation (2) by 3 gives:

$$39 - y = 9x \qquad (4)$$

Rearranging equations (3) and (4) gives:

$$x - 8y = -20 \qquad (5)$$
$$9x + y = 39 \qquad (6)$$

Multiplying equation (6) by 8 gives:

$$72x + 8y = 312 \qquad (7)$$

Adding equations (5) and (7) gives:

$$73x + 0 = 292$$
$$\boldsymbol{x} = \frac{292}{73} = \mathbf{4}$$

Problem 9. The final length, l_2 of a piece of wire heated through θ °C is given by the formula $l_2 = l_1(1 + \alpha\theta)$. Make the coefficient of expansion, α, the subject

Rearranging gives: $\quad l_1(1 + \alpha\theta) = l_2$

Removing the bracket gives: $\quad l_1 + l_1\alpha\theta = l_2$

Rearranging gives: $\quad l_1\alpha\theta = l_2 - l_1$

Dividing both sides by $l_1\theta$ gives:

$$\frac{l_1\alpha\theta}{l_1\theta} = \frac{l_2 - l_1}{l_1\theta} \quad \text{i.e.} \quad \boldsymbol{\alpha = \frac{l_2 - l_1}{l_1\theta}}$$

Problem 10. A formula for the distance moved by a body is given by: $s = \frac{1}{2}(v + u)t$. Rearrange the formula to make u the subject

Rearranging gives: $\quad \frac{1}{2}(v + u)t = s$

Multiplying both sides by 2 gives: $\quad (v + u)t = 2s$

Dividing both sides by t gives:

$$\frac{(v + u)t}{t} = \frac{2s}{t}$$

i.e. $\quad v + u = \frac{2s}{t}$

Hence $\quad \boldsymbol{u = \frac{2s}{t} - v}$ or $\boldsymbol{u = \frac{2s - vt}{t}}$

Problem 11. A formula for kinetic energy is $k = \frac{1}{2}mv^2$. Transpose the formula to make v the subject

Rearranging gives: $\quad \frac{1}{2}mv^2 = k$

Whenever the prospective new subject is a squared term, that term is isolated on the LHS, and then the square root of both sides of the equation is taken.

Multiplying both sides by 2 gives: $\quad mv^2 = 2k$

Dividing both sides by m gives: $\quad \frac{mv^2}{m} = \frac{2k}{m}$

i.e. $\quad v^2 = \frac{2k}{m}$

Taking the square root of both sides gives:

$$\sqrt{v^2} = \sqrt{\frac{2k}{m}}$$

i.e. $\quad \boldsymbol{v = \sqrt{\frac{2k}{m}}}$

Problem 12. In a right angled triangle having sides x, y and hypotenuse z, Pythagoras' theorem states $z^2 = x^2 + y^2$. Transpose the formula to find x

Rearranging gives: $\quad x^2 + y^2 = z^2$

and $\quad x^2 = z^2 - y^2$

Taking the square root of both sides gives:

$$\boldsymbol{x = \sqrt{z^2 - y^2}}$$

Problem 13. Given $t = 2\pi\sqrt{\frac{l}{g}}$, find g in terms of t, l and π

Whenever the prospective new subject is within a square root sign, it is best to isolate that term on the LHS and then to square both sides of the equation.

Rearranging gives: $\quad 2\pi\sqrt{\frac{l}{g}} = t$

Dividing both sides by 2π gives: $\quad \sqrt{\frac{l}{g}} = \frac{t}{2\pi}$

Squaring both sides gives: $\quad \frac{l}{g} = \left(\frac{t}{2\pi}\right)^2 = \frac{t^2}{4\pi^2}$

Cross-multiplying, i.e. multiplying each term by $4\pi^2 g$, gives:

$$4\pi^2 l = gt^2$$

or $\quad gt^2 = 4\pi^2 l$

Dividing both sides by t^2 gives: $\quad \frac{gt^2}{t^2} = \frac{4\pi^2 l}{t^2}$

i.e. $\quad \boldsymbol{g = \frac{4\pi^2 l}{t^2}}$

Problem 14. The impedance of an a.c. circuit is given by $Z = \sqrt{R^2 + X^2}$. Make the reactance, X, the subject

11

Quadratic equations

11.1 Introduction to quadratic equations

As stated in Chapter 8, an **equation** is a statement that two quantities are equal and to **'solve an equation'** means 'to find the value of the unknown'. The value of the unknown is called the **root** of the equation.

A **quadratic equation** is one in which the highest power of the unknown quantity is 2. For example, $x^2 - 3x + 1 = 0$ is a quadratic equation.

There are four methods of **solving quadratic equations**.

These are: (i) by factorisation (where possible)

(ii) by 'completing the square'

(iii) by using the 'quadratic formula'

or (iv) graphically (see Chapter 30)

11.2 Solution of quadratic equations by factorisation

Multiplying out $(2x+1)(x-3)$ gives $2x^2-6x+x-3$, i.e. $2x^2 - 5x - 3$. The reverse process of moving from $2x^2 - 5x - 3$ to $(2x + 1)(x - 3)$ is called **factorising**.

If the quadratic expression can be factorised this provides the simplest method of solving a quadratic equation.

For example, if $2x^2 - 5x - 3 = 0$, then,

by factorising: $(2x + 1)(x - 3) = 0$

Hence either $(2x + 1) = 0$ i.e. $x = -\frac{1}{2}$

or $(x - 3) = 0$ i.e. $x = 3$

The technique of factorising is often one of 'trial and error'.

Problem 1. Solve the equations:
(a) $x^2 + 2x - 8 = 0$ (b) $3x^2 - 11x - 4 = 0$
by factorisation

(a) $x^2 + 2x - 8 = 0$. The factors of x^2 are x and x. These are placed in brackets thus: $(x \quad)(x \quad)$

The factors of -8 are $+8$ and -1, or -8 and $+1$, or $+4$ and -2, or -4 and $+2$. The only combination to give a middle term of $+2x$ is $+4$ and -2, i.e.

$$x^2 + 2x - 8 = (x + 4)(x - 2)$$

(Note that the product of the two inner terms added to the product of the two outer terms must equal the middle term, $+2x$ in this case.)

The quadratic equation $x^2 + 2x - 8 = 0$ thus becomes $(x + 4)(x - 2) = 0$.

Since the only way that this can be true is for either the first or the second, or both factors to be zero, then

either $(x + 4) = 0$ i.e. $x = -4$

or $(x - 2) = 0$ i.e. $x = 2$

Hence the roots of $x^2 + 2x - 8 = 0$ are $x = -4$ and 2

(b) $3x^2 - 11x - 4 = 0$

The factors of $3x^2$ are $3x$ and x. These are placed in brackets thus: $(3x \quad)(x \quad)$

The factors of -4 are -4 and $+1$, or $+4$ and -1, or -2 and 2.

Remembering that the product of the two inner terms added to the product of the two outer terms must equal $-11x$, the only combination to give this is $+1$ and -4, i.e.

$$3x^2 - 11x - 4 = (3x + 1)(x - 4)$$

12

Logarithms

12.1 Introduction to logarithms

With the use of calculators firmly established, logarithmic tables are now rarely used for calculation. However, the theory of logarithms is important, for there are several scientific and engineering laws that involve the rules of logarithms.

If a number y can be written in the form a^x, then the index x is called the 'logarithm of y to the base of a',

i.e. $\boxed{\textbf{if } y = a^x \quad \textbf{then} \quad x = \log_a y}$

Thus, since $1000 = 10^3$, then $3 = \log_{10} 1000$

Check this using the 'log' button on your calculator.

(a) Logarithms having a base of 10 are called **common logarithms** and $\log_{10}$ is usually abbreviated to lg. The following values may be checked by using a calculator:

$$\lg 17.9 = 1.2528\ldots,$$

$$\lg 462.7 = 2.6652\ldots$$

and $\lg 0.0173 = -1.7619\ldots$

(b) Logarithms having a base of e (where 'e' is a mathematical constant approximately equal to 2.7183) are called **hyperbolic, Napierian** or **natural logarithms**, and $\log_e$ is usually abbreviated to ln.

The following values may be checked by using a calculator:

$$\ln 3.15 = 1.1474\ldots,$$

$$\ln 362.7 = 5.8935\ldots$$

and $\ln 0.156 = -1.8578\ldots$

For more on Napierian logarithms see Chapter 13.

12.2 Laws of logarithms

There are three laws of logarithms, which apply to any base:

(i) To multiply two numbers:

$$\boxed{\log (A \times B) = \log A + \log B}$$

The following may be checked by using a calculator:

$$\lg 10 = 1,$$

also $\lg 5 + \lg 2 = 0.69897\ldots + 0.301029\ldots = 1$

Hence $\lg(5 \times 2) = \lg 10 = \lg 5 + \lg 2$

(ii) To divide two numbers:

$$\boxed{\log \left(\frac{A}{B}\right) = \log A - \log B}$$

The following may be checked using a calculator:

$$\ln \left(\frac{5}{2}\right) = \ln 2.5 = 0.91629\ldots$$

Also $\ln 5 - \ln 2 = 1.60943\ldots - 0.69314\ldots = 0.91629\ldots$

Hence $\ln \left(\frac{5}{2}\right) = \ln 5 - \ln 2$

(iii) To raise a number to a power:

$$\boxed{\lg A^n = n \log A}$$

The following may be checked using a calculator:

$$\lg 5^2 = \lg 25 = 1.39794\ldots$$

13

Exponential functions

13.1 The exponential function

An exponential function is one which contains e^x, e being a constant called the exponent and having an approximate value of 2.7183. The exponent arises from the natural laws of growth and decay and is used as a base for natural or Napierian logarithms.

13.2 Evaluating exponential functions

The value of e^x may be determined by using:

(a) a calculator, or
(b) the power series for e^x (see Section 13.3), or
(c) tables of exponential functions.

The most common method of evaluating an exponential function is by using a scientific notation **calculator**, this now having replaced the use of tables. Most scientific notation calculators contain an e^x function which enables all practical values of e^x and e^{-x} to be determined, correct to 8 or 9 significant figures. For example,

$$e^1 = 2.7182818$$
$$e^{2.4} = 11.023176$$
$$e^{-1.618} = 0.19829489$$

each correct to 8 significant figures.

In practical situations the degree of accuracy given by a calculator is often far greater than is appropriate. The accepted convention is that the final result is stated to one significant figure greater than the least significant measured value. Use your calculator to check the following values:

$$e^{0.12} = 1.1275, \text{ correct to 5 significant figures}$$
$$e^{-1.47} = 0.22993, \text{ correct to 5 decimal places}$$
$$e^{-0.431} = 0.6499, \text{ correct to 4 decimal places}$$
$$e^{9.32} = 11\,159, \text{ correct to 5 significant figures}$$
$$e^{-2.785} = 0.0617291, \text{ correct to 7 decimal places}$$

Problem 1. Using a calculator, evaluate, correct to 5 significant figures:

(a) $e^{2.731}$ (b) $e^{-3.162}$ (c) $\frac{5}{3}e^{5.253}$

(a) $e^{2.731} = 15.348227\ldots = \mathbf{15.348}$, correct to 5 significant figures.

(b) $e^{-3.162} = 0.04234097\ldots = \mathbf{0.042341}$, correct to 5 significant figures.

(c) $\frac{5}{3}e^{5.253} = \frac{5}{3}(191.138825\ldots) = \mathbf{318.56}$, correct to 5 significant figures.

Problem 2. Use a calculator to determine the following, each correct to 4 significant figures:

(a) $3.72e^{0.18}$ (b) $53.2e^{-1.4}$ (c) $\frac{5}{122}e^7$

(a) $3.72e^{0.18} = (3.72)(1.197217\ldots) = \mathbf{4.454}$, correct to 4 significant figures.

(b) $53.2e^{-1.4} = (53.2)(0.246596\ldots) = \mathbf{13.12}$, correct to 4 significant figures.

(c) $\frac{5}{122}e^7 = \frac{5}{122}(1096.6331\ldots) = \mathbf{44.94}$, correct to 4 significant figures.

Problem 3. Evaluate the following correct to 4 decimal places, using a calculator:

(a) $0.0256(e^{5.21} - e^{2.49})$

(b) $5\left(\frac{e^{0.25} - e^{-0.25}}{e^{0.25} + e^{-0.25}}\right)$

(a) $0.0256(e^{5.21} - e^{2.49})$

$= 0.0256(183.094058\ldots - 12.0612761\ldots)$

$= \mathbf{4.3784}$, correct to 4 decimal places

(b) $\ln 0.06213 = -2.7785263\ldots = \mathbf{-2.7785}$, correct to 5 significant figures.

(c) $3.2 \ln 762.923 = 3.2(6.6371571\ldots) = \mathbf{21.239}$, correct to 5 significant figures.

> **Problem 12.** Use a calculator to evaluate the following, each correct to 5 significant figures:
>
> (a) $\frac{1}{4} \ln 4.7291$ (b) $\frac{\ln 7.8693}{7.8693}$
>
> (c) $\frac{5.29 \ln 24.07}{e^{-0.1762}}$

(a) $\frac{1}{4} \ln 4.7291 = \frac{1}{4}(1.5537349\ldots) = \mathbf{0.38843}$,

correct to 5 significant figures.

(b) $\frac{\ln 7.8693}{7.8693} = \frac{2.06296911\ldots}{7.8693} = \mathbf{0.26215}$,

correct to 5 significant figures.

(c) $\frac{5.29 \ln 24.07}{e^{-0.1762}} = \frac{5.29(3.18096625\ldots)}{0.83845027\ldots}$

$= \mathbf{20.070}$, correct to 5 significant figures.

> **Problem 13.** Evaluate the following:
>
> (a) $\frac{\ln e^{2.5}}{\lg 10^{0.5}}$ (b) $\frac{4e^{2.23} \lg 2.23}{\ln 2.23}$ (correct to 3 decimal places)

(a) $\frac{\ln e^{2.5}}{\lg 10^{0.5}} = \frac{2.5}{0.5} = \mathbf{5}$

(b) $\frac{4e^{2.23} \lg 2.23}{\ln 2.23}$

$= \frac{4(9.29986607\ldots)(0.34830486\ldots)}{0.80200158\ldots}$

$= \mathbf{16.156}$, correct to 3 decimal places

> **Problem 14.** Solve the equation $7 = 4e^{-3x}$ to find x, correct to 4 significant figures

Rearranging $7 = 4e^{-3x}$ gives: $\frac{7}{4} = e^{-3x}$

Taking the reciprocal of both sides gives:

$$\frac{4}{7} = \frac{1}{e^{-3x}} = e^{3x}$$

Taking Napierian logarithms of both sides gives:

$$\ln\left(\frac{4}{7}\right) = \ln(e^{3x})$$

Since $\log_e e^{\alpha} = \alpha$, then $\ln\left(\frac{4}{7}\right) = 3x$

Hence $x = \frac{1}{3} \ln\left(\frac{4}{7}\right) = \frac{1}{3}(-0.55962) = \mathbf{-0.1865}$, correct to 4 significant figures.

> **Problem 15.** Given $20 = 60(1 - e^{-t/2})$ determine the value of t, correct to 3 significant figures

Rearranging $20 = 60(1 - e^{-t/2})$ gives:

$$\frac{20}{60} = 1 - e^{-1/2}$$

and

$$e^{-t/2} = 1 - \frac{20}{60} = \frac{2}{3}$$

Taking the reciprocal of both sides gives:

$$e^{t/2} = \frac{3}{2}$$

Taking Napierian logarithms of both sides gives:

$$\ln e^{t/2} = \ln \frac{3}{2}$$

i.e. $\frac{t}{2} = \ln \frac{3}{2}$

from which, $t = 2 \ln \frac{3}{2} = \mathbf{0.881}$, correct to 3 significant figures.

> **Problem 16.** Solve the equation $3.72 = \ln\left(\frac{5.14}{x}\right)$ to find x

From the definition of a logarithm, since

$3.72 = \left(\frac{5.14}{x}\right)$ then $e^{3.72} = \frac{5.14}{x}$

Rearranging gives: $x = \frac{5.14}{e^{3.72}} = 5.14e^{-3.72}$

i.e. $\mathbf{x = 0.1246}$,

correct to 4 significant figures

For example, the sum to infinity of the GP

$1+\frac{1}{2}+\frac{1}{4}+\ldots$ is

$$S_\infty = \frac{1}{1-\frac{1}{2}}, \text{ since } a = 1 \text{ and } r = \frac{1}{2}, \text{ i.e. } S_\infty = 2$$

14.5 Worked problems on geometric progressions

> Problem 9. Determine the tenth term of the series 3, 6, 12, 24, ...

3, 6, 12, 24, ... is a geometric progression with a common ratio r of 2.

The n'th term of a GP is ar^{n-1}, where a is the first term. Hence the 10th term is:

$$(3)(2)^{10-1} = (3)(2)^9 = 3(512) = \mathbf{1536}$$

> Problem 10. Find the sum of the first 7 terms of the series, $\frac{1}{2}, 1\frac{1}{2}, 4\frac{1}{2}, 13\frac{1}{2}, \ldots$

$$\frac{1}{2},\ 1\frac{1}{2},\ 4\frac{1}{2},\ 13\frac{1}{2},\ \ldots$$

is a GP with a common ratio $r = 3$

The sum of n terms, $S_n = \dfrac{a(r^n-1)}{(r-1)}$

$$\text{Hence } S_7 = \frac{\frac{1}{2}(3^7-1)}{(3-1)} = \frac{\frac{1}{2}(2187-1)}{2} = \mathbf{546\frac{1}{2}}$$

> Problem 11. The first term of a geometric progression is 12 and the fifth term is 55. Determine the 8'th term and the 11'th term

The 5th term is given by $ar^4 = 55$, where the first term $a = 12$

Hence $r^4 = \dfrac{55}{a} = \dfrac{55}{12}$ and

$$r = \sqrt[4]{\frac{55}{12}} = 1.4631719\ldots$$

The 8th term is

$$ar^7 = (12)(1.4631719\ \ldots)^7 = \mathbf{172.3}$$

The 11th term is

$$ar^{10} = (12)(1.4631719\ \ldots)^{10} = \mathbf{539.7}$$

> Problem 12. Which term of the series: 2187, 729, 243, ... is $\frac{1}{9}$?

2187, 729, 243, ... is a GP with a common ratio $r = \frac{1}{3}$ and first term $a = 2187$

The n'th term of a GP is given by: ar^{n-1}

Hence $\quad \dfrac{1}{9} = (2187)\left(\dfrac{1}{3}\right)^{n-1}$ from which

$$\left(\frac{1}{3}\right)^{n-1} = \frac{1}{(9)(2187)} = \frac{1}{3^2 3^7} = \frac{1}{3^9} = \left(\frac{1}{3}\right)^9$$

Thus $(n-1) = 9$, from which, $n = 9 + 1 = 10$

i.e. $\mathbf{\frac{1}{9}}$ **is the 10th term of the GP**

> Problem 13. Find the sum of the first 9 terms of the series: 72.0, 57.6, 46.08, ...

The common ratio,

$$r = \frac{ar}{a} = \frac{57.6}{72.0} = 0.8$$

$$\left(\text{also } \frac{ar^2}{ar} = \frac{46.08}{57.6} = 0.8\right)$$

The sum of 9 terms,

$$S_9 = \frac{a(1-r^n)}{(1-r)} = \frac{72.0(1-0.8^9)}{(1-0.8)}$$
$$= \frac{72.0(1-0.1342)}{0.2} = \mathbf{311.7}$$

> Problem 14. Find the sum to infinity of the series 3, 1, $\frac{1}{3}$, ...

3, 1, $\frac{1}{3}$, ... is a GP of common ratio, $r = \frac{1}{3}$

The sum to infinity,

$$S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{1}{3}} = \frac{3}{\frac{2}{3}} = \frac{9}{2} = \mathbf{4\frac{1}{2}}$$

Now try the following exercise

Exercise 53 Further problems on geometric progressions

1. Find the 10th term of the series 5, 10, 20, 40, ... [2560]
2. Determine the sum of the first 7 terms of the series 0.25, 0.75, 2.25, 6.75, ... [273.25]
3. The first term of a geometric progression is 4 and the 6th term is 128. Determine the 8th and 11th terms. [512, 4096]
4. Which term of the series 3, 9, 27, ... is 59 049? [10^{th}]
5. Find the sum of the first 7 terms of the series 2, 5, $12\frac{1}{2}$, ... (correct to 4 significant figures). [812.5]
6. Determine the sum to infinity of the series 4, 2, 1, ... [8]
7. Find the sum to infinity of the series $2\frac{1}{2}, -1\frac{1}{4}, \frac{5}{8}, \ldots$ $\left[1\frac{2}{3}\right]$

14.6 Further worked problems on geometric progressions

Problem 15. In a geometric progression the sixth term is 8 times the third term and the sum of the seventh and eighth terms is 192. Determine (a) the common ratio, (b) the first term, and (c) the sum of the fifth to eleventh terms, inclusive

(a) Let the GP be $a, ar, ar^2, ar^3, \ldots, ar^{n-1}$

The 3rd term $= ar^2$ and the sixth term $= ar^5$

The 6th term is 8 times the 3rd

Hence $ar^5 = 8\,ar^2$ from which, $r^3 = 8$ and $r = \sqrt[3]{8}$

i.e. **the common ratio $r = 2$**

(b) The sum of the 7th and 8th terms is 192. Hence $ar^6 + ar^7 = 192$. Since $r = 2$, then

$$64a + 128a = 192$$
$$192a = 192,$$

from which, **a, the first term $= 1$**

(c) The sum of the 5th to 11th terms (inclusive) is given by:

$$S_{11} - S_4 = \frac{a(r^{11}-1)}{(r-1)} - \frac{a(r^4-1)}{(r-1)}$$
$$= \frac{1(2^{11}-1)}{(2-1)} - \frac{1(2^4-1)}{(2-1)}$$
$$= (2^{11}-1) - (2^4-1)$$
$$= 2^{11} - 2^4 = 2408 - 16 = \mathbf{2032}$$

Problem 16. A hire tool firm finds that their net return from hiring tools is decreasing by 10% per annum. If their net gain on a certain tool this year is £400, find the possible total of all future profits from this tool (assuming the tool lasts for ever)

The net gain forms a series:

$$£400 + £400 \times 0.9 + £400 \times 0.9^2 + \ldots,$$

which is a GP with $a = 400$ and $r = 0.9$

The sum to infinity,

$$S_\infty = \frac{a}{(1-r)} = \frac{400}{(1-0.9)}$$
$$= \mathbf{£4000 = total\ future\ profits}$$

Problem 17. If £100 is invested at compound interest of 8% per annum, determine (a) the value after 10 years, (b) the time, correct to the nearest year, it takes to reach more than £300

(a) Let the GP be $a, ar, ar^2, \ldots ar^n$

The first term $a = £100$ and

The common ratio $r = 1.08$

where, for example, 4! denotes $4 \times 3 \times 2 \times 1$ and is termed 'factorial 4'.

Thus,

$$^5C_3 = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)}$$
$$= \frac{120}{6 \times 2} = 10$$

For example, the five letters A, B, C, D, E can be arranged in groups of three as follows: ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE, i.e. there are ten groups. The above calculation $^5C_3$ produces the answer of 10 combinations without having to list all of them.

A **permutation** is the number of ways of selecting $r \le n$ objects from n distinguishable objects when order of selection is important. A permutation is denoted by ^nP_r or $_nP_r$

where $\quad ^nP_r = n(n-1)(n-2)\ldots(n-r+1)$

or $\quad \boldsymbol{^nP_r = \dfrac{n!}{(n-r)!}}$

Thus, $\quad ^4P_2 = 4(3) = 12$

or $\quad ^4P_2 = \dfrac{4!}{(4-2)!} = \dfrac{4!}{2!}$
$$= \frac{4 \times 3 \times 2}{2} = 12$$

Problem 19. Evaluate: (a) $^7C_4$ (b) $^{10}C_6$

(a) $\quad ^7C_4 = \dfrac{7!}{4!(7-4)!} = \dfrac{7!}{4!3!}$
$$= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2}{(4 \times 3 \times 2)(3 \times 2)} = \mathbf{35}$$

(b) $\quad ^{10}C_6 = \dfrac{10!}{6!(10-6)!} = \dfrac{10!}{6!4!} = \mathbf{210}$

Problem 20. Evaluate: (a) $^6P_2$ (b) $^9P_5$

(a) $\quad ^6P_2 = \dfrac{6!}{(6-2)!} = \dfrac{6!}{4!}$
$$= \frac{6 \times 5 \times 4 \times 3 \times 2}{4 \times 3 \times 2} = \mathbf{30}$$

(b) $\quad ^9P_5 = \dfrac{9!}{(9-5)!} = \dfrac{9!}{4!}$
$$= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4!} = \mathbf{15\,120}$$

Now try the following exercise

Exercise 55 Further problems on permutations and combinations

Evaluate the following:

1. (a) $^9C_6$ (b) $^3C_1$ [(a) 84 (b) 3]
2. (a) $^6C_2$ (b) $^8C_5$ [(a) 15 (b) 56]
3. (a) $^4P_2$ (b) $^7P_4$ [(a) 12 (b) 840]
4. (a) $^{10}P_3$ (b) $^8P_5$ [(a) 720 (b) 6720]

Problem 10.
(a) Expand $\dfrac{1}{(4-x)^2}$ in ascending powers of x as far as the term in x^3, using the binomial theorem.

(b) What are the limits of x for which the expansion in (a) is true?

(a) $$\frac{1}{(4-x)^2} = \frac{1}{\left[4\left(1-\frac{x}{4}\right)\right]^2} = \frac{1}{4^2\left(1-\frac{x}{4}\right)^2}$$
$$= \frac{1}{16}\left(1-\frac{x}{4}\right)^{-2}$$

Using the expansion of $(1+x)^n$

$$\frac{1}{(4-x)^2} = \frac{1}{16}\left(1-\frac{x}{4}\right)^{-2}$$
$$= \frac{1}{16}\left[1+(-2)\left(-\frac{x}{4}\right) + \frac{(-2)(-3)}{2!}\left(-\frac{x}{4}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(-\frac{x}{4}\right)^3 + \cdots\right]$$
$$= \mathbf{\frac{1}{16}\left(1+\frac{x}{2}+\frac{3x^2}{16}+\frac{x^3}{16}+\cdots\right)}$$

(b) The expansion in (a) is true provided $\left|\dfrac{x}{4}\right| < 1$,

i.e. $\mathbf{|x| < 4}$ or $\mathbf{-4 < x < 4}$

Problem 11. Use the binomial theorem to expand $\sqrt{4+x}$ in ascending powers of x to four terms. Give the limits of x for which the expansion is valid

$$\sqrt{4+x} = \sqrt{4\left(1+\frac{x}{4}\right)}$$
$$= \sqrt{4}\sqrt{1+\frac{x}{4}}$$
$$= 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$

Using the expansion of $(1+x)^n$,

$$2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$
$$= 2\left[1+\left(\frac{1}{2}\right)\left(\frac{x}{4}\right) + \frac{(1/2)(-1/2)}{2!}\left(\frac{x}{4}\right)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}\left(\frac{x}{4}\right)^3 + \cdots\right]$$
$$= 2\left(1+\frac{x}{8}-\frac{x^2}{128}+\frac{x^3}{1024}-\cdots\right)$$
$$= \mathbf{2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}-\cdots}$$

This is valid when $\left|\dfrac{x}{4}\right| < 1$,

i.e. $\mathbf{\left|\dfrac{x}{4}\right| < 4}$ or $\mathbf{-4 < x < 4}$

Problem 12. Expand $\dfrac{1}{\sqrt{1-2t}}$ in ascending powers of t as far as the term in t^3.
State the limits of t for which the expression is valid

$$\frac{1}{\sqrt{1-2t}}$$
$$= (1-2t)^{-\frac{1}{2}}$$
$$= 1+\left(-\frac{1}{2}\right)(-2t) + \frac{(-1/2)(-3/2)}{2!}(-2t)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}(-2t)^3 + \cdots$$

using the expansion for $(1+x)^n$

$$= \mathbf{1+t+\frac{3}{2}t^2+\frac{5}{2}t^3+\cdots}$$

The expression is valid when $|2t| < 1$,

i.e. $\mathbf{|t| < \dfrac{1}{2}}$ or $\mathbf{-\dfrac{1}{2} < t < \dfrac{1}{2}}$

Problem 13. Simplify $\dfrac{\sqrt[3]{1-3x}\sqrt{1+x}}{\left(1+\dfrac{x}{2}\right)^3}$ given that powers of x above the first may be neglected

6. Expand $(2+3x)^{-6}$ to three terms. For what values of x is the expansion valid?

$$\left[\frac{1}{64}\left(1-9x+\frac{189}{4}x^2\right) \quad |x|<\frac{2}{3}\right]$$

7. When x is very small show that:

(a) $\dfrac{1}{(1-x)^2\sqrt{1-x}} \approx 1+\dfrac{5}{2}x$

(b) $\dfrac{(1-2x)}{(1-3x)^4} \approx 1+10x$

(c) $\dfrac{\sqrt{1+5x}}{\sqrt[3]{1-2x}} \approx 1+\dfrac{19}{6}x$

8. If x is very small such that x^2 and higher powers may be neglected, determine the power series for

$$\frac{\sqrt{x+4}\;\sqrt[3]{8-x}}{\sqrt[5]{(1+x)^3}}. \qquad \left[4-\frac{31}{15}x\right]$$

9. Express the following as power series in ascending powers of x as far as the term in x^2. State in each case the range of x for which the series is valid.

(a) $\sqrt{\dfrac{1-x}{1+x}}$ (b) $\dfrac{(1+x)\;\sqrt[3]{1-3x^2}}{\sqrt{1+x^2}}$

$$\left[\begin{array}{l}\text{(a) } 1-x+\frac{1}{2}x^2, \quad |x|<1 \\ \text{(b) } 1-x-\frac{7}{2}x^2, \quad |x|<\frac{1}{3}\end{array}\right]$$

15.5 Practical problems involving the binomial theorem

Binomial expansions may be used for numerical approximations, for calculations with small variations and in probability theory.

Problem 15. The radius of a cylinder is reduced by 4% and its height is increased by 2%. Determine the approximate percentage change in (a) its volume and (b) its curved surface area, (neglecting the products of small quantities)

Volume of cylinder $= \pi r^2 h$

Let r and h be the original values of radius and height

The new values are $0.96r$ or $(1-0.04)r$ and $1.02\,h$ or $(1+0.02)h$

(a) New volume $= \pi[(1-0.04)r]^2[(1+0.02)h]$
$= \pi r^2 h(1-0.04)^2(1+0.02)$

Now $(1-0.04)^2 = 1-2(0.04)+(0.04)^2$
$= (1-0.08)$, neglecting powers of small terms

Hence new volume

$\approx \pi r^2 h(1-0.08)(1+0.02)$

$\approx \pi r^2 h(1-0.08+0.02)$, neglecting products of small terms

$\approx \pi r^2 h(1-0.06)$ or $0.94\pi r^2 h$, i.e. 94% of the original volume

Hence the volume is reduced by approximately 6%.

(b) Curved surface area of cylinder $= 2\pi rh$.

New surface area

$= 2\pi[(1-0.04)r][(1+0.02)h]$

$= 2\pi rh(1-0.04)(1+0.02)$

$\approx 2\pi rh(1-0.04+0.02)$, neglecting products of small terms

$\approx 2\pi rh(1-0.02)$ or $0.98(2\pi rh)$, i.e. 98% of the original surface area

Hence the curved surface area is reduced by approximately 2%.

Problem 16. The second moment of area of a rectangle through its centroid is given by $\dfrac{bl^3}{12}$. Determine the approximate change in the second moment of area if b is increased by 3.5% and l is reduced by 2.5%

New values of b and l are $(1+0.035)b$ and $(1-0.025)l$ respectively.

6. Find the angles p, q, r, s and t in Fig. 17.12(a) to (c).

$$\left[\begin{array}{r} p = 105^\circ,\ q = 35^\circ,\ r = 142^\circ, \\ s = 95^\circ,\ t = 146^\circ \end{array}\right]$$

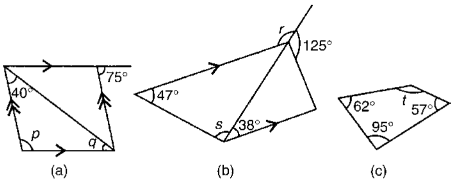


Figure 17.12

7. Name the types of quadrilateral shown in Fig. 17.13(i) to (iv), and determine (a) the area, and (b) the perimeter of each.

$$\left[\begin{array}{l} \text{(i) rhombus (a) } 14\text{ cm}^2 \quad \text{(b) } 16\text{ cm} \\ \text{(ii) parallelogram (a) } 180\text{ cm}^2 \\ \qquad\qquad\qquad\quad \text{(b) } 80\text{ mm} \\ \text{(iii) rectangle (a) } 3600\text{ mm}^2 \\ \qquad\qquad\qquad \text{(b) } 300\text{ mm} \\ \text{(iv) trapezium (a) } 190\text{ cm}^2 \quad \text{(b) } 62.91\text{ cm} \end{array}\right]$$

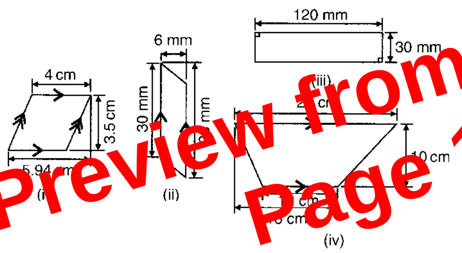


Figure 17.13

8. Calculate the area of the steel plate shown in Fig. 17.14. [6750 mm^2]

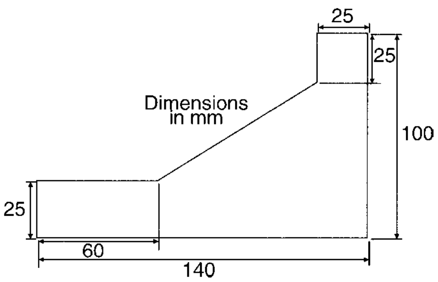


Figure 17.14

17.4 Further worked problems on areas of plane figures

Problem 7. Find the areas of the circles having (a) a radius of 5 cm, (b) a diameter of 15 mm, (c) a circumference of 70 mm

Area of a circle $= \pi r^2$ or $\dfrac{\pi d^2}{4}$

(a) Area $= \pi r^2 = \pi(5)^2 = 25\pi = \mathbf{78.54\ cm^2}$

(b) Area $= \dfrac{\pi d^2}{4} = \dfrac{\pi(15)^2}{4} = \dfrac{225\pi}{4} = \mathbf{176.7\ mm^2}$

(c) Circumference, $c = 2\pi r$, hence

$$r = \frac{c}{2\pi} = \frac{70}{2\pi} = \frac{35}{\pi}\text{ mm}$$

$$\text{Area of circle} = \pi r^2 = \pi\left(\frac{35}{\pi}\right)^2 = \frac{35^2}{\pi}$$

$$= \mathbf{389.9\ mm^2} \text{ or } \mathbf{3.899\ cm^2}$$

Problem 8. Calculate the areas of the following sectors of circles having:

(a) radius 6 cm with angle subtended at centre 50°

(b) diameter 80 mm with angle subtended at centre 107°42′

(c) radius 8 cm with angle subtended at centre 1.15 radians

Area of sector of a circle $= \dfrac{\theta^2}{360}(\pi r^2)$

or $\dfrac{1}{2}r^2\theta$ (θ in radians).

(a) Area of sector

$$= \frac{50}{360}(\pi 6^2) = \frac{50 \times \pi \times 36}{360} = 5\pi$$

$$= \mathbf{15.71\ cm^2}$$

(b) If diameter $= 80$ mm, then radius, $r = 40$ mm, and area of sector

$$= \frac{107^\circ 42'}{360}(\pi 40^2) = \frac{107\frac{42}{60}}{360}(\pi 40^2)$$

$$= \frac{107.7}{360}(\pi 40^2) = \mathbf{1504\ mm^2} \text{ or } \mathbf{15.04\ cm^2}$$

(c) Area of sector $= \frac{1}{2}r^2\theta = \frac{1}{2} \times 8^2 \times 1.15$

$$= \mathbf{36.8\ cm^2}$$

(xiii) The angle in a semicircle is a right angle (see angle BQP in Fig. 18.3).

Problem 1. Find the circumference of a circle of radius 12.0 cm

Circumference,

$$c = 2 \times \pi \times \text{radius} = 2\pi r = 2\pi(12.0) = \mathbf{75.40\ cm}$$

Problem 2. If the diameter of a circle is 75 mm, find its circumference

Circumference,

$$c = \pi \times \text{diameter} = \pi d = \pi(75) = \mathbf{235.6\ mm}$$

Problem 3. Determine the radius of a circle if its perimeter is 112 cm

Perimeter = circumference, $c = 2\pi r$

Hence **radius** $\boldsymbol{r} = \frac{c}{2\pi} = \frac{112}{2\pi} = \mathbf{17.83\ cm}$

Problem 4. In Fig. 18.4, AB is a tangent to the circle at B. If the circle radius is 40 mm and $AB = 150$ mm, calculate the length AO

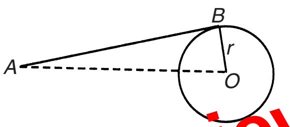


Figure 18.4

x

A tangent to a circle is at right angles to a radius drawn from the point of contact, i.e. $ABO = 90°$. Hence, using Pythagoras' theorem:

$$AO^2 = AB^2 + OB^2$$

from which, $\boldsymbol{AO} = \sqrt{AB^2 + OB^2}$

$$= \sqrt{150^2 + 40^2} = \mathbf{155.2\ mm}$$

Now try the following exercise

Exercise 65 Further problems on properties of a circle

1. Calculate the length of the circumference of a circle of radius 7.2 cm. [45.24 cm]
2. If the diameter of a circle is 82.6 mm, calculate the circumference of the circle. [259.5 mm]
3. Determine the radius of a circle whose circumference is 16.52 cm. [2.629 cm]
4. Find the diameter of a circle whose perimeter is 149.8 cm. [47.68 cm]

18.3 Arc length and area of a sector

One **radian** is defined as the angle subtended at the centre of a circle by an arc equal in length to the radius. With reference to Fig. 18.5, for arc length s,

θ radians $= s/r$ or **arc length**, $\boxed{\boldsymbol{s = r\theta}}$ (1)

where θ is in radians.

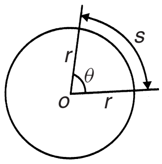


Figure 18.5

When s = whole circumference $(= 2\pi r)$ then $\theta = s/r = 2\pi r/r = 2\pi$

i.e. 2π radians $= 360°$ or

$$\boxed{\boldsymbol{\pi} \textbf{ radians} = \mathbf{180°}}$$

Thus 1 rad $= 180°/\pi = 57.30°$, correct to 2 decimal places.

Since π rad $= 180°$, then $\pi/2 = 90°$, $\pi/3 = 60°$, $\pi/4 = 45°$, and so on.

$$\textbf{Area of a sector} = \frac{\boldsymbol{\theta}}{\mathbf{360}}(\boldsymbol{\pi r^2}) \quad \text{when } \theta \text{ is in degrees}$$

$$= \frac{\theta}{2\pi}(\pi r^2) = \frac{\mathbf{1}}{\mathbf{2}}\boldsymbol{r^2\theta} \quad \text{when } \theta \text{ is in radians} \qquad (2)$$

Problem 5. Convert to radians: (a) 125° (b) 69°47′

(a) Since $180° = \pi$ rad then $1° = \pi/180$ rad, therefore

$$125° = 125\left(\frac{\pi}{180}\right)^c = \mathbf{2.182\ radians}$$

(Note that ^{c} means 'circular measure' and indicates radian measure.)

(b) $69°47' = 69\dfrac{47°}{60} = 69.783°$

$$69.783° = 69.783\left(\frac{\pi}{180}\right)^c = \mathbf{1.218\ radians}$$

Problem 6. Convert to degrees and minutes: (a) 0.749 radians (b) $3\pi/4$ radians

(a) Since π rad $= 180°$ then 1 rad $= 180°/\pi$, therefore

$$0.749 = 0.749\left(\frac{180}{\pi}\right)^\circ = 42.915°$$

$0.915° = (0.915 \times 60)' = 55'$, correct to the nearest minute, hence

$$\mathbf{0.749\ radians = 42°55'}$$

(b) Since 1 rad $= \left(\dfrac{180}{\pi}\right)^\circ$ then

$$\frac{3\pi}{4}\text{ rad} = \frac{3\pi}{4}\left(\frac{180}{\pi}\right)^\circ$$

$$= \frac{3}{4}(180)^\circ = \mathbf{135°}$$

Problem 7. Express in radians, in terms of π, (a) 150° (b) 270° (c) 37.5°

Since $180° = \pi$ rad then $1° = 180/\pi$, hence

(a) $150° = 150\left(\dfrac{\pi}{180}\right)$ rad $= \mathbf{\dfrac{5\pi}{6}\ rad}$

(b) $270° = 270\left(\dfrac{\pi}{180}\right)$ rad $= \mathbf{\dfrac{3\pi}{2}\ rad}$

(c) $37.5° = 37.5\left(\dfrac{\pi}{180}\right)$ rad $= \dfrac{75\pi}{360}$ rad $= \mathbf{\dfrac{5\pi}{24}\ rad}$

Now try the following exercise

Exercise 66 Further problems on radians and degrees

1. Convert to radians in terms of π: (a) 30° (b) 75° (c) 225° $\left[\text{(a) } \dfrac{\pi}{6} \quad \text{(b) } \dfrac{5\pi}{12} \quad \text{(c) } \dfrac{5\pi}{4}\right]$

2. Convert to radians: (a) 48° (b) 84°51′ (c) 232°15’
   [(a) 0.838 (b) 1.481 (c) 4.054]

3. Convert to degrees: (a) $\dfrac{5\pi}{6}$ rad (b) $\dfrac{4\pi}{9}$ rad (c) $\dfrac{7\pi}{12}$ rad [(a) 150° (b) 80° (c) 105°]

4. Convert to degrees and minutes:
   (a) 0.0125 rad (b) 2.69 rad (c) 7.241 rad
   [(a) 0° 43′ (b) 154° 8′ (c) 414°53′]

18.4 Worked problems on arc length and sector of a circle

Problem 8. Find the length of arc of a circle of radius 5.5 cm when the angle subtended at the centre is 1.20 radians

From equation (1), length of arc, $s = r\theta$, where θ is in radians, hence

$$s = (5.5)(1.20) = \mathbf{6.60\ cm}$$

Problem 9. Determine the diameter and circumference of a circle if an arc of length 4.75 cm subtends an angle of 0.91 radians

Since $s = r\theta$ then $r = \dfrac{s}{\theta} = \dfrac{4.75}{0.91} = 5.22$ cm.

Diameter $= 2 \times$ radius $= 2 \times 5.22 = \mathbf{10.44\ cm}$.

Circumference, $c = \pi d = \pi(10.44) = \mathbf{32.80\ cm}$.

Problem 10. If an angle of 125° is subtended by an arc of a circle of radius 8.4 cm, find the length of (a) the minor arc, and (b) the major arc, correct to 3 significant figures

Since $180° = \pi$ rad then $1° = \left(\dfrac{\pi}{180}\right)$ rad

and $125° = 125\left(\dfrac{\pi}{180}\right)$ rad

Length of minor arc,

$$s = r\theta = (8.4)(125)\left(\frac{\pi}{180}\right) = \mathbf{18.3\ cm}$$

correct to 3 significant figures.

Length of major arc $=$ (circumference $-$ minor arc) $= 2\pi(8.4) - 18.3 = \mathbf{34.5\ cm}$, correct to 3 significant figures.

(Alternatively, major arc $= r\theta = 8.4(360 - 125)(\pi/180) = \mathbf{34.5\ cm}$.)

a height of 3.5 m, with a diameter of 15 m. Calculate the surface area of material needed to make the marquee assuming 12% of the material is wasted in the process. [393.4 m^2]

5. Determine (a) the volume and (b) the total surface area of the following solids:

   (i) a cone of radius 8.0 cm and perpendicular height 10 cm
   (ii) a sphere of diameter 7.0 cm
   (iii) a hemisphere of radius 3.0 cm
   (iv) a 2.5 cm by 2.5 cm square pyramid of perpendicular height 5.0 cm
   (v) a 4.0 cm by 6.0 cm rectangular pyramid of perpendicular height 12.0 cm
   (vi) a 4.2 cm by 4.2 cm square pyramid whose sloping edges are each 15.0 cm
   (vii) a pyramid having an octagonal base of side 5.0 cm and perpendicular height 20 cm.

(i)	(a) 670 cm^3	(b) 523 cm^2
(ii)	(a) 180 cm^3	(b) 154 cm^2
(iii)	(a) 56.5 cm^3	(b) 84.8 cm^2
(iv)	(a) 10.4 cm^3	(b) 32.0 cm^2
(v)	(a) 96.0 cm^3	(b) 146 cm^2
(vi)	(a) 86.5 cm^3	(b) 142 cm^2
(vii)	(a) 805 cm^3	(b) 539 cm^2

6. The volume of a sphere is 325 cm^3. Determine its diameter. [8.53 cm]

7. A metal sphere weighing 24 kg is melted down and recast into a solid cone of base radius 8.0 cm. If the density of the metal is 8000 kg/m^3 determine (a) the diameter of the metal sphere and (b) the perpendicular height of the cone, assuming that 15% of the metal is lost in the process. [(a) 17.9 cm (b) 38.0 cm]

8. Find the volume of a regular hexagonal pyramid if the perpendicular height is 16.0 cm and the side of base is 3.0 cm. [125 cm^3]

9. A buoy consists of a hemisphere surmounted by a cone. The diameter of the cone and hemisphere is 2.5 m and the slant height of the cone is 4.0 m. Determine the volume and surface area of the buoy. [10.3 m^3, 25.5 m^2]

10. A petrol container is in the form of a central cylindrical portion 5.0 m long with a hemispherical section surmounted on each end. If the diameters of the hemisphere and cylinder are both 1.2 m determine the capacity of the tank in litres (1 litre = 1000 cm^3). [6560 litre]

11. Figure 19.9 shows a metal rod section. Determine its volume and total surface area. [657.1 cm^3, 1027 cm^2]

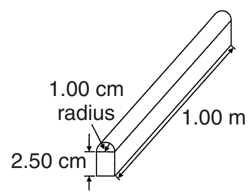


Figure 19.9

19.4 Volumes and surface areas of frusta of pyramids and cones

The **frustum** of a pyramid or cone is the portion remaining when a part containing the vertex is cut off by a plane parallel to the base.

The **volume of a frustum of a pyramid or cone** is given by the volume of the whole pyramid or cone minus the volume of the small pyramid or cone cut off.

The **surface area of the sides of a frustum of a pyramid or cone** is given by the surface area of the whole pyramid or cone minus the surface area of the small pyramid or cone cut off. This gives the lateral surface area of the frustum. If the total surface area of the frustum is required then the surface area of the two parallel ends are added to the lateral surface area.

There is an alternative method for finding the volume and surface area of a **frustum of a cone**. With reference to Fig. 19.10:

$$\textbf{Volume} = \frac{1}{3}\pi h(R^2 + Rr + r^2)$$

$$\textbf{Curved surface area} = \pi l(R + r)$$

$$\textbf{Total surface area} = \pi l(R + r) + \pi r^2 + \pi R^2$$

(d) $\dfrac{\text{Volume of frustum}}{\text{Volume of sphere}} = \dfrac{311.0}{904.8} \times 100\%$

$= \mathbf{34.37\%}$

> Problem 26. A spherical storage tank is filled with liquid to a depth of 20 cm. If the internal diameter of the vessel is 30 cm, determine the number of litres of liquid in the container (1 litre = 1000 cm^3)

The liquid is represented by the shaded area in the section shown in Fig. 19.19. The volume of liquid comprises a hemisphere and a frustum of thickness 5 cm.

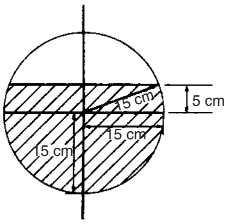


Figure 19.19

Hence volume of liquid

$$= \frac{2}{3}\pi r^3 + \frac{\pi h}{6}[h^2 + 3r_1^2 + 3r_2^2]$$

where $r_2 = 30/2 = 15$ cm and

$$r_1 = \sqrt{15^2 - 5^2} = 14.14 \text{ cm}$$

Volume of liquid

$$= \frac{2}{3}\pi(15)^3 + \frac{\pi(5)}{6}[5^2 + 3(14.14)^2 + 3(15)^2]$$

$$= 7069 + 3403 = 10\,470 \text{ cm}^3$$

Since 1 litre = 1000 cm^3, the number of litres of liquid

$$= \frac{10\,470}{1000} = \mathbf{10.47\ litres}$$

Now try the following exercise

Exercise 72 Further problems on frustums and zones of spheres

1. Determine the volume and surface area of a frustum of a sphere of diameter 47.85 cm, if the radii of the ends of the frustum are 14.0 cm and 22.0 cm and the height of the frustum is 10.0 cm
[11 210 cm^3, 1503 cm^2]

2. Determine the volume (in cm^3) and the surface area (in cm^2) of a frustum of a sphere if the diameter of the ends are 80.0 mm and 120.0 mm and the thickness is 30.0 mm. [259.2 cm^3, 118.3 cm^2]

3. A sphere has a radius of 6.50 cm. Determine its volume and surface area. A frustum of the sphere is formed by two parallel planes, one through the diameter and the other at a distance h from the diameter. If the curved surface area of the frustum is to be $\frac{1}{5}$ of the surface area of the sphere, find the height h and the volume of the frustum.
$\begin{bmatrix} 1150 \text{ cm}^3, 531 \text{ cm}^2, \\ 2.60 \text{ cm}, 326.7 \text{ cm}^3 \end{bmatrix}$

4. A sphere has a diameter of 32.0 mm. Calculate the volume (in cm^3) of the frustum of the sphere contained between two parallel planes distances 12.0 mm and 10.00 mm from the centre and on opposite sides of it. [14.84 cm^3]

5. A spherical storage tank is filled with liquid to a depth of 30.0 cm. If the inner diameter of the vessel is 45.0 cm determine the number of litres of liquid in the container (1litre = 1000 cm^3).
[35.34 litres]

19.6 Prismoidal rule

The prismoidal rule applies to a solid of length x divided by only three equidistant plane areas, A_1, A_2 and A_3 as shown in Fig. 19.20 and is merely an extension of Simpson's rule (see Chapter 20) — but for volumes.

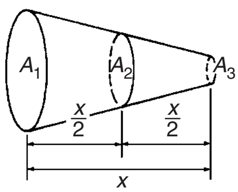


Figure 19.20

In Problems 9 to 14, determine the acute angle in degrees (correct to 2 decimal places), degrees and minutes, and in radians (correct to 3 decimal places).

9. $\sin^{-1} 0.2341$ [13.54°, 13°32′, 0.236 rad]

10. $\cos^{-1} 0.8271$ [34.20°, 34°12′, 0.597 rad]

11. $\tan^{-1} 0.8106$ [39.03°, 39°2′, 0.681 rad]

12. $\sec^{-1} 1.6214$ [51.92°, 51°55′, 0.906 rad]

13. $\text{cosec}^{-1} 2.4891$ [23.69°, 23°41′, 0.413 rad]

14. $\cot^{-1} 1.9614$ [27.01°, 27°1′, 0.471 rad]

In Problems 15 to 18, evaluate correct to 4 significant figures.

15. $4\cos 56°19' - 3\sin 21°57'$ [1.097]

16. $\dfrac{11.5\tan 49°11' - \sin 90°}{3\cos 45°}$ [5.805]

17. $\dfrac{5\sin 86°3'}{3\tan 14°29' - 2\cos 31°9'}$ [−5.325]

18. $\dfrac{6.4\,\text{cosec}\, 29°5' - \sec 81°}{2\cot 12°}$ [0.7199]

19. Determine the acute angle, in degrees and minutes, correct to the nearest minute, given by: $\sin^{-1}\left(\dfrac{4.32\sin 42°16'}{7.86}\right)$ [21°42′]

20. If $\tan x = 1.5276$, determine $\sec x$, $\text{cosec}\, x$, and $\cot x$. (Assume x is an acute angle) [1.8258, 1.1952, 0.6546]

In Problems 21 to 23 evaluate correct to 4 significant figures.

21. $\dfrac{(\sin 34°27')(\cos 69°2')}{(2\tan 53°39')}$ [0.07448]

22. $3\cot 14°15' \sec 23°9'$ [12.85]

23. $\dfrac{\text{cosec}\, 27°19' + \sec 45°29'}{1 - \text{cosec}\, 27°19' \sec 45°29'}$ [−1.710]

24. Evaluate correct to 4 decimal places:

(a) sine(−125°) (b) tan(−241°)
(c) cos(−49°15′)

[(a) −0.8192 (b) −1.8040 (c) 0.6528]

25. Evaluate correct to 5 significant figures:

(a) cosec(−143°) (b) cot(−252°)
(c) sec(−67°22′)

[(a) −1.6616 (b) −0.32492 (c) 2.5985]

21.8 Trigonometric approximations for small angles

If angle x is a small angle (i.e. less than about 5°) and is expressed in radians, then the following trigonometric approximations may be shown to be true:

(i) $\mathbf{\sin x \approx x}$
(ii) $\mathbf{\tan x \approx x}$
(iii) $\mathbf{\cos x \approx 1 - \dfrac{x^2}{2}}$

For example, let $x = 1°$, i.e. $1 \times \dfrac{\pi}{180} = 0.01745$ radians, correct to 5 decimal places. By calculator, $\sin 1° = 0.01745$ and $\tan 1° = 0.01746$, showing that: $\sin x = x \approx \tan x$ when $x = 0.01745$ radians. Also, $\cos 1° = 0.99985$; when $x = 1°$, i.e. 0.001745 radians,

$$1 - \frac{x^2}{2} = 1 - \frac{0.01745^2}{2} = 0.99985,$$

correct to 5 decimal places, showing that

$$\cos x = 1 - \frac{x^2}{2} \text{ when } x = 0.01745 \text{ radians.}$$

Similarly, let $x = 5°$, i.e. $5 \times \dfrac{\pi}{180} = 0.08727$ radians, correct to 5 decimal places.

By calculator, $\sin 5° = 0.08716$, thus $\sin x \approx x$,
$\tan 5° = 0.08749$, thus $\tan x \approx x$,
and $\cos 5° = 0.99619$;

since $x = 0.08727$ radians, $1 - \dfrac{x^2}{2} = 1 - \dfrac{0.08727^2}{2} = 0.99619$, showing that:

$$\cos x = 1 - \frac{x^2}{2} \text{ when } x = 0.0827 \text{ radians.}$$

If $\sin x \approx x$ for small angles, then $\dfrac{\sin x}{x} \approx \mathbf{1}$, and this relationship can occur in engineering considerations.

If phasor OR makes one revolution (i.e. 2π radians) in T seconds, then the angular velocity, $\omega = 2\pi/T$ rad/s,

from which, $\boxed{\boldsymbol{T = 2\pi/\omega} \textbf{ seconds}}$

T is known as the **periodic time**.

The number of complete cycles occurring per second is called the **frequency, *f***

$$\text{Frequency} = \frac{\text{number of cycles}}{\text{second}} = \frac{1}{T} = \frac{\omega}{2\pi}\text{ Hz}$$

i.e. $\boxed{\boldsymbol{f = \frac{\omega}{2\pi}} \textbf{ Hz}}$

Hence **angular velocity,** $\boxed{\boldsymbol{\omega = 2\pi f} \textbf{ rad/s}}$

Amplitude is the name given to the maximum or peak value of a sine wave, as explained in Section 22.4. The amplitude of the sine wave shown in Fig. 22.26 has an amplitude of 1.

A sine or cosine wave may not always start at 0°. To show this a periodic function is represented by $y = \sin(\omega t \pm \alpha)$ or $y = \cos(\omega t \pm \alpha)$, where α is a phase displacement compared with $y = \sin A$ or $y = \cos A$. A graph of $y = \sin(\omega t - \alpha)$ **lags** $y = \sin\omega t$ by angle α, and a graph of $y = \sin(\omega t + \alpha)$ **leads** $y = \sin\omega t$ by angle α.

The angle ωt is measured in **radians**

$$\left[\text{i.e. } \left(\omega\frac{\text{rad}}{\text{s}}\right)(t\ s) = \omega t \text{ radians}\right]$$

hence angle α should also be in radians.

The relationship between degrees and radians is:

$$360^\circ = 2\pi \text{ radians or } \boxed{\mathbf{180^\circ = \boldsymbol{\pi} \textbf{ radians}}}$$

Hence $1 \text{ rad} = \dfrac{180}{\pi} = 57.30^\circ$ and, for example,

$$71^\circ = 71 \times \frac{\pi}{180} = 1.239 \text{ rad}$$

Summarising, given a general sinusoidal function $\boldsymbol{y = A\sin(\omega t \pm \alpha)}$, then:

(i) A = amplitude

(ii) ω = angular velocity $= 2\pi f$ rad/s

(iii) $\dfrac{2\pi}{\omega}$ = periodic time T seconds

(iv) $\dfrac{\omega}{2\pi}$ = frequency, f hertz

(v) α = angle of lead or lag (compared with $y = A\sin\omega t$)

Problem 11. An alternating current is given by $i = 30\sin(100\pi t + 0.27)$ amperes. Find the amplitude, periodic time, frequency and phase angle (in degrees and minutes)

$i = 30\sin(100\pi t + 0.27)A$, hence **amplitude = 30 A.**

Angular velocity $\omega = 100\pi$, hence

periodic time, $\displaystyle T = \frac{2\pi}{\omega} = \frac{2\pi}{100\pi} = \frac{1}{50}$

$= \mathbf{0.02}$ **s** or **20 ms**

Frequency, $\displaystyle f = \frac{1}{T} = \frac{1}{0.02} = \mathbf{50}$ **Hz**

Phase angle, $\displaystyle \alpha = 0.27 \text{ rad} = \left(0.27 \times \frac{180}{\pi}\right)^\circ$

$= \mathbf{15.47^\circ}$ or $\mathbf{15^\circ 28'}$ **leading**

$\boldsymbol{i = 30\sin(100\pi t)}$

Problem 12. An oscillating mechanism has a maximum displacement of 2.5 m and a frequency of 60 Hz. At time $t = 0$ the displacement is 90 cm. Express the displacement in the general form $A\sin(\omega t \pm \alpha)$

Amplitude = maximum displacement = 2.5 m

Angular velocity, $\omega = 2\pi f = 2\pi(60) = 120\pi$ rad/s

Hence displacement $= 2.5\sin(120\pi t + \alpha)$ m

When $t = 0$, displacement = 90 cm = 0.90 m

Hence, $0.90 = 2.5\sin(0 + \alpha)$

i.e. $\displaystyle \sin\alpha = \frac{0.90}{2.5} = 0.36$

Hence $\alpha = \sin^{-1} 0.36 = 21.10^\circ$

$= 21^\circ 6' = 0.368$ rad

Thus, **displacement** $= \mathbf{2.5\sin(120\boldsymbol{\pi} t + 0.368)}$ **m**

23

Cartesian and polar co-ordinates

23.1 Introduction

There are two ways in which the position of a point in a plane can be represented. These are

(a) by **Cartesian co-ordinates**, i.e. (x, y), and

(b) by **polar co-ordinates**, i.e. (r, θ), where r is a 'radius' from a fixed point and θ is an angle from a fixed point.

23.2 Changing from Cartesian into polar co-ordinates

In Fig. 23.1, if lengths x and y are known, then the length of r can be obtained from Pythagoras' theorem (see Chapter 21) since OPQ is a right-angled triangle.

Hence $\quad r^2 = (x^2 + y^2)$

from which, $\quad \boxed{r = \sqrt{x^2 + y^2}}$

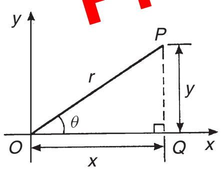


Figure 23.1

From trigonometric ratios (see Chapter 21),

$$\tan\theta = \frac{y}{x}$$

from which $\quad \boxed{\theta = \tan^{-1}\frac{y}{x}}$

$r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$ are the two formulae we need to change from Cartesian to polar co-ordinates. The angle θ, which may be expressed in degrees or radians, must **always** be measured from the positive x-axis, i.e. measured from the line OQ in Fig. 23.1. It is suggested that when changing from Cartesian to polar co-ordinates a diagram should always be sketched.

Problem 1. Change the Cartesian co-ordinates (3, 4) into polar co-ordinates.

A diagram representing the point (3, 4) is shown in Fig. 23.2.

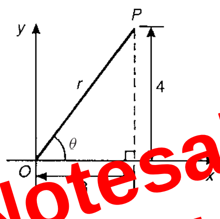


Figure 23.2

From Pythagoras' theorem, $r = \sqrt{3^2 + 4^2} = 5$ (note that -5 has no meaning in this context). By trigonometric ratios, $\theta = \tan^{-1}\frac{4}{3} = 53.13°$ or 0.927 rad [note that $53.13° = 53.13 \times (\pi/180)$ rad $= 0.927$ rad.]

Hence (3, 4) in Cartesian co-ordinates corresponds to (5, 53.13°) or (5, 0.927 rad) in polar co-ordinates.

Problem 2. Express in polar co-ordinates the position $(-4, 3)$

A diagram representing the point using the Cartesian co-ordinates $(-4, 3)$ is shown in Fig. 23.3.
From Pythagoras' theorem, $r = \sqrt{4^2 + 3^2} = 5$

By trigonometric ratios, $\alpha = \tan^{-1}\frac{3}{4} = 36.87°$ or 0.644 rad.

Hence $\quad \theta = 180° - 36.87° = 143.13°$

or $\quad \theta = \pi - 0.644 = 2.498$ rad.

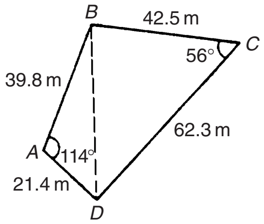


Figure 24.19

A diagonal drawn from B to D divides the quadrilateral into two triangles.

Area of quadrilateral $ABCD$

$= \text{area of triangle } ABD + \text{area of triangle } BCD$

$= \frac{1}{2}(39.8)(21.4)\sin 114^\circ + \frac{1}{2}(42.5)(62.3)\sin 56^\circ$

$= 389.04 + 1097.5 = \mathbf{1487\ m^2}$

Now try the following exercise

Exercise 92 Further problems on practical situations involving trigonometry

1. Three forces acting on a fixed point are represented by the sides of a triangle of dimensions 7.2 cm, 9.6 cm and 11.0 cm. Determine the angles between the lines of action and the three forces. [80.42°, 59.38°, 40.20°]

2. A vertical aerial AB, 9.60 m high, stands on ground which is inclined 12° to the horizontal. A stay connects the top of the aerial A to a point C on the ground 10.0 m downhill from B, the foot of the aerial. Determine (a) the length of the stay, and (b) the angle the stay makes with the ground. [(a) 15.23 m (b) 38.07°]

3. A reciprocating engine mechanism is shown in Fig. 24.20. The crank AB is 12.0 cm long and the connecting rod BC is 32.0 cm long. For the position shown determine the length of AC and the angle between the crank and the connecting rod. [40.25 cm, 126.05°]

4. From Fig. 22.20, determine how far C moves, correct to the nearest millimetre when angle CAB changes from 40° to 160°, B moving in an anticlockwise direction. [19.8 cm]

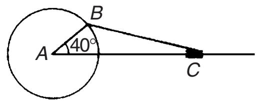


Figure 24.20

5. A surveyor, standing W 25° S of a tower measures the angle of elevation of the top of the tower as 46°30′. From a position E 23° S from the tower the elevation of the top is 37°15′. Determine the height of the tower if the distance between the two observations is 75 m. [36.2 m]

6. Calculate, correct to 3 significant figures, the co-ordinates x and y to locate the hole centre at P shown in Fig. 24.21. [$x = 69.3$ mm, $y = 142$ mm]

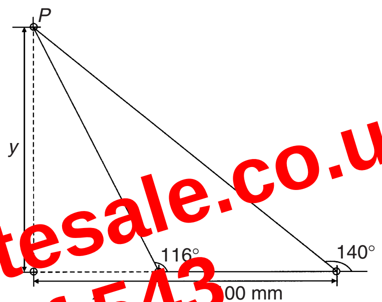


Figure 24.21

7. An idler gear, 30 mm in diameter, has to be fitted between a 70 mm diameter driving gear and a 90 mm diameter driven gear as shown in Fig. 24.22. Determine the value of angle θ between the centre lines. [130°]

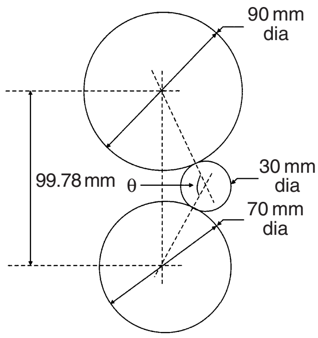


Figure 24.22

Factorising gives $(5\sin t + 2)(\sin t - 1) = 0$. Hence $5\sin t + 2 = 0$, from which, $\sin t = -\frac{2}{5} = -0.4000$, or $\sin t - 1 = 0$, from which, $\sin t = 1$.
$t = \sin^{-1}(-0.4000) = 203.58°$ or $336.42°$, since sine is negative in the third and fourth quadrants, or $t = \sin^{-1} 1 = 90°$. Hence

$$\mathbf{t = 90°, 203.58° \text{ or } 336.42°}$$

as shown in Fig. 25.7.

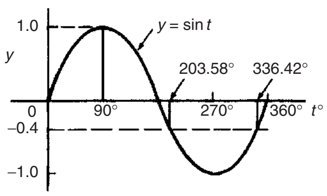


Figure 25.7

> Problem 14. Solve: $18\sec^2 A - 3\tan A = 21$ for values of A between 0° and 360°

$1 + \tan^2 A = \sec^2 A$. Substituting for $\sec^2 A$ in $18\sec^2 A - 3\tan A = 21$ gives

$$18(1 + \tan^2 A) - 3\tan A = 21$$

i.e. $$18 + 18\tan^2 A - 3\tan A - 21 = 0$$

$$18\tan^2 A - 3\tan A - 3 = 0$$

Factorising gives $(6\tan A - 3)(3\tan A + 1) = 0$
Hence $6\tan A - 3 = 0$, from which, $\tan A = \frac{3}{6} = 0.5000$ or $3\tan A + 1 = 0$, from which, $\tan A = -\frac{1}{3} = -0.3333$. Thus $A = \tan^{-1}(0.5000) = 26.57°$ or $206.57°$, since tangent is positive in the first and third quadrants, or $A = \tan^{-1}(-0.3333) = 161.57°$ or $341.57°$, since tangent is negative in the second and fourth quadrants. Hence

$$\mathbf{A = 26.57°, 161.57°, 206.57° \text{ or } 341.57°}$$

> Problem 15. Solve: $3\,\text{cosec}^2\theta - 5 = 4\cot\theta$ in the range $0 < \theta < 360°$

$\cot^2\theta + 1 = \text{cosec}^2\theta$. Substituting for $\text{cosec}^2\theta$ in $3\,\text{cosec}^2\theta - 5 = 4\cot\theta$ gives:

$$3(\cot^2\theta + 1) - 5 = 4\cot\theta$$

$$3\cot^2\theta + 3 - 5 = 4\cot\theta$$

$$3\cot^2\theta - 4\cot\theta - 2 = 0$$

Since the left-hand side does not factorise the quadratic formula is used.

Thus, $$\cot\theta = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{4 \pm \sqrt{16 + 24}}{6} = \frac{4 \pm \sqrt{40}}{6}$$

$$= \frac{10.3246}{6} \text{ or } -\frac{2.3246}{6}$$

Hence $\cot\theta = 1.7208$ or -0.3874,
$\theta = \cot^{-1} 1.7208 = 30.17°$ or $210.17°$, since cotangent is positive in the first and third quadrants, or $\theta = \cot^{-1}(-0.3874) = 111.18°$ or $291.18°$, since cotangent is negative in the second and fourth quadrants.

Hence, $\mathbf{\theta = 30.17°, 111.18°, 210.17° \text{ or } 291.18°}$

Now try the following exercise

Exercise 97 Further problems on trigonometric equations

Solve the following equations for angles between 0° and 360°

1. $12\sin^2\theta - 6 = \cos\theta$
   $\left[\begin{array}{l}\theta = 48.18°, 138.58°, \\ 221.42° \text{ or } 311.82°\end{array}\right]$

2. $16\sec x - 2 = 14\tan^2 x$
   $[x = 52.93° \text{ or } 307.07°]$

3. $4\cot^2 A - 6\,\text{cosec}\,A + 6 = 0$
   $[A = 90°]$

4. $5\sec t + 2\tan^2 t = 3$
   $[t = 107.83° \text{ or } 252.17°]$

5. $2.9\cos^2 a - 7\sin a + 1 = 0$
   $[a = 27.83° \text{ or } 152.17°]$

6. $3\,\text{cosec}^2\beta = 8 - 7\cot\beta$
   $\left[\begin{array}{l}\beta = 60.17°, 161.02°, \\ 240.17° \text{ or } 341.02°\end{array}\right]$

Multiple choice questions on chapters 17–26

All questions have only one correct answer (answers on page 526).

1. In the right-angled triangle ABC shown in Figure M2.1, sine A is given by:

   (a) b/a (b) c/b (c) b/c (d) a/b

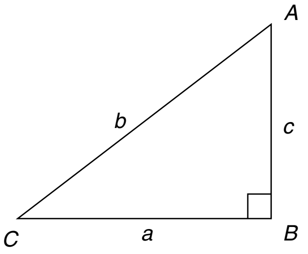


Figure M2.1

2. In the right-angled triangle ABC shown in Figure M2.1, cosine C is given by:

   (a) a/b (b) c/b (c) a/c (d) b/a

3. In the right-angled triangle shown in Figure M2.1, tangent A is given by:

   (a) b/c (b) a/c (c) a/b (d) c/a

4. $\dfrac{3\pi}{4}$ radians is equivalent to:

   (a) 135° (b) 270° (c) 45° (d) 67.5°

5. In the triangular template ABC shown in Figure M2.2, the length AC is:

   (a) 6.17 cm (b) 11.17 cm

   (c) 9.22 cm (d) 12.40 cm

6. $(-4, 3)$ in polar co-ordinates is:

   (a) (5, 2.498 rad) (b) (7, 36.87°)

   (c) (5, 36.87°) (d) (5, 323.13°)

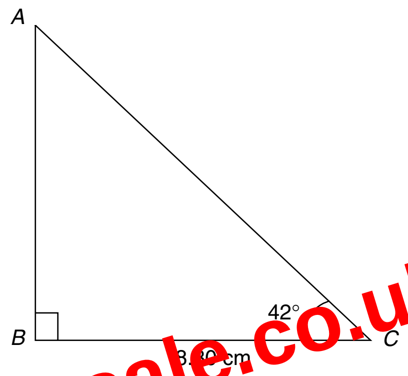


Figure M2.2

7. Correct to 3 decimal places, $\sin(-2.6 \text{ rad})$ is:

   (a) 0.516 (b) −0.045 (c) −0.516 (d) 0.045

8. For the right-angled triangle PQR shown in Figure M2.3, angle R is equal to:

   (a) 41.41° (b) 48.59° (c) 36.87° (d) 53.13°

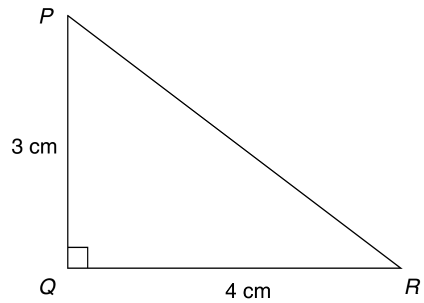


Figure M2.3

9. A hollow shaft has an outside diameter of 6.0 cm and an inside diameter of 4.0 cm. The cross-sectional area of the shaft is:

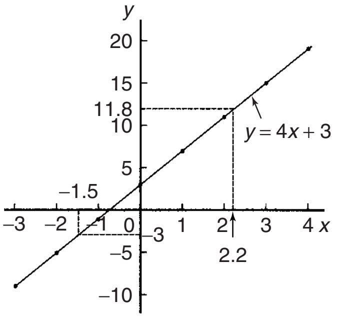


Figure 27.4

A table of co-ordinates is produced for each graph.

(a) $y = x$

x	−4	−3	−2	−1	0	1	2	3	4
y	−4	−3	−2	−1	0	1	2	3	4

(b) $y = x + 2$

x	−4	−3	−2	−1	0	1	2	3	4
y	−2	−1	0	1	2	3	4	5	6

(c) $y = x + 5$

x	−4	−3	−2	−1	0	1	2	3	4
y	1	2	3	4	5	6	7	8	9

(d) $y = x - 3$

x	−4	−3	−2	−1	0	1	2	3	4
y	−7	−6	−5	−4	−3	−2	−1	0	1

The co-ordinates are plotted and joined for each graph. The results are shown in Fig. 27.5. Each of the straight lines produced are parallel to each other, i.e. the slope or gradient is the same for each.

To find the gradient of any straight line, say, $y = x - 3$ a horizontal and vertical component needs to be constructed. In Fig. 27.5, AB is constructed vertically at $x = 4$ and BC constructed horizontally at $y = -3$.

$$\text{The gradient of } AC = \frac{AB}{BC} = \frac{1-(-3)}{4-0}$$

$$= \frac{4}{4} = 1$$

i.e. the gradient of the straight line $y = x - 3$ is 1. The actual positioning of AB and BC is unimportant

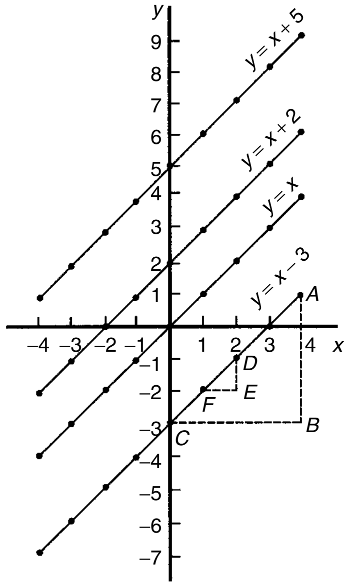


Figure 27.5

for the gradient is also given by, for example,

$$\frac{DE}{EF} = \frac{-1-(-2)}{2-1} = \frac{1}{1} = 1$$

The slope or gradient of each of the straight lines in Fig. 27.5 is thus 1 since they are all parallel to each other.

Problem 3. Plot the following graphs on the same axes between the values $x = -3$ to $x = +3$ and determine the gradient and y-axis intercept of each.

(a) $y = 3x$ (b) $y = 3x + 7$

(c) $y = -4x + 4$ (d) $y = -4x - 5$

A table of co-ordinates is drawn up for each equation.

(a) $y = 3x$

x	−3	−2	−1	0	1	2	3
y	−9	−6	−3	0	3	6	9

(b) $y = 3x + 7$

x	−3	−2	−1	0	1	2	3
y	−2	1	4	7	10	13	16

x	2	0	−3
$\frac{1}{2}x + 2\frac{1}{2}$	$3\frac{1}{2}$	$2\frac{1}{2}$	1

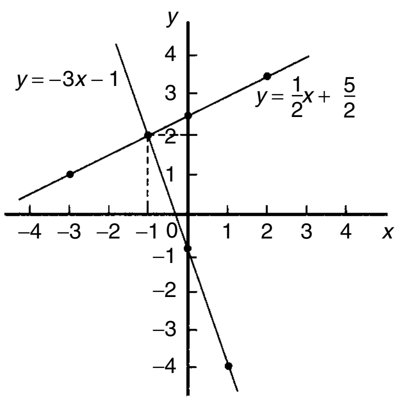


Figure 27.8

The graphs are plotted as shown in Fig. 27.8.

The two straight lines are seen to intersect at (−1, 2).

Now try the following exercise

Exercise 103 Further problems on straight line graphs

1. Corresponding values obtained experimentally for two quantities are:

x	−2.0	−0.5	0	1.0	2.5	3.0	5.0
y	−13.0	−5.5	−3.0	2.0	9.5	12.0	22.0

Use a horizontal scale for x of 1 cm $= \frac{1}{2}$ unit and a vertical scale for y of 1 cm = 2 units and draw a graph of x against y. Label the graph and each of its axes. By interpolation, find from the graph the value of y when x is 3.5

[14.5]

2. The equation of a line is $4y = 2x + 5$. A table of corresponding values is produced and is shown below. Complete the table and plot a graph of y against x. Find the gradient of the graph.

x	−4	−3	−2	−1	0	1	2	3	4
y		−0.25			1.25				3.25

$\left[\frac{1}{2}\right]$

3. Determine the gradient and intercept on the y-axis for each of the following equations:

(a) $y = 4x - 2$ (b) $y = -x$

(c) $y = -3x - 4$ (d) $y = 4$

$\left[\begin{matrix}(a)\ 4, -2 & (b)\ -1, 0\\ (c)\ -3, -4 & (d)\ 0, 4\end{matrix}\right]$

4. Find the gradient and intercept on the y-axis for each of the following equations:

(a) $2y - 1 = 4x$ (b) $6x - 2y = 5$

(c) $3(2y - 1) = \dfrac{x}{4}$

$\left[(a)\ 2, \frac{1}{2} \quad (b)\ 3, -2\frac{1}{2} \quad (c)\ \frac{1}{24}, \frac{1}{2}\right]$

5. Determine the gradient and y-axis intercept for each of the following equations and sketch the graphs:

(a) $y = 6x - 3$ (b) $y = 3x$ (c) $y = 7$

(d) $2x + 3y + 5 = 0$

$\left[\begin{matrix}(a)\ 6, -3 & (b)\ 3, 0\\ (c)\ 0, 7 & (d)\ -\frac{2}{3}, -1\frac{2}{3}\end{matrix}\right]$

6. Determine the gradient of the straight line graphs passing through the co-ordinates:

(a) (2, 7) and (−3, 4)

(b) (−4, −1) and (−5, 3)

(c) $\left(\frac{1}{4}, -\frac{3}{4}\right)$ and $\left(-\frac{1}{2}, \frac{5}{8}\right)$

$\left[(a)\ \frac{3}{5} \quad (b)\ -4 \quad (c)\ -1\frac{5}{6}\right]$

7. State which of the following equations will produce graphs which are parallel to one another:

(a) $y - 4 = 2x$ (b) $4x = -(y + 1)$

(c) $x = \frac{1}{2}(y + 5)$ (d) $1 + \frac{1}{2}y = \frac{3}{2}x$

(e) $2x = \frac{1}{2}(7 - y)$

[(a) and (c), (b) and (e)]

8. Draw a graph of $y - 3x + 5 = 0$ over a range of $x = -3$ to $x = 4$. Hence

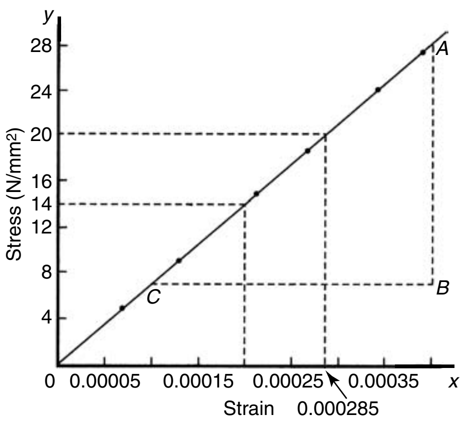


Figure 27.12

Since 1 $m^2 = 10^6$ mm^2, 70 000 N/mm^2 is equivalent to $70\,000 \times 10^6$ N/m^2, i.e. **70×10^9 N/m^2 (or Pascals).**

From Fig. 27.12:

(b) the value of the strain at a stress of 20 N/mm^2 is **0.000285**, and

(c) the value of the stress when the strain is 0.00020 is **14 N/mm^2.**

> Problem 11. The following values of resistance R ohms and corresponding voltage V volts are obtained from a test on a filament lamp.
>
> | R ohms | 30 | 48.5 | 73 | 107 | 128 |
> |---|---|---|---|---|---|
> | V volts | 16 | 29 | 52 | 76 | 94 |
>
> Choose suitable scales and plot a graph with R representing the vertical axis and V the horizontal axis. Determine (a) the gradient of the graph, (b) the R axis intercept value, (c) the equation of the graph, (d) the value of resistance when the voltage is 60 V, and (e) the value of the voltage when the resistance is 40 ohms. (f) If the graph were to continue in the same manner, what value of resistance would be obtained at 110 V?

The co-ordinates (16, 30), (29, 48.5), and so on, are shown plotted in Fig. 27.13 where the best straight line is drawn through the points.

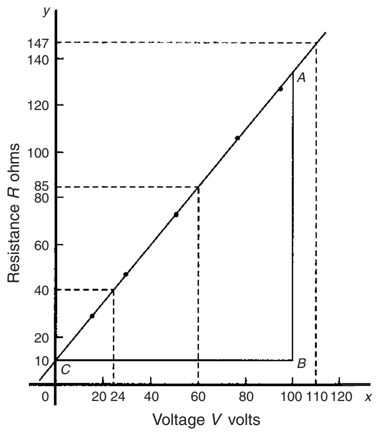


Figure 27.13

(a) The slope or gradient of the straight line AC is given by:

$$\frac{AB}{BC} = \frac{135 - 10}{100 - 0} = \frac{125}{100} = \mathbf{1.25}$$

(Note that the vertical line AB and the horizontal line BC may be constructed anywhere along the length of the straight line. However, calculations are made easier if the horizontal line BC is carefully chosen, in this case, 100).

(b) The R-axis intercept is at $\mathbf{R = 10}$ **ohms** (by extrapolation).

(c) The equation of a straight line is $y = mx + c$, when y is plotted on the vertical axis and x on the horizontal axis. m represents the gradient and c the y-axis intercept. In this case, R corresponds to y, V corresponds to x, $m = 1.25$ and $c = 10$. Hence the equation of the graph is $\mathbf{R = (1.25\,V + 10)\ \Omega}$

From Fig. 27.13,

(d) when the voltage is 60 V, the resistance is **85 Ω**

(e) when the resistance is 40 ohms, the voltage is **24 V**, and

(f) by extrapolation, when the voltage is 110 V, the resistance is **147 Ω**.

> Problem 12. Experimental tests to determine the breaking stress σ of rolled copper at various temperatures t gave the following results.

28

Reduction of non-linear laws to linear form

28.1 Determination of law

Frequently, the relationship between two variables, say x and y, is not a linear one, i.e. when x is plotted against y a curve results. In such cases the non-linear equation may be modified to the linear form, $y = mx + c$, so that the constants, and thus the law relating the variables can be determined. This technique is called **'determination of law'**.

Some examples of the reduction of equations to linear form include:

(i) $y = ax^2 + b$ compares with $Y = mX + c$, where $m = a$, $c = b$ and $X = x^2$.

Hence y is plotted vertically against x^2 horizontally to produce a straight line graph of gradient 'a' and y-axis intercept 'b'

(ii) $y = \dfrac{a}{x} + b$

y is plotted vertically against $\dfrac{1}{x}$ horizontally to produce a straight line graph of gradient 'a' and y-axis intercept 'b'

(iii) $y = ax^2 + bx$

Dividing both sides by x gives $\dfrac{y}{x} = ax + b$.

Comparing with $Y = mX + c$ shows that $\dfrac{y}{x}$ is plotted vertically against x horizontally to produce a straight line graph of gradient 'a' and $\dfrac{y}{x}$ axis intercept 'b'

Problem 1. Experimental values of x and y, shown below, are believed to be related by the law $y = ax^2 + b$. By plotting a suitable graph verify this law and determine approximate values of a and b

x	1	2	3	4	5
y	9.8	15.2	24.2	36.5	53.0

If y is plotted against x a curve results and it is not possible to determine the values of constants a and b from the curve. Comparing $y = ax^2 + b$ with $Y = mX + c$ shows that y is to be plotted vertically against x^2 horizontally. A table of values is drawn up as shown below.

x	1	2	3	4	5
x^2	1	4	9	16	25
y	9.8	15.2	24.2	36.5	53.0

A graph of y against x^2 is shown in Fig. 28.1, with the best straight line drawn through the points. Since a straight line graph results, the law is verified.

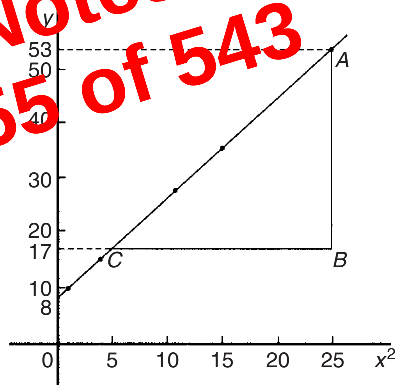


Figure 28.1

From the graph, gradient

$$a = \frac{AB}{BC} = \frac{53 - 17}{25 - 5} = \frac{36}{20} = \mathbf{1.8}$$

and the y-axis intercept,

$$\boldsymbol{b = 8.0}$$

Hence the law of the graph is:

$$\boldsymbol{y = 1.8x^2 + 8.0}$$

Problem 2. Values of load L newtons and distance d metres obtained experimentally are shown in the following table

Load, L N	32.3	29.6	27.0	23.2
distance, d m	0.75	0.37	0.24	0.17

Load, L N	18.3	12.8	10.0	6.4
distance, d m	0.12	0.09	0.08	0.07

Verify that load and distance are related by a law of the form $L = \frac{a}{d} + b$ and determine approximate values of a and b. Hence calculate the load when the distance is 0.20 m and the distance when the load is 20 N.

Comparing $L = \frac{a}{d} + b$ i.e. $L = a\left(\frac{1}{d}\right) + b$ with $Y = mX + c$ shows that L is to be plotted vertically against $\frac{1}{d}$ horizontally. Another table of values is drawn up as shown below.

L	32.3	29.6	27.0	23.2	18.3	12.8	10.0	6.4
d	0.75	0.37	0.24	0.17	0.12	0.09	0.08	0.07
$\frac{1}{d}$	1.33	2.70	4.17	5.88	8.33	11.11	12.50	14.29

A graph of L against $\frac{1}{d}$ is shown in Fig. 28.2. A straight line can be drawn through the points, which verifies that load and distance are related by a law of the form $L = \frac{a}{d} + b$

Gradient of straight line,

$$a = \frac{AB}{BC} = \frac{31 - 11}{2 - 12} = \frac{20}{-10} = \mathbf{-2}$$

L-axis intercept,

$$b = \mathbf{35}$$

Hence the law of the graph is

$$L = -\frac{2}{d} + 35$$

When the distance $d = 0.20$ m, load

$$L = \frac{-2}{0.20} + 35 = \mathbf{25.0\ N}$$

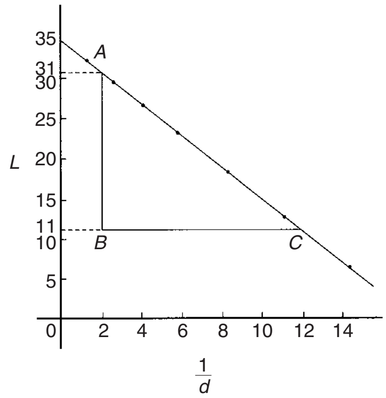


Figure 28.2

Rearranging $L = -\frac{2}{d} + 35$ gives:

$$\frac{2}{d} = 35 - L \quad \text{and} \quad d = \frac{2}{35 - L}$$

Hence when the load $L = 20$ N, distance

$$d = \frac{2}{35 - 20} = \frac{2}{15} = \mathbf{0.13\ m}$$

Problem 3. The solubility s of potassium chlorate is shown by the following table:

t °C	10	20	30	40	50	60	80	100
s	4.9	7.6	11.1	15.4	20.4	26.4	40.6	58.0

The relationship between s and t is thought to be of the form $s = 3 + at + bt^2$. Plot a graph to test the supposition and use the graph to find approximate values of a and b. Hence calculate the solubility of potassium chlorate at 70 °C

Rearranging $s = 3 + at + bt^2$ gives $s - 3 = at + bt^2$ and $\frac{s-3}{t} = a + bt$ or $\frac{s-3}{t} = bt + a$ which is of the form $Y = mX + c$, showing that $\frac{s-3}{t}$ is to be plotted vertically and t horizontally. Another table of values is drawn up as shown below.

t	10	20	30	40	50	60	80	100
s	4.9	7.6	11.1	15.4	20.4	26.4	40.6	58.0
$\frac{s-3}{t}$	0.19	0.23	0.27	0.31	0.35	0.39	0.47	0.55

a constant. Determine (a) the value of constant c and (b) the safe load for a span of 3.0 m. [(a) 950 (b) 317 kN]

9. The following results give corresponding values of two quantities x and y which are believed to be related by a law of the form $y = ax^2 + bx$ where a and b are constants.

y	33.86	55.54	72.80	84.10	111.4	168.1
x	3.4	5.2	6.5	7.3	9.1	12.4

Verify the law and determine approximate values of a and b.

Hence determine (i) the value of y when x is 8.0 and (ii) the value of x when y is 146.5

[$a = 0.4$, $b = 8.6$ (i) 94.4 (ii) 11.2]

28.2 Determination of law involving logarithms

Examples of reduction of equations to linear form involving logarithms include:

(i) $y = ax^n$

Taking logarithms to a base of 10 of both sides gives:

$$\lg y = \lg(ax^n) = \lg a + \lg x^n$$

i.e. $\lg y = n \lg x + \lg a$

by the laws of logarithms which compares with

$$Y = mX + c$$

and shows that $\lg y$ is plotted vertically against $\lg x$ horizontally to produce a straight line graph of gradient n and $\lg y$-axis intercept $\lg a$

(ii) $y = ab^x$

Taking logarithms to a base of 10 of the both sides gives:

$$\lg y = \lg(ab^x)$$

i.e. $\lg y = \lg a + \lg b^x$

i.e. $\lg y = x \lg b + \lg a$

by the laws of logarithms

or $\lg y = (\lg b)x + \lg a$

which compares with

$$Y = mX + c$$

and shows that $\lg y$ is plotted vertically against x horizontally to produce a straight line graph of gradient $\lg b$ and $\lg y$-axis intercept $\lg a$

(iii) $y = ae^{bx}$

Taking logarithms to a base of e of both sides gives:

$$\ln y = \ln(ae^{bx})$$

i.e. $\ln y = \ln a + \ln e^{bx}$

i.e. $\ln y = \ln a + bx \ln e$

i.e. $\ln y = bx + \ln a$

(since $\ln e = 1$), which compares with

$$Y = mX + c$$

and shows that $\ln y$ is plotted vertically against x horizontally to produce a straight line graph of gradient b and $\ln y$-axis intercept $\ln a$

Problem 4. The current flowing in, and the power dissipated by, a resistor are measured experimentally for various values and the results are as shown below.

Current, I amperes	2.2	3.6	4.1	5.6	6.8
Power, P watts	116	311	403	753	1110

Show that the law relating current and power is of the form $P = RI^n$, where R and n are constants, and determine the law

Taking logarithms to a base of 10 of both sides of $P = RI^n$ gives:

$$\lg P = \lg(RI^n) = \lg R + \lg I^n = \lg R + n \lg I$$

by the laws of logarithms

i.e. $\lg P = n \lg I + \lg R$,

constants. Determine the approximate values of I and T

Taking Napierian logarithms of both sides of $i = Ie^{t/T}$ gives

$$\ln i = \ln(Ie^{t/T}) = \ln I + \ln e^{t/T} = \ln I + \frac{t}{T}\ln e$$

i.e. $$\ln i = \ln I + \frac{t}{T} \quad (\text{since } \ln e = 1)$$

or $$\ln i = \left(\frac{1}{T}\right)t + \ln I$$

which compares with $y = mx + c$, showing that $\ln i$ is plotted vertically against t horizontally. (For methods of evaluating Napierian logarithms see Chapter 13.) Another table of values is drawn up as shown below

t	100	160	210	275	320	390
i	203	61.14	22.49	6.13	2.49	0.615
$\ln i$	5.31	4.11	3.11	1.81	0.91	−0.49

A graph of $\ln i$ against t is shown in Fig. 28.7 and since a straight line results the law $i = Ie^{t/T}$ is verified.

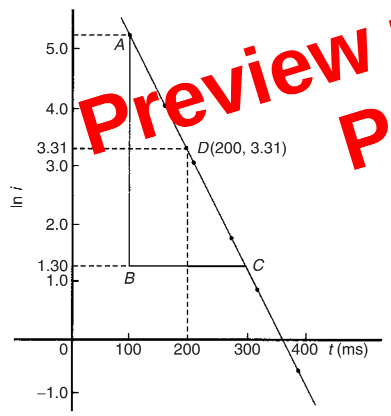


Figure 28.7

Gradient of straight line,

$$\frac{1}{T} = \frac{AB}{BC} = \frac{5.30 - 1.30}{100 - 300} = \frac{4.0}{-200} = -0.02$$

Hence $T = \dfrac{1}{-0.02} = \mathbf{-50}$

Selecting any point on the graph, say point D, where $t = 200$ and $\ln i = 3.31$, and substituting into

$$\ln i = \left(\frac{1}{T}\right)t + \ln I$$

gives: $$3.31 = -\frac{1}{50}(200) + \ln I$$

from which, $\ln I = 3.31 + 4.0 = 7.31$

and $I = \text{antilog } 7.31\ (= e^{7.31}) = 1495$

or **1500** correct to 3 significant figures.

Hence the law of the graph is, $i = 1500\, e^{-t/50}$

Now try the following exercise

Exercise 106 Further problems on reducing non-linear laws to linear form

In Problems 1 to 3, x and y are two related variables and all other letters denote constants. For the stated laws to be verified it is necessary to plot graphs of the variables in a modified form. State for each (a) what should be plotted on the vertical axis, (b) what should be plotted on the horizontal axis, (c) the gradient and (d) the vertical axis intercept.

1. $y = ba^x$
   [(a) lg y (b) x (c) lg a (d) lg b]
2. $y = kx^l$
   [(a) lg y (b) lg x (c) l (d) lg k]
3. $\dfrac{y}{m} = e^{nx}$
   [(a) ln y (b) x (c) n (d) ln m]
4. The luminosity I of a lamp varies with the applied voltage V and the relationship between I and V is thought to be $I = kV^n$. Experimental results obtained are:

I candelas	1.92	4.32	9.72
V volts	40	60	90

I candelas	15.87	23.52	30.72
V volts	115	140	160

   Verify that the law is true and determine the law of the graph. Determine also the luminosity when 75 V is applied across the lamp.
   [$I = 0.0012\ V^2$, 6.75 candelas]

The law relating x and y is believed to be $y = ax^b$, where a and b are constants. Verify that this law is true and determine the approximate values of a and b

If $y = ax^b$ then $\lg y = b \lg x + \lg a$, from above, which is of the form $Y = mX + c$, showing that to produce a straight line graph $\lg y$ is plotted vertically against $\lg x$ horizontally. x and y may be plotted directly on to log–log graph paper as shown in Fig. 29.2. The values of y range from 0.45 to 82.46 and 3 cycles are needed (i.e. 0.1 to 1, 1 to 10 and 10 to 100). The values of x range from 0.41 to 3.95 and 2 cycles are needed (i.e. 0.1 to 1 and 1 to 10). Hence 'log 3 cycle $\times$ 2 cycle' is used as shown in Fig. 29.2 where the axes are marked and the points plotted. Since the points lie on a straight line the law $y = ax^b$ is verified.

To evaluate constants a and b:

Method 1. Any two points on the straight line, say points A and C, are selected, and AB and BC are measured (say in centimetres).

Then, gradient, $\boldsymbol{b} = \dfrac{AB}{BC} = \dfrac{11.5 \text{ units}}{5 \text{ units}} = \mathbf{2.3}$

Since $\lg y = b \lg x + \lg a$, when $x = 1$, $\lg x = 0$ and $\lg y = \lg a$.

The straight line crosses the ordinate $x = 1.0$ at $y = 3.5$.

Hence $\lg a = \lg 3.5$, i.e. $\boldsymbol{a} = \mathbf{3.5}$

Method 2. Any two points on the straight line, say points A and C, are selected. A has coordinates (2, 17.25) and C has coordinates (0.5, 0.7).

Since $\quad y = ax^b$ then $17.25 = a(2)^b \qquad (1)$

and $\qquad 0.7 = a(0.5)^b \qquad (2)$

i.e. two simultaneous equations are produced and may be solved for a and b.

Dividing equation (1) by equation (2) to eliminate a gives:

$$\frac{17.25}{0.7} = \frac{(2)^b}{(0.5)^b} = \left(\frac{2}{0.5}\right)^b$$

i.e. $\quad 24.643 = (4)^b$

Taking logarithms of both sides gives $\lg 24.643 = b \lg 4$, i.e.

$$b = \frac{\lg 24.643}{\lg 4}$$

$$= 2.3, \text{ correct to 2 significant figures.}$$

Substituting $b = 2.3$ in equation (1) gives: $17.25 = a(2)^{2.3}$, i.e.

$$a = \frac{17.25}{(2)^{2.3}} = \frac{17.25}{4.925}$$

$$= 3.5, \text{ correct to 2 significant figures.}$$

Hence the law of the graph is: $\boldsymbol{y = 3.5x^{2.3}}$

Problem 2. The power dissipated by a resistor was measured for varying values of current flowing in the resistor and the results are as shown:

Current, I amperes	1.4	4.7	6.8	9.1	11.2	13.1
Power, P watts	49	552	1156	2070	3136	4290

Prove that the law relating current and power is of the form $P = RI^n$, where R and n are constants, and determine the law. Hence calculate the power when the current is 12 amperes and the current when the power is 1000 watts

Since $P = RI^n$ then $\lg P = n \lg I + \lg R$, which is of the form $Y = mX + c$, showing that to produce a straight line graph $\lg P$ is plotted vertically against $\lg I$ horizontally. Power values range from 49 to 4290, hence 3 cycles of log–log graph paper are needed (10 to 100, 100 to 1000 and 1000 to 10 000). Current values range from 1.4 to 11.2, hence 2 cycles of log–log graph paper are needed (1 to 10 and 10 to 100). Thus 'log 3 cycles $\times$ 2 cycles' is used as shown in Fig. 29.3 (or, if not available, graph paper having a larger number of cycles per axis can be used). The co-ordinates are plotted and a straight line results which proves that the law relating current and power is of the form $P = RI^n$. Gradient of straight line,

$$n = \frac{AB}{BC} = \frac{14 \text{ units}}{7 \text{ units}} = 2$$

30

Graphical solution of equations

30.1 Graphical solution of simultaneous equations

Linear simultaneous equations in two unknowns may be solved graphically by:

(i) plotting the two straight lines on the same axes, and

(ii) noting their point of intersection.

The co-ordinates of the point of intersection give the required solution.

Problem 1. Solve graphically the simultaneous equations:

$$2x - y = 4$$
$$x + y = 5$$

Rearranging each equation into $y = mx + c$ form gives:

$$y = 2x - 4 \qquad (1)$$
$$y = -x + 5 \qquad (2)$$

Only three co-ordinates need be calculated for each graph since both are straight lines.

x	0	1	2
$y = 2x - 4$	-4	-2	0

x	0	1	2
$y = -x + 5$	5	4	3

Each of the graphs is plotted as shown in Fig. 30.1. The point of intersection is at (3, 2) and since this is the only point which lies simultaneously on both lines then $\boldsymbol{x = 3, y = 2}$ is the solution of the simultaneous equations.

Problem 2. Solve graphically the equations:

$$1.20x + y = 1.80$$
$$x - 5.0y = 8.50$$

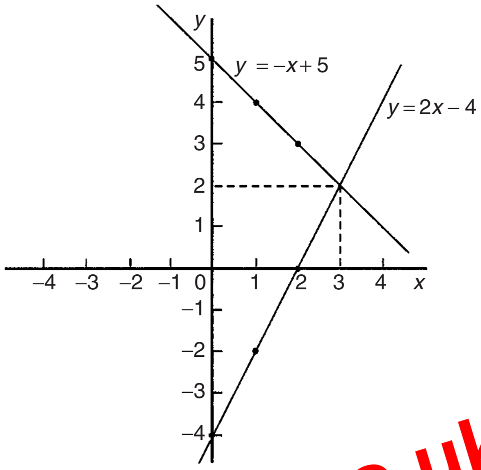


Figure 30.1

Rearranging each equation into $y = mx + c$ form gives:

$$y = -1.20x + 1.80 \qquad (1)$$

$$y = \frac{x}{5.0} - \frac{8.5}{5.0}$$

i.e. $\quad y = 0.20x - 1.70 \qquad (2)$

Three co-ordinates are calculated for each equation as shown below:

x	0	1	2
$y = -1.20x + 1.80$	1.80	0.60	-0.60

x	0	1	2
$y = 0.20x - 1.70$	-1.70	-1.50	-1.30

The two lines are plotted as shown in Fig. 30.2. The point of intersection is (2.50, -1.20). Hence the solution of the simultaneous equation is $\boldsymbol{x = 2.50, y = -1.20}$.

(It is sometimes useful initially to sketch the two straight lines to determine the region where the point of intersection is. Then, for greater accuracy, a graph having a smaller range of values can be drawn to 'magnify' the point of intersection).

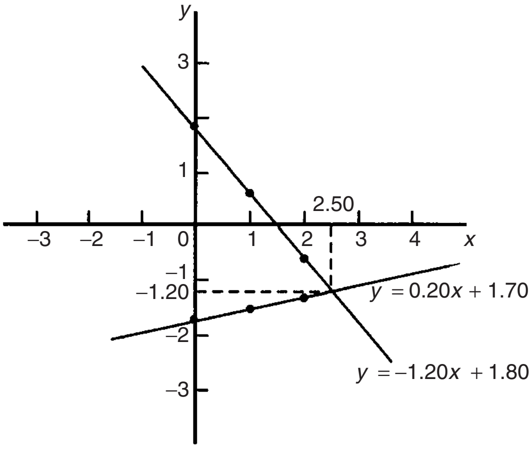


Figure 30.2

Now try the following exercise

Exercise 110 Further problems on the graphical solution of simultaneous equations

In Problems 1 to 5, solve the simultaneous equations graphically.

1. $x + y = 2$

   $3y - 2x = 1$ $\quad [x = 1,\ y = 1]$

2. $y = 5 - x$

   $x - y = 2$ $\quad [x = 3\frac{1}{2},\ y = 1\frac{1}{2}]$

3. $3x + 4y = 5$

   $2x - 5y + 12 = 0$ $\quad [x = -1,\ y = 2]$

4. $1.4x - 7.06 = 3.2y$

   $2.1x - 6.7y = 12.87$

   $[x = 2.3,\ y = -1.2]$

5. $3x - 2y = 0$

   $4x + y + 11 = 0$ $\quad [x = -2,\ y = -3]$

6. The friction force F Newton's and load L Newton's are connected by a law of the form $F = aL + b$, where a and b are constants. When $F = 4$ Newton's, $L = 6$ Newton's and when $F = 2.4$ Newton's, $L = 2$ Newton's. Determine graphically the values of a and b.

   $[a = 0.4,\ b = 1.6]$

30.2 Graphical solution of quadratic equations

A general **quadratic equation** is of the form $y = ax^2 + bx + c$, where a, b and c are constants and a is not equal to zero.

A graph of a quadratic equation always produces a shape called a **parabola**. The gradient of the curve between 0 and A and between B and C in Fig. 30.3 is positive, whilst the gradient between A and B is negative. Points such as A and B are called **turning points**. At A the gradient is zero and, as x increases, the gradient of the curve changes from positive just before A to negative just after. Such a point is called a **maximum value**. At B the gradient is also zero, and, as x increases, the gradient of the curve changes from negative just before B to positive just after. Such a point is called a **minimum value**.

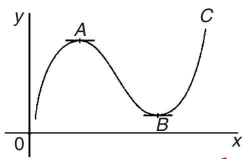


Figure 30.3

Quadratic graphs

(i) $y = ax^2$

Graphs of $y = x^2$, $y = 3x^2$ and $y = \frac{1}{2}x^2$ are shown in Fig. 30.4.

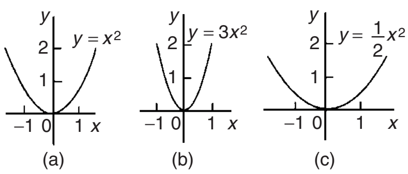


Figure 30.4

All have minimum values at the origin (0, 0).

Graphs of $y = -x^2$, $y = -3x^2$ and $y = -\frac{1}{2}x^2$ are shown in Fig. 30.5.

All have maximum values at the origin (0, 0).

When $y = ax^2$,

(a) curves are symmetrical about the y-axis,

(b) the magnitude of 'a' affects the gradient of the curve, and

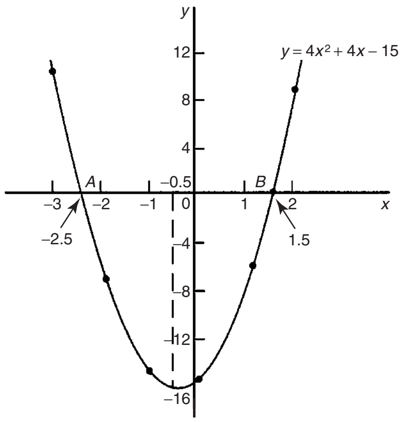


Figure 30.8

has a turning point at $(-0.5, -16)$ and the nature of the point is a **minimum**.

An alternative graphical method of solving $4x^2 + 4x - 15 = 0$ is to rearrange the equation as $4x^2 = -4x + 15$ and then plot two separate graphs — in this case $y = 4x^2$ and $y = -4x + 15$. Their points of intersection give the roots of equation $4x^2 = -4x + 15$, i.e. $4x^2 + 4x - 15 = 0$. This is shown in Fig. 30.9, where the roots are $x = -2.5$ and $x = 1.5$ as before.

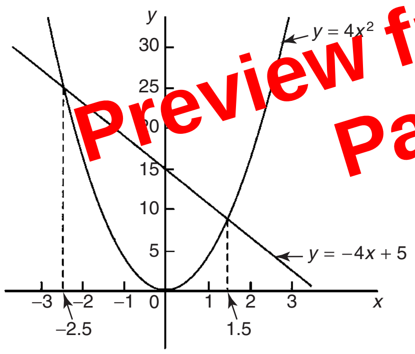


Figure 30.9

Problem 4. Solve graphically the quadratic equation $-5x^2 + 9x + 7.2 = 0$ given that the solutions lie between $x = -1$ and $x = 3$. Determine also the co-ordinates of the turning point and state its nature

Let $y = -5x^2 + 9x + 7.2$. A table of values is drawn up as shown below. A graph of $y = -5x^2 + 9x + 7.2$

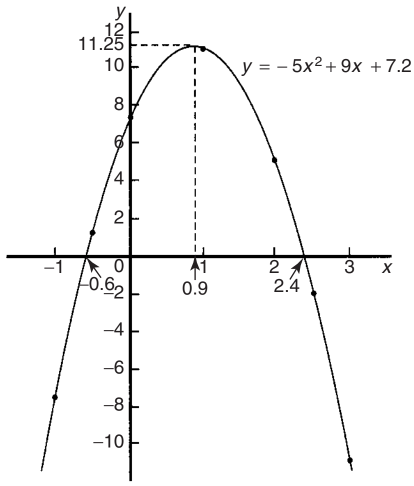


Figure 30.10

is shown plotted in Fig. 30.10. The graph crosses the x-axis (i.e. where $y = 0$) at $\boldsymbol{x = -0.6}$ **and** $\boldsymbol{x = 2.4}$ and these are the solutions of the quadratic equation $-5x^2 + 9x + 7.2 = 0$. The turning point is a **maximum** having co-ordinates **(0.9, 11.25)**.

x	−1	−0.5	0	1
$-5x^2$ $+9x$ $+7.2$	−5 −9 7.2	−1.25 −4.5 7.2	0 0 7.2	−5 9 7.2
$y = -5x^2 + 9x + 7.2$	−6.8	1.45	7.2	11.2

x	2	2.5	3
$-5x^2$ $+9x$ $+7.2$	−20 18 7.2	−31.25 22.5 7.2	−45 27 7.2
$y = -5x^2 + 9x + 7.2$	5.2	−1.55	−10.8

Problem 5. Plot a graph of: $y = 2x^2$ and hence solve the equations: (a) $2x^2 - 8 = 0$ and (b) $2x^2 - x - 3 = 0$

A graph of $y = 2x^2$ is shown in Fig. 30.11.

(a) Rearranging $2x^2 - 8 = 0$ gives $2x^2 = 8$ and the solution of this equation is obtained from the points of intersection of $y = 2x^2$

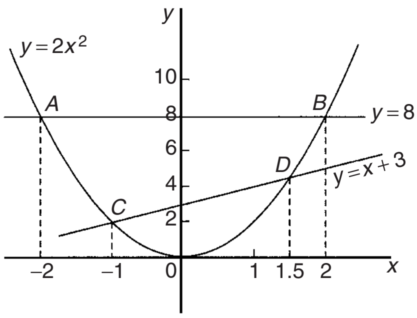


Figure 30.11

and $y = 8$, i.e. at co-ordinates $(-2, 8)$ and $(2, 8)$, shown as A and B, respectively, in Fig. 30.11. Hence the solutions of $2x^2 - 8 = 0$ are $\boldsymbol{x = -2}$ **and** $\boldsymbol{x = +2}$

(b) Rearranging $2x^2 - x - 3 = 0$ gives $2x^2 = x + 3$ and the solution of this equation is obtained from the points of intersection of $y = 2x^2$ and $y = x + 3$, i.e. at C and D in Fig. 30.11. Hence the solutions of $2x^2 - x - 3 = 0$ are $\boldsymbol{x = -1}$ **and** $\boldsymbol{x = 1.5}$

Problem 6. Plot the graph of $y = -2x^2 + 3x + 6$ for values of x from $x = -2$ to $x = 4$. Use the graph to find the roots of the following equations:

(a) $-2x^2 + 3x + 6 = 0$

(b) $-2x^2 + 3x + 2 = 0$

(c) $-2x^2 + 3x + 9 = 0$

(d) $-2x^2 + x + 5 = 0$

A table of values is drawn up as shown below.

x	−2	−1	0	1	2	3	4
$-2x^2$	−8	−2	0	−2	−8	−18	−32
$+3x$	−6	−3	0	3	6	9	12
$+6$	6	6	6	6	6	6	6
y	−8	1	6	7	4	−3	−14

A graph of $-2x^2 + 3x + 6$ is shown in Fig. 30.12.

(a) The parabola $y = -2x^2 + 3x + 6$ and the straight line $y = 0$ intersect at A and B, where $\boldsymbol{x = -1.13}$ **and** $\boldsymbol{x = 2.63}$ and these are the roots of the equation $-2x^2 + 3x + 6 = 0$

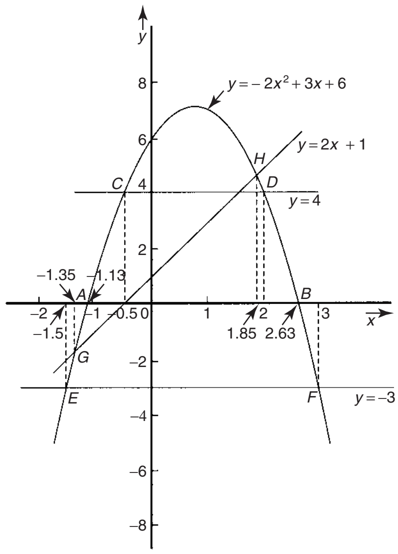


Figure 30.12

(b) Comparing

$$y = -2x^2 + 3x + 6 \quad (1)$$

with $\quad 0 = -2x^2 + 3x + 2 \quad (2)$

shows that if 4 is added to both sides of equation (2), the right-hand side of both equations will be the same.

Hence $4 = -2x^2 + 3x + 6$. The solution of this equation is found from the points of intersection of the line $y = 4$ and the parabola $y = -2x^2 + 3x + 6$, i.e. points C and D in Fig. 30.12. Hence the roots of $-2x^2 + 3x + 2 = 0$ are $\boldsymbol{x = -0.5}$ **and** $\boldsymbol{x = 2}$

(c) $-2x^2 + 3x + 9 = 0$ may be rearranged as $-2x^2 + 3x + 6 = -3$, and the solution of this equation is obtained from the points of intersection of the line $y = -3$ and the parabola $y = -2x^2 + 3x + 6$, i.e. at points E and F in Fig. 30.12. Hence the roots of $-2x^2 + 3x + 9 = 0$ are $\boldsymbol{x = -1.5}$ **and** $\boldsymbol{x = 3}$

(d) Comparing

$$y = -2x^2 + 3x + 6 \quad (3)$$

with $\quad 0 = -2x^2 + x + 5 \quad (4)$

shows that if $2x + 1$ is added to both sides of equation (4) the right-hand side of both equations will be the same. Hence equation (4) may be written as $2x + 1 = -2x^2 + 3x + 6$. The solution of this equation is found from the

Now try the following exercise

Exercise 116 Further problems on inverse functions

Determine the inverse of the functions given in Problems 1 to 4.

1. $f(x) = x + 1$ $\quad [f^{-1}(x) = x - 1]$
2. $f(x) = 5x - 1$ $\quad \left[f^{-1}(x) = \frac{1}{5}(x + 1)\right]$
3. $f(x) = x^3 + 1$ $\quad [f^{-1}(x) = \sqrt[3]{x - 1}]$
4. $f(x) = \frac{1}{x} + 2$ $\quad \left[f^{-1}(x) = \frac{1}{x - 2}\right]$

Determine the principal value of the inverse functions in problems 5 to 11.

5. $\arcsin(-1)$ $\quad \left[-\frac{\pi}{2} \text{ or } -1.5708 \text{ rad}\right]$
6. $\arccos 0.5$ $\quad \left[\frac{\pi}{3} \text{ or } 1.0472 \text{ rad}\right]$
7. $\arctan 1$ $\quad \left[\frac{\pi}{4} \text{ or } 0.7854 \text{ rad}\right]$
8. $\text{arccot}\, 2$ $\quad [0.4636 \text{ rad}]$
9. $\text{arccosec}\, 2.5$ $\quad [0.4115 \text{ rad}]$
10. $\text{arcsec}\, 1.5$ $\quad [0.8411 \text{ rad}]$
11. $\arcsin\left(\frac{1}{\sqrt{2}}\right)$ $\quad \left[\frac{\pi}{4} \text{ or } 0.7854 \text{ rad}\right]$
12. Evaluate x, correct to 3 decimal places: $x = \arcsin\frac{1}{3} + \arccos\frac{4}{5} - \arctan\frac{8}{9}$ $\quad [0.257]$
13. Evaluate y, correct to 4 significant figures: $y = 3\,\text{arcsec}\sqrt{2} - 4\,\text{arccosec}\sqrt{2} + 5\,\text{arccot}\, 2$ $\quad [1.533]$

Answers to Exercise 114

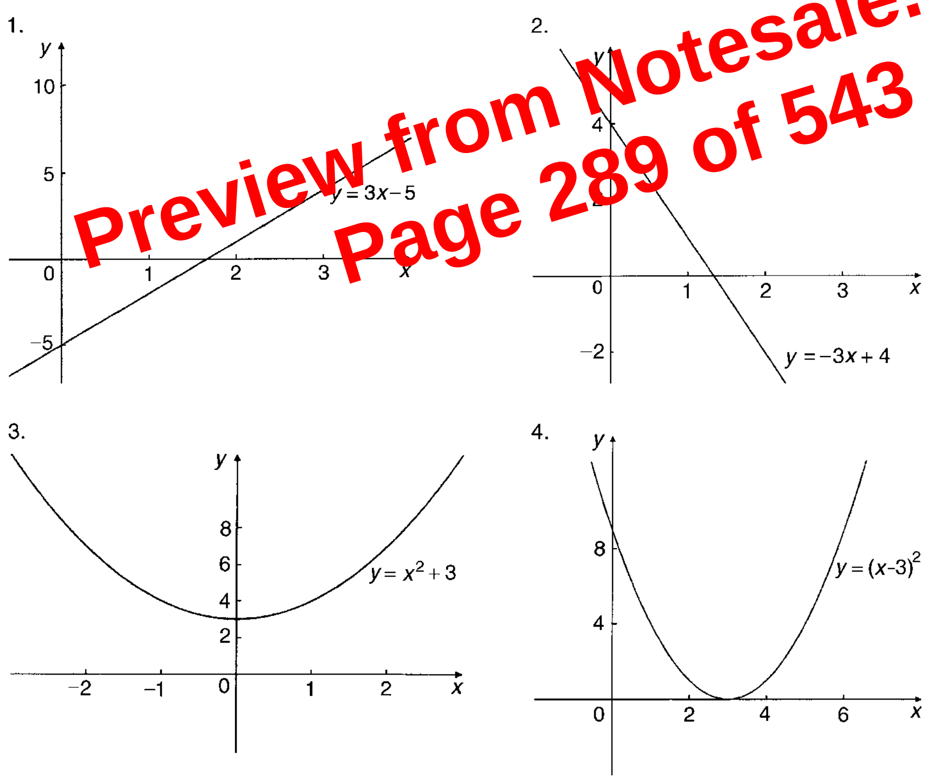


Figure 31.33 Graphical solutions to Exercise 114, page 272.

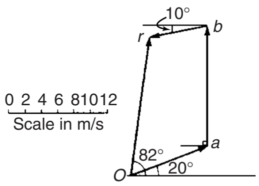


Figure 32.6

32.3 Resolution of vectors

A vector can be resolved into two component parts such that the vector addition of the component parts is equal to the original vector. The two components usually taken are a horizontal component and a vertical component. For the vector shown as $\boldsymbol{F}$ in Fig. 32.7, the horizontal component is $F\cos\theta$ and the vertical component is $F\sin\theta$.

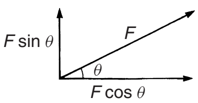


Figure 32.7

For the vectors $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ shown in Fig. 32.8, the horizontal component of vector addition is:

$$H = F_1\cos\theta_1 + F_2\cos\theta_2$$

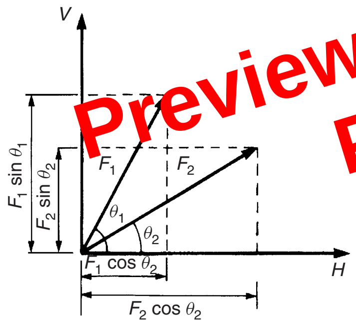


Figure 32.8

and the vertical component of vector addition is:

$$V = F_1\sin\theta_1 + F_2\sin\theta_2$$

Having obtained H and V, the magnitude of the resultant vector R is given by: $\boldsymbol{\sqrt{H^2+V^2}}$ and its angle to the horizontal is given by $\mathbf{tan^{-1}}\,\dfrac{\boldsymbol{V}}{\boldsymbol{H}}$

Problem 3. Resolve the acceleration vector of 17 m/s^2 at an angle of 120° to the horizontal into a horizontal and a vertical component

For a vector $\boldsymbol{A}$ at angle θ to the horizontal, the horizontal component is given by $A\cos\theta$ and the vertical component by $A\sin\theta$. Any convention of signs may be adopted, in this case horizontally from left to right is taken as positive and vertically upwards is taken as positive.

Horizontal component $H = 17\cos 120° =$ $\mathbf{-8.50\ m/s^2}$, acting from left to right.

Vertical component $V = 17\sin 120° = \mathbf{14.72\ m/s^2}$, acting vertically upwards.

These component vectors are shown in Fig. 32.9.

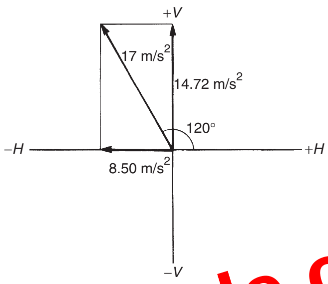


Figure 32.9

Problem 4. Calculate the resultant force of the two forces given in Problem 1

With reference to Fig. 32.4(a):

Horizontal component of force,

$$H = 7\cos 0° + 4\cos 45°$$
$$= 7 + 2.828 = \mathbf{9.828\ N}$$

Vertical component of force,

$$V = 7\sin 0° + 4\sin 45°$$
$$= 0 + 2.828 = \mathbf{2.828\ N}$$

The magnitude of the resultant of vector addition

$$= \sqrt{H^2+V^2} = \sqrt{9.828^2 + 2.828^2}$$
$$= \sqrt{104.59} = \mathbf{10.23\ N}$$

The direction of the resultant of vector addition

$$= \tan^{-1}\left(\frac{V}{H}\right) = \tan^{-1}\left(\frac{2.828}{9.828}\right) = \mathbf{16.05°}$$

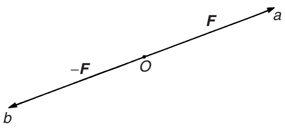


Figure 32.11

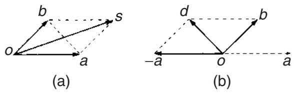


Figure 32.12

Fig. 32.12(a) shows that the second diagonal of the 'parallelogram' method of vector addition gives the magnitude and direction of vector subtraction of $\boldsymbol{oa}$ from $\boldsymbol{ob}$.

> Problem 6. Accelerations of $a_1 = 1.5$ m/s^2 at 90° and $a_2 = 2.6$ m/s^2 at 145° act at a point. Find $\boldsymbol{a_1} + \boldsymbol{a_2}$ and $\boldsymbol{a_1} - \boldsymbol{a_2}$ by:
> (i) drawing a scale vector diagram and
> (ii) by calculation

(i) The scale vector diagram is shown in Fig. 32.13. By measurement,

$$\boldsymbol{a_1 + a_2 = 3.7\ \text{m/s}^2\ \text{at}\ 126°}$$

$$\boldsymbol{a_1 - a_2 = 2.1\ \text{m/s}^2\ \text{at}\ 0°}$$

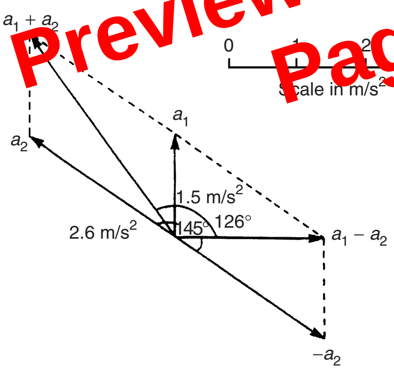


Figure 32.13

(ii) Resolving horizontally and vertically gives:

Horizontal component of $\boldsymbol{a_1} + \boldsymbol{a_2}$,

$$H = 1.5 \cos 90° + 2.6 \cos 145° = -2.13$$

Vertical component of $\boldsymbol{a_1} + \boldsymbol{a_2}$,

$$V = 1.5 \sin 90° + 2.6 \sin 145° = 2.99$$

Magnitude of

$$\boldsymbol{a_1} + \boldsymbol{a_2} = \sqrt{(-2.13)^2 + 2.99^2}$$
$$= \mathbf{3.67\ m/s^2}$$

Direction of $\boldsymbol{a_1} + \boldsymbol{a_2} = \tan^{-1}\left(\dfrac{2.99}{-2.13}\right)$ and must lie in the second quadrant since H is negative and V is positive.

$\tan^{-1}\left(\dfrac{2.99}{-2.13}\right) = -54.53°$, and for this to be in the second quadrant, the true angle is 180° displaced, i.e. $180° - 54.53°$ or $125.47°$

Thus $\quad \boldsymbol{a_1 + a_2 = 3.67\ \text{m/s}^2\ \text{at}\ 125.47°}$.

Horizontal component of $\boldsymbol{a_1} - \boldsymbol{a_2}$, that is, $\boldsymbol{a_1} + (-\boldsymbol{a_2})$

$$= 1.5 \cos 90° + 2.6 \cos(145° - 180°)$$
$$= 2.6 \cos(-35°) = 2.13$$

Vertical component of $\boldsymbol{a_1} - \boldsymbol{a_2}$, that is, $\boldsymbol{a_1} + (-\boldsymbol{a_2})$

$$= 1.5 \sin 90° + 2.6 \sin(-35°) = 0$$

Magnitude of $\boldsymbol{a_1} - \boldsymbol{a_2} = \sqrt{2.13^2 + 0^2}$
$= 2.13$ m/s^2

Direction of $\boldsymbol{a_1} - \boldsymbol{a_2} = \tan^{-1}\left(\dfrac{0}{2.13}\right) = 0°$

Thus $\boldsymbol{a_1 - a_2 = 2.13\ \text{m/s}^2\ \text{at}\ 0°}$

> Problem 7. Calculate the resultant of (i) $\boldsymbol{v_1} - \boldsymbol{v_2} + \boldsymbol{v_3}$ and (ii) $\boldsymbol{v_2} - \boldsymbol{v_1} - \boldsymbol{v_3}$ when $\boldsymbol{v_1} = 22$ units at 140°, $\boldsymbol{v_2} = 40$ units at 190° and $\boldsymbol{v_3} = 15$ units at 290°

(i) The vectors are shown in Fig. 32.14.

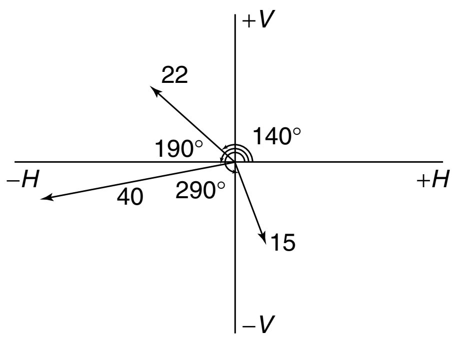


Figure 32.14

The horizontal component of $\boldsymbol{v_1 - v_2 + v_3}$

$$= (22\cos 140^\circ) - (40\cos 190^\circ) + (15\cos 290^\circ)$$
$$= (-16.85) - (-39.39) + (5.13)$$
$$= \mathbf{27.67\ units}$$

The vertical component of $\boldsymbol{v_1 - v_2 + v_3}$

$$= (22\sin 140^\circ) - (40\sin 190^\circ) + (15\sin 290^\circ)$$
$$= (14.14) - (-6.95) + (-14.10)$$
$$= \mathbf{6.99\ units}$$

The magnitude of the resultant, R, which can be represented by the mathematical symbol for 'the **modulus** of' as $|v_1 - v_2 + v_3|$ is given by:

$$|R| = \sqrt{27.67^2 + 6.99^2} = \mathbf{28.54\ units}$$

The direction of the resultant, $\boldsymbol{R}$, which can be represented by the mathematical symbol for 'the **argument** of' as $\arg(v_1 - v_2 + v_3)$ is given by:

$$\mathbf{arg}\ \boldsymbol{R} = \tan^{-1}\left(\frac{6.99}{27.67}\right) = 14.18^\circ$$

Thus $\boldsymbol{v_1 - v_2 + v_3} = \mathbf{28.54\ units\ at\ 14.18^\circ}$

(ii) The horizontal component of $\boldsymbol{v_2 - v_1 - v_3}$

$$= (40\cos 190^\circ) - (22\cos 140^\circ) - (15\cos 290^\circ)$$
$$= (-39.39) - (-16.85) - (5.13)$$
$$= \mathbf{-27.67\ units}$$

The vertical component of $\boldsymbol{v_2 - v_1 - v_3}$

$$= (40\sin 190^\circ) - (22\sin 140^\circ) - (15\sin 290^\circ)$$
$$= (-6.95) - (14.14) - (-14.10)$$
$$= \mathbf{-6.99\ units}$$

Let $\boldsymbol{R} = v_2 - v_1 - v_3$ then

$$|R| = \sqrt{(-27.67)^2 + (-6.99)^2} = 28.54 \text{ units}$$

and $\quad \mathbf{arg}\ \boldsymbol{R} = \tan^{-1}\left(\frac{-6.99}{-27.67}\right)$

and must lie in the third quadrant since both H and V are negative quantities.

$$\tan^{-1}\left(\frac{-6.99}{-27.67}\right) = 14.18^\circ,$$

hence the required angle is $180^\circ + 14.18^\circ = 194.18^\circ$

Thus $\boldsymbol{v_2 - v_1 - v_3} = \mathbf{28.54\ units\ at\ 194.18^\circ}$

This result is as expected, since $\boldsymbol{v_2 - v_1 - v_3} = -\boldsymbol{(v_1 - v_2 + v_3)}$ and the vector 28.54 units at 194.18° is minus times the vector 28.54 units at 14.18°

Now try the following exercise

Exercise 118 Further problems on vectors subtraction

1. Forces of $\boldsymbol{F_1} = 40$ N at 45° and $\boldsymbol{F_2} = 30$ N at 125° act at a point. Determine by drawing and by calculation (a) $\boldsymbol{F_1 + F_2}$ (b) $\boldsymbol{F_1 - F_2}$

   [(a) 54.0 N at 78.16° (b) 45.64 N at 4.66°]

2. Calculate the resultant of (a) $\boldsymbol{v_1 + v_2 - v_3}$ (b) $\boldsymbol{v_3 - v_2 + v_1}$ when $\boldsymbol{v_1} = 15$ m/s at 85°, $\boldsymbol{v_2} = 25$ m/s at 175° and $\boldsymbol{v_3} = 12$ m/s at 235°

   [(a) 31.71 m/s at 121.81° (b) 19.55 m/s at 8.63°]

(a) use of the cosine rule (and then sine rule to calculate angle ϕ), or

(b) determining horizontal and vertical components of lengths oa and ab in Fig. 33.5, and then using Pythagoras' theorem to calculate ob.

In the above example, by calculation, $y_R = 6.083$ and angle $\phi = 25.28°$ or 0.441 rad. Thus the resultant may be expressed in sinusoidal form as $y_R = 6.083\sin(\omega t - 0.441)$. If the resultant phasor, $y_R = y_1 - y_2$ is required, then y_2 is still 3 units long but is drawn in the opposite direction, as shown in Fig. 33.7, and y_R is determined by calculation.

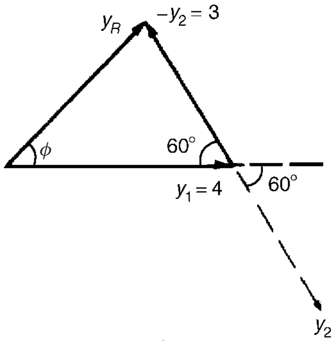


Figure 33.7

Resolution of phasors by calculation is demonstrated in worked problems 4 to 6.

> Problem 4. Given $y_1 = 2\sin\omega t$ and $y_2 = 3\sin(\omega t + \pi/4)$, obtain an expression for the resultant $y_R = y_1 + y_2$, (a) by drawing, and (b) by calculation

(a) When time $t = 0$ the position of phasors y_1 and y_2 are as shown in Fig. 33.8(a). To obtain the resultant, y_1 is drawn horizontally, 2 units long, y_2 is drawn 3 units long at an angle of $\pi/4$ rads or 45° and joined to the end of y_1 as shown in Fig. 33.8(b). y_R is measured as 4.6 units long and angle ϕ is measured as 27° or 0.47 rad. Alternatively, y_R is the diagonal of the parallelogram formed as shown in Fig. 33.8(c).

Hence, by drawing, $\boldsymbol{y_R = 4.6\sin(\omega t + 0.47)}$

(b) From Fig. 33.8(b), and using the cosine rule:

$$y_R^2 = 2^2 + 3^2 - [2(2)(3)\cos 135°]$$
$$= 4 + 9 - [-8.485] = 21.49$$

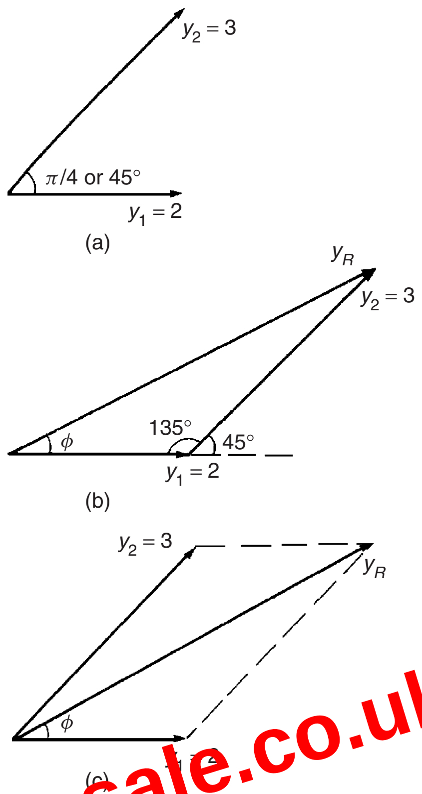


Figure 33.8

Hence $y_R = \sqrt{21.49} = 4.64$

Using the sine rule: $\dfrac{3}{\sin\phi} = \dfrac{4.64}{\sin 135°}$ from

which $\sin\phi = \dfrac{3\sin 135°}{4.64} = 0.4572$

Hence $\phi = \sin^{-1} 0.4572 = 27.21°$ or 0.475 rad.

By calculation, $\boldsymbol{y_R = 4.64\sin(\omega t + 0.475)}$

> Problem 5. Two alternating voltages are given by $v_1 = 15\sin\omega t$ volts and $v_2 = 25\sin(\omega t - \pi/6)$ volts. Determine a sinusoidal expression for the resultant $v_R = v_1 + v_2$ by finding horizontal and vertical components

The relative positions of v_1 and v_2 at time $t = 0$ are shown in Fig. 33.9(a) and the phasor diagram is shown in Fig. 33.9(b).

$Z = 2 + j3$ lies in the first quadrant as shown in Fig. 34.5.

Modulus, $|Z| = r = \sqrt{2^2 + 3^2} = \sqrt{\mathbf{13}}$ or **3.606**, correct to 3 decimal places.

Argument, $\arg Z = \theta = \tan^{-1} \dfrac{3}{2}$

$$= \mathbf{56.31^\circ} \text{ or } \mathbf{56^\circ 19'}$$

In polar form, $2 + j3$ is written as $\mathbf{3.606\angle 56.31^\circ}$ or $\mathbf{3.606\angle 56^\circ 19'}$

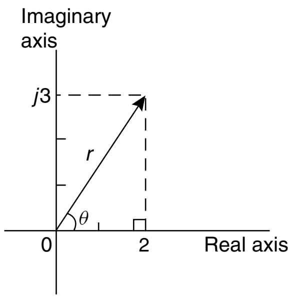


Figure 34.5

> Problem 10. Express the following complex numbers in polar form:
>
> (a) $3 + j4$ (b) $-3 + j4$
>
> (c) $-3 - j4$ (d) $3 - j4$

(a) $3 + j4$ is shown in Fig. 34.6 and lies in the first quadrant.

Modulus, $r = \sqrt{3^2 + 4^2} = 5$ and argument $\theta = \tan^{-1} \dfrac{4}{3} = 53.13^\circ$ or $53^\circ 8'$

Hence $\mathbf{3 + j4 = 5\angle 53.13^\circ}$

(b) $-3 + j4$ is shown in Fig. 34.6 and lies in the second quadrant.

Modulus, $r = 5$ and angle $\alpha = 53.13^\circ$, from part (a).

Argument $= 180^\circ - 53.13^\circ = 126.87^\circ$ (i.e. the argument must be measured from the positive real axis).

Hence $\mathbf{-3 + j4 = 5\angle 126.87^\circ}$

(c) $-3 - j4$ is shown in Fig. 34.6 and lies in the third quadrant.

Modulus, $r = 5$ and $\alpha = 53.13^\circ$, as above.

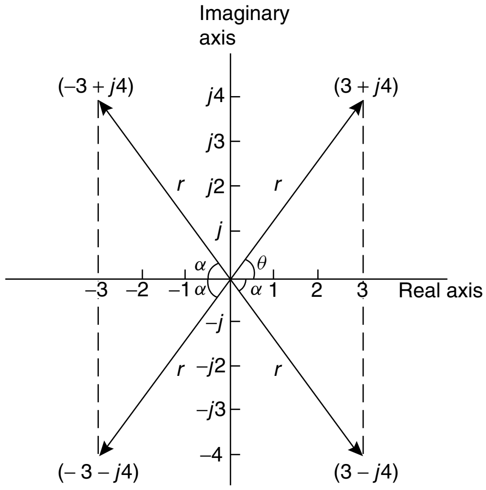


Figure 34.6

Hence the argument $= 180^\circ + 53.13^\circ = 233.13^\circ$, which is the same as -126.87°

Hence $\mathbf{(-3 - j4) = 5\angle 233.13^\circ}$ **or** $\mathbf{5\angle -126.87^\circ}$

(By convention the **principal value** is normally used, i.e. the numerically least value, such that $-\pi < \theta < \pi$).

(d) $3 - j4$ is shown in Fig. 34.6 and lies in the fourth quadrant.

Modulus, $r = 5$ and angle $\alpha = 53.13^\circ$, as above.

Hence $\mathbf{(3 - j4) = 5\angle -53.13^\circ}$

> Problem 11. Convert (a) $4\angle 30^\circ$ (b) $7\angle -145^\circ$ into $a + jb$ form, correct to 4 significant figures

(a) $4\angle 30^\circ$ is shown in Fig. 34.7(a) and lies in the first quadrant.

Using trigonometric ratios,

$x = 4 \cos 30^\circ = 3.464$ and $y = 4 \sin 30^\circ$
$= 2.000$

Hence $\mathbf{4\angle 30^\circ = 3.464 + j2.000}$

(b) $7\angle -145^\circ$ is shown in Fig. 34.7(b) and lies in the third quadrant.

Angle $\alpha = 180^\circ - 145^\circ = 35^\circ$

Hence $x = 7 \cos 35^\circ = 5.734$

and $y = 7 \sin 35^\circ = 4.015$

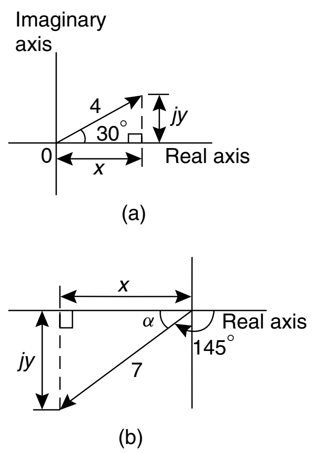


Figure 34.7

Hence $7\angle-145^\circ = -5.734 - j4.015$

Alternatively

$$7\angle-145^\circ = 7\cos(-145^\circ) + j7\sin(-145^\circ)$$
$$= \mathbf{-5.734 - j4.015}$$

34.7 Multiplication and division in polar form

If $Z_1 = r_1\angle\theta_1$ and $Z_2 = r_2\angle\theta_2$ then:

(i) $Z_1Z_2 = r_1r_2\angle(\theta_1+\theta_2)$ and

(ii) $\dfrac{Z_1}{Z_2} = \dfrac{r_1}{r_2}\angle(\theta_1-\theta_2)$

> Problem 12. Determine, in polar form:
>
> (a) $8\angle25^\circ \times 4\angle60^\circ$
>
> (b) $3\angle16^\circ \times 5\angle-44^\circ \times 2\angle80^\circ$

(a) $8\angle25^\circ \times 4\angle60^\circ = (8\times4)\angle(25^\circ+60^\circ) = \mathbf{32\angle85^\circ}$

(b) $3\angle16^\circ \times 5\angle-44^\circ \times 2\angle80^\circ$
$= (3\times5\times2)\angle[16^\circ+(-44^\circ)+80^\circ] = \mathbf{30\angle52^\circ}$

> Problem 13. Evaluate in polar form:
>
> (a) $\dfrac{16\angle75^\circ}{2\angle15^\circ}$ (b) $\dfrac{10\angle\frac{\pi}{4} \times 12\angle\frac{\pi}{2}}{6\angle-\frac{\pi}{3}}$

(a) $\dfrac{16\angle75^\circ}{2\angle15^\circ} = \dfrac{16}{2}\angle(75^\circ-15^\circ) = \mathbf{8\angle60^\circ}$

(b) $\dfrac{10\angle\frac{\pi}{4} \times 12\angle\frac{\pi}{2}}{6\angle-\frac{\pi}{3}} = \dfrac{10\times12}{6}\angle\left(\dfrac{\pi}{4}+\dfrac{\pi}{2}-\left(-\dfrac{\pi}{3}\right)\right)$

$= \mathbf{20\angle\dfrac{13\pi}{12}}$ or $\mathbf{20\angle-\dfrac{11\pi}{12}}$ or
$\mathbf{20\angle195^\circ}$ or $\mathbf{20\angle-165^\circ}$

> Problem 14. Evaluate, in polar form:
> $2\angle30^\circ + 5\angle-45^\circ - 4\angle120^\circ$

Addition and subtraction in polar form is not possible directly. Each complex number has to be converted into Cartesian form first.

$$2\angle30^\circ = 2(\cos30^\circ + j\sin30^\circ)$$
$$= 2\cos30^\circ + j2\sin30^\circ = 1.732 + j1.000$$
$$5\angle-45^\circ = 5(\cos(-45^\circ) + j\sin(-45^\circ))$$
$$= 5\cos(-45^\circ) + j5\sin(-45^\circ)$$
$$= 3.536 - j3.536$$
$$4\angle120^\circ = 4(\cos120^\circ + j\sin120^\circ)$$
$$= 4\cos120^\circ + j4\sin120^\circ$$
$$= -2.000 + j3.464$$

Hence $2\angle30^\circ + 5\angle-45^\circ - 4\angle120^\circ$

$= (1.732 + j1.000) + (3.536 - j3.536) - (-2.000 + j3.464)$

$= 7.268 - j6.000$, which lies in the fourth quadrant

$= \sqrt{7.268^2 + 6.000^2}\angle\tan^{-1}\left(\dfrac{-6.000}{7.268}\right)$

$= \mathbf{9.425\angle-39.54^\circ}$ or $\mathbf{9.425\angle-39^\circ32'}$

Now try the following exercise

> **Exercise 124 Further problems on polar form**
>
> 1. Determine the modulus and argument of (a) $2 + j4$ (b) $-5 - j2$ (c) $j(2 - j)$.
> [(a) 4.472, 63.43° (b) 5.385, −158.20° (c) 2.236, 63.43°]

35

De Moivre's theorem

35.1 Introduction

From multiplication of complex numbers in polar form,

$$(r\angle\theta) \times (r\angle\theta) = r^2\angle 2\theta$$

Similarly, $(r\angle\theta)\times(r\angle\theta)\times(r\angle\theta) = r^3\angle 3\theta$, and so on.

In general, **de Moivre's theorem** states:

$$\boxed{[\boldsymbol{r\angle\theta}]^{\boldsymbol{n}} = \boldsymbol{r^n\angle n\theta}}$$

The theorem is true for all positive, negative and fractional values of n. The theorem is used to determine powers and roots of complex numbers.

35.2 Powers of complex numbers

For example, $[3\angle 20^\circ]^4 = 3^4\angle(4\times 20^\circ) = 81\angle 80^\circ$ by de Moivre's theorem.

> Problem 1. Determine, in polar form:
> (a) $[2\angle 35^\circ]^5$ (b) $(-2 + j3)^6$

(a) $$[2\angle 35^\circ]^5 = 2^5\angle(5\times 35^\circ),$$ from De Moivre's theorem $$= \mathbf{32\angle 175^\circ}$$

(b) $$(-2 + j3) = \sqrt{(-2)^2 + (3)^2}\angle\tan\frac{3}{-2}$$
$$= \sqrt{13}\angle 123.69^\circ,\ \text{since } -2 + j3$$ lies in the second quadrant

$$(-2 + j3)^6 = [\sqrt{13}\angle 123.69^\circ]^6$$
$$= \sqrt{13}^6\angle(6\times 123.69^\circ),$$ by De Moivre's theorem
$$= 2197\angle 742.14^\circ$$
$$= 2197\angle 382.14^\circ$$
(since $742.14 \equiv 742.14^\circ - 360^\circ = 382.14^\circ$)
$$= \mathbf{2197\angle 22.14^\circ}$$
(since $382.14^\circ \equiv 382.14^\circ - 360^\circ = 22.14^\circ$)

> Problem 2. Determine the value of $(-7 + j5)^4$, expressing the result in polar and rectangular forms

$$(-7 + j5) = \sqrt{(-7)^2 + 5^2}\angle\tan^{-1}\frac{5}{-7}$$
$$= \sqrt{74}\angle 144.46^\circ$$

(Note, by considering the Argand diagram, $-7 + j5$ must represent an angle in the second quadrant and **not** in the fourth quadrant).

Applying de Moivre's theorem:

$$(-7 + j5)^4 = [\sqrt{74}\angle 144.46^\circ]^4$$
$$= \sqrt{74}^4\angle 4\times 144.46^\circ$$
$$= 5476\angle 577.84^\circ$$
$$= \mathbf{5476\angle 217.84^\circ} \text{ or}$$
$$\mathbf{5476\angle 217^\circ 15'} \text{ in polar form.}$$

Since $r\angle\theta = r\cos\theta + jr\sin\theta$,

$$5476\angle 217.84^\circ = 5476\cos 217.84^\circ + j5476\sin 217.84^\circ$$
$$= -4325 - j3359$$

i.e. $$\mathbf{(-7 + j5)^4 = -4325 - j3359}$$ in rectangular form.

Now try the following exercise

> **Exercise 126 Further problems on powers of complex numbers**
>
> 1. Determine in polar form (a) $[1.5\angle 15^\circ]^5$ (b) $(1 + j2)^6$
>
> [(a) $7.594\angle 75^\circ$ (b) $125\angle 20.62^\circ$]

2. Determine in polar and Cartesian forms (a) $[3\angle 41^\circ]^4$ (b) $(-2-j)^5$
$$\begin{bmatrix}\text{(a) } 81\angle 164^\circ, -77.86+j22.33 \\ \text{(b) } 55.90\angle -47.17^\circ, 38-j41\end{bmatrix}$$

3. Convert $(3-j)$ into polar form and hence evaluate $(3-j)^7$, giving the answer in polar form.
$$[\sqrt{10}\angle -18.43^\circ, \quad 3162\angle -129.03^\circ]$$

In Problems 4 to 7, express in both polar and rectangular forms:

4. $(6+j5)^3$
$$[476.4\angle 119.42^\circ, -234+j415]$$

5. $(3-j8)^5$
$$[45\,530\angle 12.78^\circ, 44\,400+j10\,070]$$

6. $(-2+j7)^4$
$$[2809\angle 63.78^\circ, 1241+j2520]$$

7. $(-16-j9)^6$ $\begin{bmatrix}(38.27\times 10^6)\angle 176.15^\circ, \\ 10^6(-38.18+j2.570)\end{bmatrix}$

35.3 Roots of complex numbers

The **square root** of a complex number is determined by letting $n=\frac{1}{2}$ in De Moivre's theorem,

i.e. $$\sqrt{r\angle\theta}=[r\angle\theta]^{1/2}=r^{1/2}\angle\frac{1}{2}\theta=\sqrt{r}\angle\frac{\theta}{2}$$

There are two square roots of a real number, equal in size but opposite in sign.

Problem 3. Determine the two square roots of the complex number $(5+j12)$ in polar and Cartesian forms and show the roots on an Argand diagram

$$(5+112)=\sqrt{5^2+12^2}\angle\tan^{-1}\frac{12}{5}=13\angle 67.38^\circ$$

When determining square roots two solutions result. To obtain the second solution one way is to express $13\angle 67.38^\circ$ also as $13\angle(67.38^\circ+360^\circ)$, i.e. $13\angle 427.38^\circ$. When the angle is divided by 2 an angle less than 360° is obtained.

Hence

$$\begin{aligned}\sqrt{5^2+12^2}&=\sqrt{13\angle 67.38^\circ}\text{ and }\sqrt{13\angle 427.38^\circ}\\&=[13\angle 67.38^\circ]^{1/2}\text{ and }[13\angle 427.38^\circ]^{1/2}\\&=13^{1/2}\angle\left(\tfrac{1}{2}\times 67.38^\circ\right)\text{ and}\\&\qquad 13^{1/2}\angle\left(\tfrac{1}{2}\times 427.38^\circ\right)\\&=\sqrt{13}\angle 33.69^\circ\text{ and }\sqrt{13}\angle 213.69^\circ\\&=3.61\angle 33.69^\circ\text{ and }3.61\angle 213.69^\circ\end{aligned}$$

Thus, in polar form, the two roots are: $3.61\angle 33.69^\circ$ and $3.61\angle -146.31^\circ$

$$\begin{aligned}\sqrt{13}\angle 33.69^\circ&=\sqrt{13}(\cos 33.69^\circ+j\sin 33.69^\circ)\\&=3.0+j2.0\\\sqrt{13}\angle 213.69^\circ&=\sqrt{13}(\cos 213.69^\circ+j\sin 213.69^\circ)\\&=-3.0-j2.0\end{aligned}$$

Thus, in Cartesian form the two roots are: $\pm(3.0+j2.0)$.

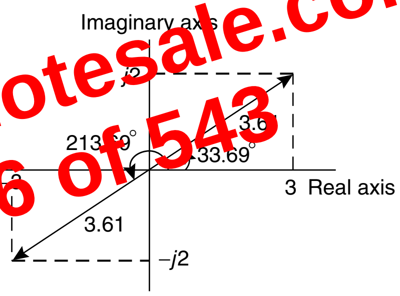


Figure 35.1

From the Argand diagram shown in Fig. 35.1 the two roots are seen to be 180° apart, which is always true when finding square roots of complex numbers.

In general, **when finding the n^{th} root of a complex number, there are n solutions**. For example, there are three solutions to a cube root, five solutions to a fifth root, and so on. In the solutions to the roots of a complex number, the modulus, r, is always the same, but the arguments, θ, are different. It is shown in Problem 3 that arguments are symmetrically spaced on an Argand diagram and are $\dfrac{360^\circ}{n}$ apart, where n is the number of the roots required. Thus if one of the solutions to the cube

(d) The amount of money spent on food can only be expressed correct to the nearest pence, the amount being **counted**. Hence, these data are **discrete**.

Now try the following exercise

Exercise 128 Further problems on discrete and continuous data

In Problems 1 and 2, state whether data relating to the topics given are discrete or continuous.

1. (a) The amount of petrol produced daily, for each of 31 days, by a refinery.
   (b) The amount of coal produced daily by each of 15 miners.
   (c) The number of bottles of milk delivered daily by each of 20 milkmen.
   (d) The size of 10 samples of rivets produced by a machine.
   [(a) continuous (b) continuous (c) discrete (d) continuous]
2. (a) The number of people visiting an exhibition on each of 5 days.
   (b) The time taken by each of 12 athletes to run 100 metres.
   (c) The value of stamps sold in a day by each of 20 post offices.
   (d) The number of defective items produced in each of 10 one-hour periods by a machine.
   [(a) discrete (b) continuous (c) discrete (d) discrete]

36.2 Presentation of ungrouped data

Ungrouped data can be presented diagrammatically in several ways and these include:

(a) **pictograms**, in which pictorial symbols are used to represent quantities (see Problem 2),

(b) **horizontal bar charts**, having data represented by equally spaced horizontal rectangles (see Problem 3), and

(c) **vertical bar charts**, in which data are represented by equally spaced vertical rectangles (see Problem 4).

Trends in ungrouped data over equal periods of time can be presented diagrammatically by a **percentage component bar chart**. In such a chart, equally spaced rectangles of any width, but whose height corresponds to 100%, are constructed. The rectangles are then subdivided into values corresponding to the percentage relative frequencies of the members (see Problem 5).

A **pie diagram** is used to show diagrammatically the parts making up the whole. In a pie diagram, the area of a circle represents the whole, and the areas of the sectors of the circle are made proportional to the parts which make up the whole (see Problem 6).

Problem 2. The number of television sets repaired in a workshop by a technician in six, one-month periods is as shown below. Present these data as a pictogram.

Month	January	February	March
Number repaired	11	6	15

Month	April	May	June
Number repaired	9	13	8

Each symbol shown in Fig. 36.1 represents two television sets repaired. Thus, in January, $5\frac{1}{2}$ symbols are used to represent the 11 sets repaired, in February, 3 symbols are used to represent the 6 sets repaired, and so on.

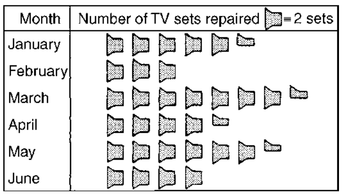


Figure 36.1

Problem 3. The distance in miles travelled by four salesmen in a week are as shown below.

Salesmen	P	Q	R	S
Distance traveled (miles)	413	264	597	143

Use a horizontal bar chart to represent these data diagrammatically

Equally spaced horizontal rectangles of any width, but whose length is proportional to the distance travelled, are used. Thus, the length of the rectangle for salesman P is proportional to 413 miles, and so on. The horizontal bar chart depicting these data is shown in Fig. 36.2.

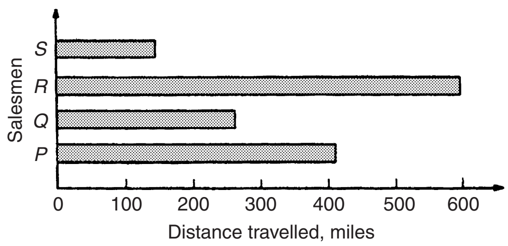


Figure 36.2

Problem 4. The number of issues of tools or materials from a store in a factory is observed for seven, one-hour periods in a day, and the results of the survey are as follows:

Period	1	2	3	4	5	6	7
Number of issues	34	17	9	5	27	13	6

Present these data on a vertical bar chart.

In a vertical bar chart, equally spaced vertical rectangles of any width, but whose height is proportional to the quantity being represented, are used. Thus the height of the rectangle for period 1 is proportional to 34 units, and so on. The vertical bar chart depicting these data is shown in Fig. 36.3.

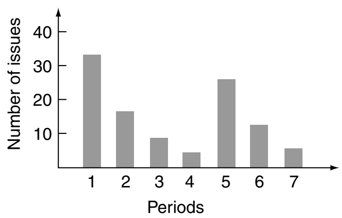


Figure 36.3

Problem 5. The numbers of various types of dwellings sold by a company annually over a three-year period are as shown below. Draw percentage component bar charts to present these data.

	Year 1	Year 2	Year 3
4-roomed bungalows	24	17	7
5-roomed bungalows	38	71	118
4-roomed houses	44	50	53
5-roomed houses	64	82	147
6-roomed houses	30	30	25

A table of percentage relative frequency values, correct to the nearest 1%, is the first requirement. Since,

$$\text{percentage relative frequency} = \frac{\text{frequency of member} \times 100}{\text{total frequency}}$$

then for 4-roomed bungalows in year 1:

$$\text{percentage relative frequency} = \frac{24 \times 100}{24 + 38 + 44 + 64 + 30} = 12\%$$

The percentage relative frequencies of the other types of dwellings for each of the three years are similarly calculated and the results are as shown in the table below.

	Year 1	Year 2	Year 3
4-roomed bungalows	12%	7%	2%
5-roomed bungalows	19%	28%	34%
4-roomed houses	22%	20%	15%
5-roomed houses	32%	33%	42%
6-roomed houses	15%	12%	7%

The percentage component bar chart is produced by constructing three equally spaced rectangles of any width, corresponding to the three years. The heights of the rectangles correspond to 100% relative frequency, and are subdivided into the values in the table of percentages shown above. A key is used (different types of shading or different colour schemes) to indicate corresponding percentage values in the rows of the table of percentages. The percentage component bar chart is shown in Fig. 36.4.

Problem 6. The retail price of a product costing £2 is made up as follows: materials 10 p, labour 20 p, research and development 40 p, overheads 70 p, profit 60 p. Present these data on a pie diagram

A circle of any radius is drawn, and the area of the circle represents the whole, which in this case is

37

Measures of central tendency and dispersion

37.1 Measures of central tendency

A single value, which is representative of a set of values, may be used to give an indication of the general size of the members in a set, the word **'average'** often being used to indicate the single value.

The statistical term used for 'average' is the arithmetic mean or just the **mean**. Other measures of central tendency may be used and these include the **median** and the **modal** values.

37.2 Mean, median and mode for discrete data

Mean

The **arithmetic mean value** is found by adding together the values of the members of a set and dividing by the number of members in the set. Thus, the mean of the set of numbers: {4, 5, 6, 9} is:

$$\frac{4+5+6+9}{4}, \quad \text{i.e.} \quad 6$$

In general, the mean of the set: $\{x_1, x_2, x_3, \ldots, x_n\}$ is

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}, \text{ written as } \frac{\sum x}{n}$$

where $\sum$ is the Greek letter 'sigma' and means 'the sum of', and $\bar{x}$ (called x-bar) is used to signify a mean value.

Median

The **median value** often gives a better indication of the general size of a set containing extreme values. The set: {7, 5, 74, 10} has a mean value of 24, which is not really representative of any of the values of the members of the set. The median value is obtained by:

(a) **ranking** the set in ascending order of magnitude, and

(b) selecting the value of the **middle member** for sets containing an odd number of members, or finding the value of the mean of the two middle members for sets containing an even number of members.

For example, the set: {7, 5, 74, 10} is ranked as {5, 7, 10, 74}, and since it contains an even number of members (four in this case), the mean of 7 and 10 is taken, giving a median value of 8.5. Similarly, the set: {3, 81, 15, 7, 14} is ranked as {3, 7, 14, 15, 81} and the median value is the value of the middle member, i.e. 14.

Mode

The **modal value**, or **mode**, is the most commonly occurring value in a set. If two values occur with the same frequency, the set is 'bi-modal'. The set: {5, 6, 8, 2, 5, 4, 6, 5, 3} has a modal value of 5, since the member having a value of 5 occurs three times.

> Problem 1. Determine the mean, median and mode for the set:
>
> {2, 3, 7, 5, 5, 13, 1, 7, 4, 8, 3, 4, 3}

The mean value is obtained by adding together the values of the members of the set and dividing by the number of members in the set.

Thus, **mean value**,

$$\bar{x} = \frac{2+3+7+5+5+13+1 \atop +7+4+8+3+4+3}{13} = \frac{65}{13} = \mathbf{5}$$

To obtain the median value the set is ranked, that is, placed in ascending order of magnitude, and since the set contains an odd number of members the value of the middle member is the median value. Ranking the set gives:

{1, 2, 3, 3, 3, 4, 4, 5, 5, 7, 7, 8, 13}

The middle term is the seventh member, i.e. 4, thus the **median value is 4**.

The **modal value** is the value of the most commonly occurring member and is **3**, which occurs three times, all other members only occurring once or twice.

Problem 2. The following set of data refers to the amount of money in £s taken by a news vendor for 6 days. Determine the mean, median and modal values of the set:

{27.90, 34.70, 54.40, 18.92, 47.60, 39.68}

$$\textbf{Mean value} = \frac{27.90 + 34.70 + 54.40 + 18.92 + 47.60 + 39.68}{6} = \pounds\mathbf{37.20}$$

The ranked set is:

{18.92, 27.90, 34.70, 39.68, 47.60, 54.40}

Since the set has an even number of members, the mean of the middle two members is taken to give the median value, i.e.

$$\textbf{median value} = \frac{34.70 + 39.68}{2} = \pounds\mathbf{37.19}$$

Since no two members have the same value, this set has **no mode**.

Now try the following exercise

Exercise 131 Further problems on mean, median and mode for discrete data

In Problems 1 to 4, determine the mean, median and modal values for the sets given.

1. {3, 8, 10, 7, 5, 14, 2, 9, 8}

   [mean 7.33, median 8, mode 8]

2. {26, 31, 21, 29, 32, 26, 25, 28}

   [mean 27.25, median 27, mode 26]

3. {4.72, 4.71, 4.74, 4.73, 4.72, 4.71, 4.73, 4.72}

   [mean 4.7225, median 4.72, mode 4.72]

4. {73.8, 126.4, 40.7, 141.7, 28.5, 237.4, 157.9}

   [mean 115.2, median 126.4, no mode]

37.3 Mean, median and mode for grouped data

The mean value for a set of grouped data is found by determining the sum of the (frequency × class mid-point values) and dividing by the sum of the frequencies,

$$\text{i.e. mean value } \bar{x} = \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{f_1 + f_2 + \cdots + f_n} = \frac{\sum(fx)}{\sum f}$$

where f is the frequency of the class having a mid-point value of x, and so on.

Problem 3. The frequency distribution for the value of resistance in ohms of 48 resistors is as shown. Determine the mean value of resistance.

20.5–20.9 3, 21.0–21.4 10, 21.5–21.9 11,

22.0–22.4 13, 22.5–22.9 9, 23.0–23.4 2

The class mid-point/frequency values are:

20.7 3, 21.2 10, 21.7 11, 22.2 13,

22.7 9 and 23.2 2

For grouped data, the mean value is given by:

$$\bar{x} = \frac{\sum(fx)}{\sum f}$$

where f is the class frequency and x is the class mid-point value. Hence mean value,

$$\bar{x} = \frac{(3 \times 20.7) + (10 \times 21.2) + (11 \times 21.7) + (13 \times 22.2) + (9 \times 22.7) + (2 \times 23.2)}{48}$$

$$= \frac{1052.1}{48} = 21.919\ldots$$

i.e. **the mean value is 21.9 ohms**, correct to 3 significant figures.

38

Probability

38.1 Introduction to probability

The **probability** of something happening is the likelihood or chance of it happening. Values of probability lie between 0 and 1, where 0 represents an absolute impossibility and 1 represents an absolute certainty. The probability of an event happening usually lies somewhere between these two extreme values and is expressed either as a proper or decimal fraction. Examples of probability are:

that a length of copper wire has zero resistance at 100 °C	0
that a fair, six-sided dice will stop with a 3 upwards	$\frac{1}{6}$ or 0.1667
that a fair coin will land with a head upwards	$\frac{1}{2}$ or 0.5
that a length of copper wire has some resistance at 100 °C	1

If p is the probability of an event happening and q is the probability of the same event not happening, then the total probability is $p + q$ and is equal to unity, since it is an absolute certainty that the event either does or does not occur, i.e. $p + q = 1$

Expectation

The **expectation**, E, of an event happening is defined in general terms as the product of the probability p of an event happening and the number of attempts made, n, i.e. $\boldsymbol{E = pn}$.

Thus, since the probability of obtaining a 3 upwards when rolling a fair dice is $\frac{1}{6}$, the expectation of getting a 3 upwards on four throws of the dice is $\frac{1}{6} \times 4$, i.e. $\frac{2}{3}$

Thus expectation is the average occurrence of an event.

Dependent event

A **dependent event** is one in which the probability of an event happening affects the probability of another ever happening. Let 5 transistors be taken at random from a batch of 100 transistors for test purposes, and the probability of there being a defective transistor, p_1, be determined. At some later time, let another 5 transistors be taken at random from the 95 remaining transistors in the batch and the probability of there being a defective transistor, p_2, be determined. The value of p_2 is different from p_1 since batch size has effectively altered from 100 to 95, i.e. probability p_2 is dependent on probability p_1. Since transistors are drawn, and then another 5 transistors drawn without replacing the first 5, the second random selection is said to be **without replacement**.

Independent event

An independent event is one in which the probability of an event happening does not affect the probability of another event happening. If 5 transistors are taken at random from a batch of transistors and the probability of a defective transistor p_1 is determined and the process is repeated after the original 5 have been replaced in the batch to give p_2, then p_1 is equal to p_2. Since the 5 transistors are replaced between draws, the second selection is said to be **with replacement**.

Conditional probability

Conditional probability is concerned with the probability of say event B occurring, given that event A has already taken place. If A and B are independent events, then the fact that event A has already occurred will not affect the probability of event B. If A and B are dependent events, then event A having occurred will effect the probability of event B.

38.2 Laws of probability

The addition law of probability

The addition law of probability is recognized by the word **'or'** joining the probabilities. If p_A is the probability of event A happening and p_B is the probability of event B happening, the probability of **event $\boldsymbol{A}$ or event $\boldsymbol{B}$** happening is given by $\boldsymbol{p_A + p_B}$ (provided events A and B are **mutually exclusive**,

3. (a) Find the probability of having a 2 upwards when throwing a fair 6-sided dice. (b) Find the probability of having a 5 upwards when throwing a fair 6-sided dice. (c) Determine the probability of having a 2 and then a 5 on two successive throws of a fair 6-sided dice.

$$\left[\text{(a) } \frac{1}{6} \quad \text{(b) } \frac{1}{6} \quad \text{(c) } \frac{1}{36}\right]$$

4. The probability of event A happening is $\frac{3}{5}$ and the probability of event B happening is $\frac{2}{3}$. Calculate the probabilities of (a) both A and B happening, (b) only event A happening, i.e. event A happening and event B not happening, (c) only event B happening, and (d) either A, or B, or A and B happening.

$$\left[\text{(a) } \frac{2}{5} \quad \text{(b) } \frac{1}{5} \quad \text{(c) } \frac{4}{15} \quad \text{(d) } \frac{13}{15}\right]$$

5. When testing 1000 soldered joints, 4 failed during a vibration test and 5 failed due to having a high resistance. Determine the probability of a joint failing due to (a) vibration, (b) high resistance, (c) vibration or high resistance and (d) vibration and high resistance.

$$\left[\begin{array}{ll}\text{(a) } \frac{1}{250} & \text{(b) } \frac{1}{200} \\ \text{(c) } \frac{9}{1000} & \text{(d) } \frac{1}{50000}\end{array}\right]$$

38.4 Further worked problems on probability

Problem 6. A batch of 40 components contains 5 which are defective. A component is drawn at random from the batch and tested and then a second component is drawn. Determine the probability that neither of the components is defective when drawn (a) with replacement, and (b) without replacement.

(a) With replacement

The probability that the component selected on the first draw is satisfactory is $\frac{35}{40}$, i.e. $\frac{7}{8}$. The component is now replaced and a second draw is made. The probability that this component is also satisfactory is $\frac{7}{8}$. Hence, the probability that both the first component drawn **and** the second component drawn are satisfactory is:

$$\frac{7}{8} \times \frac{7}{8} = \mathbf{\frac{49}{64}} \quad \text{or} \quad \mathbf{0.7656}$$

(b) Without replacement

The probability that the first component drawn is satisfactory is $\frac{7}{8}$. There are now only 34 satisfactory components left in the batch and the batch number is 39. Hence, the probability of drawing a satisfactory component on the second draw is $\frac{34}{39}$. Thus the probability that the first component drawn **and** the second component drawn are satisfactory, i.e. neither is defective, is:

$$\frac{7}{8} \times \frac{34}{39} = \mathbf{\frac{238}{312}} \quad \text{or} \quad \mathbf{0.7628}$$

Problem 7. A batch of 40 components contains 5 that are defective. If a component is drawn at random from the batch and tested and then a second component is drawn at random, calculate the probability of having one defective component, both with and without replacement.

The probability of having one defective component can be achieved in two ways. If p is the probability of drawing a defective component and q is the probability of drawing a satisfactory component, then the probability of having one defective component is given by drawing a satisfactory component and then a defective component **or** by drawing a defective component and then a satisfactory one, i.e. by $q \times p + p \times q$

With replacement:

$$p = \frac{5}{40} = \frac{1}{8} \quad \text{and} \quad q = \frac{35}{40} = \frac{7}{8}$$

Hence, probability of having one defective component is:

$$\frac{1}{8} \times \frac{7}{8} + \frac{7}{8} \times \frac{1}{8}$$

(Note that the order of the letters matter in permutations, i.e. YX is a different permutation from XY). In general, ${}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$ or $\boldsymbol{{}^nP_r = \dfrac{n!}{(n-r)!}}$ as stated in Chapter 14

For example, ${}^5P_4 = 5(4)(3)(2) = 120$ or

${}^5P_4 = \dfrac{5!}{(5-4)!} = \dfrac{5!}{1!} = (5)(4)(3)(2) = 120$

Also, ${}^3P_3 = 6$ from above; using ${}^nP_r = \dfrac{n!}{(n-r)!}$ gives ${}^3P_3 = \dfrac{3!}{(3-3)!} = \dfrac{6}{0!}$. Since this must equal 6, then $0! = 1$ (check this with your calculator).

Combinations

If selections of the three letters X, Y, Z are made without regard to the order of the letters in each group, i.e. XY is now the same as YX for example, then each group is called a **combination**. The number of possible combinations is denoted by ${}^nC_r$, where n is the total number of items and r is the number in each selection.

In general,

$$\boldsymbol{{}^nC_r = \frac{n!}{r!(n-r)!}}$$

For example,

$$ {}^5C_4 = \frac{5!}{4!(5-4)!} = \frac{5!}{4!} = \frac{5\times 4\times 3\times 2\times 1}{4\times 3\times 2\times 1} = 5$$

Problem 11. Calculate the number of permutations there are of: (a) 5 distinct objects taken 2 at a time, (b) 4 distinct objects taken 2 at a time.

(a) ${}^5P_2 = \dfrac{5!}{(5-2)!} = \dfrac{5!}{3!} = \dfrac{5\times 4\times 3\times 2}{3\times 2} = \mathbf{20}$

(b) ${}^4P_2 = \dfrac{4!}{(4-2)!} = \dfrac{4!}{2!} = \mathbf{12}$

Problem 12. Calculate the number of combinations there are of: (a) 5 distinct objects taken 2 at a time, (b) 4 distinct objects taken 2 at a time.

(a) ${}^5C_2 = \dfrac{5!}{2!(5-2)!} = \dfrac{5!}{2!\,3!}$
$= \dfrac{5\times 4\times 3\times 2\times 1}{(2\times 1)(3\times 2\times 1)} = \mathbf{10}$

(b) ${}^4C_2 = \dfrac{4!}{2!(4-2)!} = \dfrac{4!}{2!\,2!} = \mathbf{6}$

Problem 13. A class has 24 students. 4 can represent the class at an exam board. How many combinations are possible when choosing this group.

Number of combinations possible,

$$ {}^nC_r = \frac{n!}{r!(n-r)!}$$

i.e. ${}^{24}C_4 = \dfrac{24!}{4!(24-4)!} = \dfrac{24!}{4!\,20!} = \mathbf{10\,626}$

Problem 14. In how many ways can a team of eleven be picked from sixteen possible players?

Number of ways $= {}^nC_r = {}^{16}C_{11}$

$$= \frac{16!}{11!(16-11)!} = \frac{16!}{11!\,5!} = \mathbf{4368}$$

Now try the following exercise

Exercise 137 Further problems on permutations and combinations

1. Calculate the number of permutations there are of: (a) 15 distinct objects taken 2 at a time, (b) 9 distinct objects taken 4 at a time. [(a) 210 (b) 3024]

2. Calculate the number of combinations there are of: (a) 12 distinct objects taken 5 at a time, (b) 6 distinct objects taken 4 at a time. [(a) 792 (b) 15]

3. In how many ways can a team of six be picked from ten possible players? [210]

4. 15 boxes can each hold one object. In how many ways can 10 identical objects be placed in the boxes? [3003]

that (a) no blocks and (b) more than two blocks will fail to meet the specification in a batch of 9 blocks.

[(a) 0.2316 (b) 0.1408]

3. The average number of employees absent from a firm each day is 4%. An office within the firm has seven employees. Determine the probabilities that (a) no employee and (b) three employees will be absent on a particular day.

[(a) 0.7514 (b) 0.0019]

4. A manufacturer estimates that 3% of his output of a small item is defective. Find the probabilities that in a sample of 10 items (a) less than two and (b) more than two items will be defective.

[(a) 0.9655 (b) 0.0028]

5. Five coins are tossed simultaneously. Determine the probabilities of having 0, 1, 2, 3, 4 and 5 heads upwards, and draw a histogram depicting the results.

[Vertical adjacent rectangles, whose heights are proportional to 0.0313, 0.1563, 0.3125, 0.3125, 0.1563 and 0.0313]

6. If the probability of rain falling during a particular period is 2/5, find the probabilities of having 0, 1, 2, 3, 4, 5, 6 and 7 wet days in a week. Show these results on a histogram.

[Vertical adjacent rectangles, whose heights are proportional to 0.0280, 0.1306, 0.2613, 0.2903, 0.1935, 0.0774, 0.0172 and 0.0016]

7. An automatic machine produces, on average, 10% of its components outside of the tolerance required. In a sample of 10 components from this machine, determine the probability of having three components outside of the tolerance required by assuming a binomial distribution.

[0.0574]

39.2 The Poisson distribution

When the number of trials, n, in a binomial distribution becomes large (usually taken as larger than 10), the calculations associated with determining the values of the terms become laborious. If n is large and p is small, and the product np is less than 5, a very good approximation to a binomial distribution is given by the corresponding Poisson distribution, in which calculations are usually simpler.

The Poisson approximation to a binomial distribution may be defined as follows:

'the probabilities that an event will happen 0, 1, 2, 3, ..., n times in n trials are given by the successive terms of the expression

$$e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right)$$

taken from left to right'

The symbol λ is the expectation of an event happening and is equal to np.

Problem 6. If 3% of the gearwheels produced by a company are defective, determine the probabilities that in a sample of 80 gearwheels (a) two and (b) more than two will be defective.

The sample number, n, is large, the probability of a defective gearwheel, p, is small and the product np is 80×0.03, i.e. 2.4, which is less than 5. Hence a Poisson approximation to a binomial distribution may be used. The expectation of a defective gearwheel, $\lambda = np = 2.4$

The probabilities of 0, 1, 2, ... defective gearwheels are given by the successive terms of the expression

$$e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right)$$

taken from left to right, i.e. by

$$e^{-\lambda},\ \lambda e^{-\lambda},\ \frac{\lambda^2 e^{-\lambda}}{2!},\ldots \quad \text{Thus:}$$

probability of no defective gearwheels is

$$e^{-\lambda} = e^{-2.4} = 0.0907$$

probability of 1 defective gearwheel is

$$\lambda e^{-\lambda} = 2.4e^{-2.4} = 0.2177$$

Assignment 10

This assignment covers the material in Chapters 36 to 39.
The marks for each question are shown in brackets at the end of each question.

1. A company produces five products in the following proportions:

   Product A 24 Product B 16 Product C 15 Product D 11 Product E 6

   Present these data visually by drawing (a) a vertical bar chart (b) a percentage bar chart (c) a pie diagram. (13)

2. The following lists the diameters of 40 components produced by a machine, each measured correct to the nearest hundredth of a centimetre:

1.39	1.36	1.38	1.31	1.33	1.40	1.28
1.40	1.24	1.28	1.42	1.34	1.43	1.35
1.36	1.36	1.35	1.45	1.29	1.39	1.38
1.38	1.35	1.42	1.30	1.26	1.37	1.33
1.37	1.34	1.34	1.32	1.33	1.30	1.38
1.41	1.35	1.38	1.27	1.37		

(a) Using 8 classes form a frequency distribution and a cumulative frequency distribution.

(b) For the above data draw a histogram, a frequency polygon and an ogive. (21)

3. Determine for the 10 measurements of lengths shown below:

   (a) the arithmetic mean, (b) the median, (c) the mode, and (d) the standard deviation.

   28 m, 20 m, 32 m, 44 m, 28 m, 30 m, 30 m, 26 m, 28 m and 34 m (9)

4. The heights of 100 people are measured correct to the nearest centimetre with the following results:

150–157 cm	5	158–165 cm	18
166–173 cm	42	174–181 cm	27
182–189 cm	8		

   Determine for the data (a) the mean height and (b) the standard deviation. (10)

5. Determine the probabilities of:

   (a) drawing a white ball from a bag containing 6 black and 14 white balls

   (b) winning a prize in a raffle by buying 6 tickets when a total of 480 tickets are sold

   (c) selecting at random a female from a group of 12 boys and 28 girls

   (d) winning a prize in a raffle by buying 8 tickets when there are 5 prizes and a total of 800 tickets are sold. (8)

6. In a box containing 120 similar transistors 70 are satisfactory, 37 give too high a gain under normal operating conditions and the remainder give too low a gain.

   Calculate the probability that when drawing two transistors in turn, at random, **with replacement**, of having (a) two satisfactory, (b) none with low gain, (c) one with high gain and one satisfactory, (d) one with low gain and none satisfactory.

   Determine the probabilities in (a), (b) and (c) above if the transistors are drawn **without replacement.** (14)

7. A machine produces 15% defective components. In a sample of 5, drawn at random, calculate, using the binomial distribution, the probability that:

   (a) there will be 4 defective items

   (b) there will be not more than 3 defective items

   (c) all the items will be non-defective (13)

8. 2% of the light bulbs produced by a company are defective. Determine, using the Poisson distribution, the probability that in a sample of 80 bulbs:

   (a) 3 bulbs will be defective, (b) not more than 3 bulbs will be defective, (c) at least 2 bulbs will be defective. (12)

Problem 1. The mean height of 500 people is 170 cm and the standard deviation is 9 cm. Assuming the heights are normally distributed, determine the number of people likely to have heights between 150 cm and 195 cm

The mean value, $\bar{x}$, is 170 cm and corresponds to a normal standard variate value, z, of zero on the standardised normal curve. A height of 150 cm has a z-value given by $z = \dfrac{x - \bar{x}}{\sigma}$ standard deviations, i.e. $\dfrac{150 - 170}{9}$ or -2.22 standard deviations. Using a table of partial areas beneath the standardised normal curve (see Table 40.1), a z-value of -2.22 corresponds to an area of 0.4868 between the mean value and the ordinate $z = -2.22$. The negative z-value shows that it lies to the left of the $z = 0$ ordinate.

This area is shown shaded in Fig. 40.3(a). Similarly, 195 cm has a z-value of $\dfrac{195 - 170}{9}$ that is 2.78 standard deviations. From Table 40.1, this value of z corresponds to an area of 0.4973, the positive value of z showing that it lies to the right of the $z = 0$ ordinate. This area is shown shaded in Fig. 40.3(b). The total area shaded in Fig. 40.3(a) and (b) is shown in Fig. 40.3(c) and is 0.4868 + 0.4973, i.e. 0.9841 of the total area beneath the curve.

However, the area is directly proportional to probability. Thus, the probability that a person will have a height of between 150 and 195 cm is 0.9841. For a group of 500 people, 500 × 0.9841, i.e. **492 people are likely to have heights in this range**. The value of 500 × 0.9841 is 492.05, but since answers based on a normal probability distribution can only be approximate, results are usually given correct to the nearest whole number.

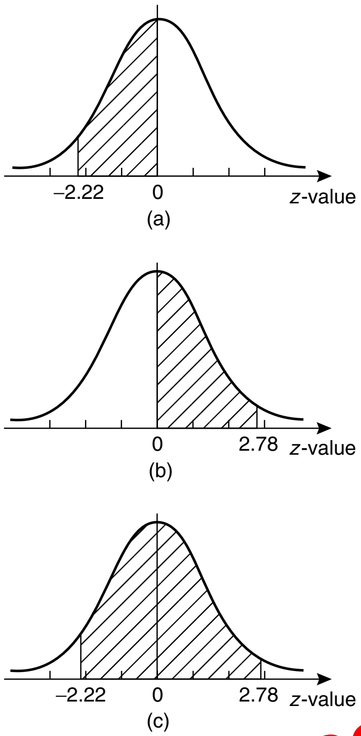


Figure 40.3

Problem 2. For the group of people given in Problem 1, find the number of people likely to have heights of less than 165 cm

A height of 165 cm corresponds to $\dfrac{165 - 170}{9}$, i.e. -0.56 standard deviations. The area between $z = 0$ and $z = -0.56$ (from Table 40.1) is 0.2123, shown shaded in Fig. 40.4(a). The total area under the standardised normal curve is unity and since the curve is symmetrical, it follows that the total area to the left of the $z = 0$ ordinate is 0.5000. Thus the area to the left of the $z = -0.56$ ordinate ('left' means 'less than', 'right' means 'more than') is 0.5000 − 0.2123, i.e. 0.2877 of the total area, which is shown shaded in Fig. 40.4(b). The area is directly proportional to probability and since the total area beneath the standardised normal curve is unity, the probability of a person's height being less than 165 cm is 0.2877. For a group of 500 people, 500 × 0.2877, i.e. **144 people are likely to have heights of less than 165 cm**.

Problem 3. For the group of people given in Problem 1 find how many people are likely to have heights of more than 194 cm

194 cm correspond to a z-value of $\dfrac{194 - 170}{9}$ that is, 2.67 standard deviations. From Table 40.1, the area between $z = 0$, $z = 2.67$ and the standardised normal curve is 0.4962, shown shaded in Fig. 40.5(a). Since the standardised normal curve is symmetrical, the total area to the right of the $z = 0$ ordinate is 0.5000, hence the shaded area shown in Fig. 40.5(b) is 0.5000 − 0.4962, i.e. 0.0038. This area represents the probability of a person having a height of more than 194 cm, and for 500 people, the

Class mid-point value (kg)	29.5	30.5	31.5	32.5	33.5
Frequency	2	4	6	8	9
Class mid-point value (kg)	34.5	35.5	36.5	37.5	38.5
Frequency	8	6	4	2	1

To test the normality of a distribution, the upper class boundary/percentage cumulative frequency values are plotted on normal probability paper. The upper class boundary values are: 30, 31, 32, ..., 38, 39. The corresponding cumulative frequency values (for 'less than' the upper class boundary values) are: 2, $(4+2) = 6$, $(6+4+2) = 12$, 20, 29, 37, 43, 47, 49 and 50. The corresponding percentage cumulative frequency values are $\frac{2}{50} \times 100 = 4$, $\frac{6}{50} \times 100 = 12$, 24, 40, 58, 74, 86, 94, 98 and 100%.

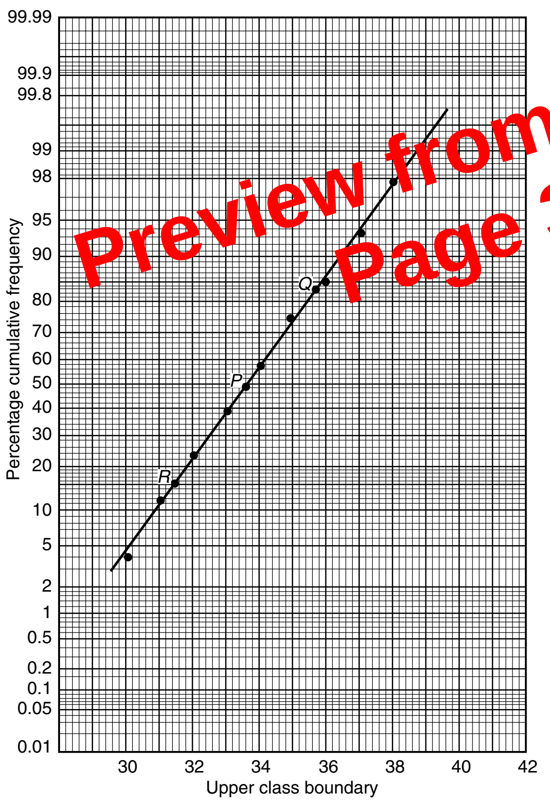


Figure 40.6

The co-ordinates of upper class boundary/percentage cumulative frequency values are plotted as shown in Fig. 40.6. When plotting these values, it will always be found that the co-ordinate for the 100% cumulative frequency value cannot be plotted, since the maximum value on the probability scale is 99.99. **Since the points plotted in Fig. 40.6 lie very nearly in a straight line, the data is approximately normally distributed**.

The mean value and standard deviation can be determined from Fig. 40.6. Since a normal curve is symmetrical, the mean value is the value of the variable corresponding to a 50% cumulative frequency value, shown as point P on the graph. This shows that **the mean value is 33.6 kg**. The standard deviation is determined using the 84% and 16% cumulative frequency values, shown as Q and R in Fig. 40.6. The variable values for Q and R are 35.7 and 31.4 respectively; thus two standard deviations correspond to $35.7 - 31.4$, i.e. 4.3, showing that the standard deviation of the distribution is approximately $\frac{4.3}{2}$ i.e. **2.15 standard deviations**.

The mean value and standard deviation of the distribution can be calculated using mean, $\bar{x} = \frac{\sum f x}{\sum f}$ and standard deviation, $\sigma = \sqrt{\frac{\sum[f(x-\bar{x})^2]}{\sum f}}$ where f is the frequency of a class and x is the class mid-point value. Using these formulae gives a mean value of the distribution of 33.6 (as obtained graphically) and a standard deviation of 2.12, showing that the graphical method of determining the mean and standard deviation give quite realistic results.

Problem 6. Use normal probability paper to determine whether the data given below is normally distributed. Use the graph and assume a normal distribution whether this is so or not, to find approximate values of the mean and standard deviation of the distribution.

Class mid-point Values	5	15	25	35	45
Frequency	1	2	3	6	9
Class mid-point Values	55	65	75	85	95
Frequency	6	2	2	1	1

the parameter may be considered to lie is called an **interval estimate**. Thus if an estimate is made of the length of an object and the result is quoted as 150 cm, this is a point estimate. If the result is quoted as 150 ± 10 cm, this is an interval estimate and indicates that the length lies between 140 and 160 cm. Generally, a point estimate does not indicate how close the value is to the true value of the quantity and should be accompanied by additional information on which its merits may be judged. A statement of the error or the precision of an estimate is often called its **reliability**. In statistics, when estimates are made of population parameters based on samples, usually interval estimates are used. The word estimate does not suggest that we adopt the approach 'let's guess that the mean value is about..', but rather that a value is carefully selected and the degree of confidence which can be placed in the estimate is given in addition.

Confidence intervals

It is stated in Section 43.3 that when samples are taken from a population, the mean values of these samples are approximately normally distributed, that is, the mean values forming the sampling distribution of means is approximately normally distributed. It is also true that if the standard deviation of each of the samples is found, then the standard deviations of all the samples are approximately normally distributed, that is, the standard deviations of the sampling distribution of standard deviations are approximately normally distributed. Parameters such as the mean or the standard deviation of a sampling distribution are called **sampling statistics**, S. Let μ_S be the mean value of a sampling statistic of the sampling distribution, that is, the mean value of the means of the samples or the mean value of the standard deviations of the samples. Also, let σ_S be the standard deviation of a sampling statistic of the sampling distribution, that is, the standard deviation of the means of the samples or the standard deviation of the standard deviations of the samples. Because the sampling distribution of the means and of the standard deviations are normally distributed, it is possible to predict the probability of the sampling statistic lying in the intervals:

mean $\pm$ 1 standard deviation,

mean $\pm$ 2 standard deviations,

or mean $\pm$ 3 standard deviations,

by using tables of the partial areas under the standardised normal curve given in Table 40.1 on page 341. From this table, the area corresponding to a z-value of $+1$ standard deviation is 0.3413, thus the area corresponding to ± 1 standard deviation is 2×0.3413, that is, 0.6826. Thus the percentage probability of a sampling statistic lying between the mean ± 1 standard deviation is 68.26%. Similarly, the probability of a sampling statistic lying between the mean ± 2 standard deviations is 95.44% and of lying between the mean ± 3 standard deviations is 99.74%

The values 68.26%, 95.44% and 99.74% are called the **confidence levels** for estimating a sampling statistic. A confidence level of 68.26% is associated with two distinct values, these being, $S - (1$ standard deviation), i.e. $S - \sigma_S$ and $S + (1$ standard deviation), i.e. $S + \sigma_S$. These two values are called the **confidence limits** of the estimate and the distance between the confidence limits is called the **confidence interval**. A confidence interval indicates the expectation or confidence of finding an estimate of the population statistic in that interval, based on a sampling statistic. The list in Table 43.1 is based on values given in Table 40.1, and gives some of the confidence levels used in practice and their associated z-values; (some of the values given are based on interpolation). When the table is used in this context, z-values are usually indicated by 'z_C' and are called the **confidence coefficients**.

Table 43.1

Confidence level, %	Confidence coefficient, z_c
99	2.58
98	2.33
96	2.05
95	1.96
90	1.645
80	1.28
50	0.6745

Any other values of confidence levels and their associated confidence coefficients can be obtained using Table 40.1.

Problem 3. Determine the confidence coefficient corresponding to a confidence level of 98.5%

98.5% is equivalent to a per unit value of 0.9850. This indicates that the area under the standardised normal curve between $-z_C$ and $+z_C$, i.e. corresponding to $2z_C$, is 0.9850 of the total area. Hence

population may be estimated by using expression (7), i.e. $s \pm z_C \sigma_S = 0.60 \pm (1.751)(0.12)$
$= 0.60 \pm 0.21\ \mu F$

Thus, **the 92% confidence interval for the estimate of the standard deviation for the batch is from 0.39 μF to 0.81 μF**.

Now try the following exercise

Exercise 145 Further problems on the estimation of population parameters based on a large sample size

1. Measurements are made on a random sample of 100 components drawn from a population of size 1546 and having a standard deviation of 2.93 mm. The mean measurement of the components in the sample is 67.45 mm. Determine the 95% and 99% confidence limits for an estimate of the mean of the population.
[66.89 and 68.01 mm, 66.72 and 68.18 mm]

2. The standard deviation of the masses of 500 blocks is 150 kg. A random sample of 40 blocks has a mean mass of 2.40 Mg.
   (a) Determine the 95% and 99% confidence intervals for estimating the mean mass of the remaining 460 blocks.
   (b) With what degree of confidence can it be said that the mean mass of the remaining 460 blocks is 2.40 ± 0.035 Mg?
[(a) 2.355 Mg to 2.445 Mg; 2.341 Mg to 2.459 Mg (b) 86%]

3. In order to estimate the thermal expansion of a metal, measurements of the change of length for a known change of temperature are taken by a group of students. The sampling distribution of the results has a mean of 12.81×10^{-4} m $^0C^{-1}$ and a standard error of the means of 0.04×10^{-4} m $^0C^{-1}$. Determine the 95% confidence interval for an estimate of the true value of the thermal expansion of the metal, correct to two decimal places.
[12.73×10^{-4} m $^0C^{-1}$ to 12.89×10^{-4} m $^0C^{-1}$]

4. The standard deviation of the time to failure of an electronic component is estimated as 100 hours. Determine how large a sample of these components must be, in order to be 90% confident that the error in the estimated time to failure will not exceed (a) 20 hours, and (b) 10 hours.
[(a) at least 68 (b) at least 271]

5. The time taken to assemble a servo-mechanism is measured for 40 operatives and the mean time is 14.63 minutes with a standard deviation of 2.45 minutes. Determine the maximum error in estimating the true mean time to assemble the servo-mechanism for all operatives, based on a 95% confidence level.
[45.6 seconds]

43.5 Estimating the mean of a population based on a small sample size

The methods used in Section 43.4 to estimate the population mean and standard deviation rely on a relatively large sample size, usually taken as 30 or more. This is because when the sample size is large the sampling distribution of a parameter is approximately normally distributed. When the sample size is small, usually taken as less than 30, the techniques used for estimating the population parameters in Section 43.4 become more and more inaccurate as the sample size becomes smaller, since the sampling distribution no longer approximates to a normal distribution. Investigations were carried out into the effect of small sample sizes on the estimation theory by W. S. Gosset in the early twentieth century and, as a result of his work, tables are available which enable a realistic estimate to be made, when sample sizes are small. In these tables, the t-value is determined from the relationship

$$t = \frac{(\bar{x} - \mu)}{s}\sqrt{N-1}$$

where $\bar{x}$ is the mean value of a sample, μ is the mean value of the population from which the sample is drawn, s is the standard deviation of the sample and N is the number of independent observations in the sample. He published his findings under the pen name of 'Student', and these tables are often referred to as the **'Student's t distribution'**.

Part 8 Differential Calculus

44

Introduction to differentiation

44.1 Introduction to calculus

Calculus is a branch of mathematics involving or leading to calculations dealing with continuously varying functions.
Calculus is a subject that falls into two parts:

(i) **differential calculus** (or **differentiation**) and
(ii) **integral calculus** (or **integration**).

Differentiation is used in calculations involving velocity and acceleration, rates of change and maximum and minimum values of curves.

44.2 Functional notation

In an equation such as $y = 3x^2 + 2x - 5$, y is said to be a function of x and may be written as $y = f(x)$.

An equation written in the form $f(x) = 3x^2 + 2x - 5$ is termed **functional notation**. The value of $f(x)$ when $x = 0$ is denoted by $f(0)$, and the value of $f(x)$ when $x = 2$ is denoted by $f(2)$ and so on. Thus when $f(x) = 3x^2 + 2x - 5$, then

$$f(0) = 3(0)^2 + 2(0) - 5 = -5$$

and $\quad f(2) = 3(2)^2 + 2(2) - 5 = 11$ and so on.

Problem 1. If $f(x) = 4x^2 - 3x + 2$ find: $f(0)$, $f(3)$, $f(-1)$ and $f(3) - f(-1)$

$$f(x) = 4x^2 - 3x + 2$$

$$f(0) = 4(0)^2 - 3(0) + 2 = \mathbf{2}$$

$$f(3) = 4(3)^2 - 3(3) + 2 = 36 - 9 + 2 = \mathbf{29}$$

$$f(-1) = 4(-1)^2 - 3(-1) + 2 = 4 + 3 + 2 = \mathbf{9}$$

$$f(3) - f(-1) = 29 - 9 = \mathbf{20}$$

Problem 2. Given that $f(x) = 5x^2 + x - 7$ determine:

(i) $f(2) \div f(1)$ (iii) $f(3 + a) - f(3)$

(ii) $f(3 + a)$ (iv) $\dfrac{f(3 + a) - f(3)}{a}$

$$f(x) = 5x^2 + x - 7$$

(i) $f(2) = 5(2)^2 + 2 - 7 = 15$

$f(1) = 5(1)^2 + 1 - 7 = -1$

$$f(2) \div f(1) = \frac{15}{-1} = \mathbf{-15}$$

(ii) $f(3 + a) = 5(3 + a)^2 + (3 + a) - 7$

$= 5(9 + 6a + a^2) + (3 + a) - 7$

$= 45 + 30a + 5a^2 + 3 + a - 7$

$= \mathbf{41 + 31a + 5a^2}$

(iii) $f(3) = 5(3)^2 + 3 - 7 = 41$

$f(3 + a) - f(3) = (41 + 31a + 5a^2) - (41)$

$= \mathbf{31a + 5a^2}$

(iv) $\dfrac{f(3 + a) - f(3)}{a} = \dfrac{31a + 5a^2}{a} = \mathbf{31 + 5a}$

45

Methods of differentiation

45.1 Differentiation of common functions

The **standard derivatives** summarised below were derived in Chapter 44 and are true for all real values of x.

y or $f(x)$	$\dfrac{dy}{dx}$ or $f'(x)$
ax^n	anx^{n-1}
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$
e^{ax}	ae^{ax}
$\ln ax$	$\dfrac{1}{x}$

The **differential coefficient of a sum or difference** is the sum or difference of the differential coefficients of the separate terms.

Thus, if $f(x) = p(x) + q(x) - r(x)$, (where f, p, q and r are functions), then $f'(x) = p'(x) + q'(x) - r'(x)$

Differentiation of common functions is demonstrated in the following worked problems.

> Problem 1. Find the differential coefficients of: (a) $y = 12x^3$ (b) $y = \dfrac{12}{x^3}$

If $y = ax^n$ then $\dfrac{dy}{dx} = anx^{n-1}$

(a) Since $y = 12x^3$, $a = 12$ and $n = 3$ thus $\dfrac{dy}{dx} = (12)(3)x^{3-1} = \mathbf{36x^2}$

(b) $y = \dfrac{12}{x^3}$ is rewritten in the standard ax^n form as $y = 12x^{-3}$ and in the general rule $a = 12$ and $n = -3$

$$\text{Thus } \frac{dy}{dx} = (12)(-3)x^{-3-1} = -36x^{-4} = -\frac{\mathbf{36}}{\mathbf{x^4}}$$

> Problem 2. Differentiate: (a) $y = 6$ (b) $y = 6x$

(a) $y = 6$ may be written as $y = 6x^0$, i.e. in the general rule $a = 6$ and $n = 0$.

Hence $\dfrac{dy}{dx} = (6)(0)x^{0-1} = \mathbf{0}$

In general, **the differential coefficient of a constant is always zero.**

(b) Since $y = 6x$, in the general rule $a = 6$ and $n = 1$

Hence $\dfrac{dy}{dx} = (6)(1)x^{1-1} = 6x^0 = \mathbf{6}$

In general, the differential coefficient of kx, where k is a constant, is always k.

> Problem 3. Find the derivatives of: (a) $y = 3\sqrt{x}$ (b) $y = \dfrac{5}{\sqrt[3]{x^4}}$

(a) $y = 3\sqrt{x}$ is rewritten in the standard differential form as $y = 3x^{1/2}$

In the general rule, $a = 3$ and $n = \dfrac{1}{2}$

$$\text{Thus } \frac{dy}{dx} = (3)\left(\frac{1}{2}\right)x^{\frac{1}{2}-1} = \frac{3}{2}x^{-\frac{1}{2}} = \frac{3}{2x^{1/2}} = \frac{\mathbf{3}}{\mathbf{2\sqrt{x}}}$$

(b) $y = \dfrac{5}{\sqrt[3]{x^4}} = \dfrac{5}{x^{4/3}} = 5x^{-4/3}$ in the standard differential form.

In the general rule, $a = 5$ and $n = -\dfrac{4}{3}$

$$\text{Thus } \frac{dy}{dx} = (5)\left(-\frac{4}{3}\right)x^{(-4/3)-1} = \frac{-20}{3}x^{-7/3} = \frac{-20}{3x^{7/3}} = \frac{\mathbf{-20}}{\mathbf{3\sqrt[3]{x^7}}}$$

Problem 4. Differentiate: $y = 5x^4 + 4x - \dfrac{1}{2x^2} + \dfrac{1}{\sqrt{x}} - 3$ with respect to x

$y = 5x^4 + 4x - \dfrac{1}{2x^2} + \dfrac{1}{\sqrt{x}} - 3$ is rewritten as
$y = 5x^4 + 4x - \dfrac{1}{2}x^{-2} + x^{-1/2} - 3$

When differentiating a sum, each term is differentiated in turn.

$$\text{Thus } \frac{dy}{dx} = (5)(4)x^{4-1} + (4)(1)x^{1-1} - \frac{1}{2}(-2)x^{-2-1} + (1)\left(-\frac{1}{2}\right)x^{(-1/2)-1} - 0$$

$$= 20x^3 + 4 + x^{-3} - \frac{1}{2}x^{-3/2}$$

$$\text{i.e } \mathbf{\frac{dy}{dx} = 20x^3 + 4 - \frac{1}{x^3} - \frac{1}{2\sqrt{x^3}}}$$

Problem 5. Find the differential coefficients of: (a) $y = 3\sin 4x$ (b) $f(t) = 2\cos 3t$ with respect to the variable

(a) When $y = 3\sin 4x$ then $\dfrac{dy}{dx} = (3)(4\cos 4x)$
$= \mathbf{12\cos 4x}$

(b) When $f(t) = 2\cos 3t$ then
$f'(t) = (2)(-3\sin 3t) = \mathbf{-6\sin 3t}$

Problem 6. Determine the derivatives of: (a) $y = 3e^{5x}$ (b) $f(\theta) = \dfrac{2}{e^{3\theta}}$ (c) $y = 6\ln 2x$

(a) When $y = 3e^{5x}$ then $\dfrac{dy}{dx} = (3)(5)e^{5x}$
$= \mathbf{15e^{5x}}$

(b) $f(\theta) = \dfrac{2}{e^{3\theta}} = 2e^{-3\theta}$, thus
$f'(\theta) = (2)(-3)e^{-3\theta} = -6e^{-3\theta} = \mathbf{\dfrac{-6}{e^{3\theta}}}$

(c) When $y = 6\ln 2x$ then $\dfrac{dy}{dx} = 6\left(\dfrac{1}{x}\right) = \mathbf{\dfrac{6}{x}}$

Problem 7. Find the gradient of the curve $y = 3x^4 - 2x^2 + 5x - 2$ at the points $(0, -2)$ and $(1, 4)$

The gradient of a curve at a given point is given by the corresponding value of the derivative. Thus, since $y = 3x^4 - 2x^2 + 5x - 2$ then the gradient $= \dfrac{dy}{dx} = 12x^3 - 4x + 5$

At the point $(0, -2)$, $x = 0$.
Thus the gradient $= 12(0)^3 - 4(0) + 5 = \mathbf{5}$

At the point $(1, 4)$, $x = 1$.
Thus the gradient $= 12(1)^3 - 4(1) + 5 = \mathbf{13}$

Problem 8. Determine the co-ordinates of the point on the graph $y = 3x^2 - 7x + 2$ where the gradient is -1

The gradient of the curve is given by the derivative.

When $y = 3x^2 - 7x + 2$ then $\dfrac{dy}{dx} = 6x - 7$

Since the gradient is -1 then $6x - 7 = -1$, from which, $x = 1$

When $x = 1$, $y = 3(1)^2 - 7(1) + 2 = -2$

Hence the gradient is -1 at the point $(1, -2)$

Now try the following exercise

Exercise 153 Further problems on differentiating common functions

In Problems 1 to 6 find the differential coefficients of the given functions with respect to the variable.

1. (a) $5x^5$ (b) $2.4x^{3.5}$ (c) $\dfrac{1}{x}$
$\left[\text{(a) } 25x^4 \quad \text{(b) } 8.4x^{2.5} \quad \text{(c) } -\dfrac{1}{x^2}\right]$

2. (a) $\dfrac{-4}{x^2}$ (b) 6 (c) $2x$
$\left[\text{(a) } \dfrac{8}{x^3} \quad \text{(b) } 0 \quad \text{(c) } 2\right]$

3. (a) $2\sqrt{x}$ (b) $3\sqrt[3]{x^5}$ (c) $\dfrac{4}{\sqrt{x}}$
$\left[\text{(a) } \dfrac{1}{\sqrt{x}} \quad \text{(b) } 5\sqrt[3]{x^2} \quad \text{(c) } -\dfrac{2}{\sqrt{x^3}}\right]$

4. (a) $\dfrac{-3}{\sqrt[3]{x}}$ (b) $(x-1)^2$ (c) $2\sin 3x$
$\left[\text{(a) } \dfrac{1}{\sqrt[3]{x^4}} \quad \text{(b) } 2(x-1) \quad \text{(c) } 6\cos 3x\right]$

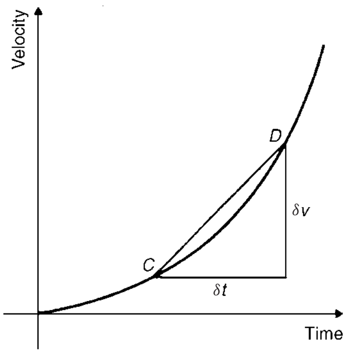


Figure 46.3

The **acceleration *a*** of the car is defined as the rate of change of velocity. A velocity/time graph is shown in Fig. 46.3. If δv is the change in v and δt the corresponding change in time, then $a = \dfrac{\delta v}{\delta t}$. As $\delta t \to 0$, the chord CD becomes a tangent, such that at point C, the acceleration is given by: $a = \dfrac{dv}{dt}$

Hence the acceleration of the car at any instant is given by the gradient of the velocity/time graph. If an expression for velocity is known in terms of time t then the acceleration is obtained by differentiating the expression.

$$\text{Acceleration} \quad a = \frac{dv}{dt}.$$

$$\text{However,} \quad v = \frac{dx}{dt}$$

$$\text{Hence} \quad a = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

The acceleration is given by the second differential coefficient of distance x with respect to time t

Summarising, if a body moves a distance x metres in a time t seconds then:

(i) **distance $x = f(t)$**

(ii) **velocity $v = f'(t)$ or $\dfrac{dx}{dt}$**, which is the gradient of the distance/time graph

(iii) **acceleration $a = \dfrac{dv}{dt} = f''$ or $\dfrac{d^2x}{dt^2}$**, which is the gradient of the velocity/time graph.

> Problem 5. The distance x metres moved by a car in a time t seconds is given by: $x = 3t^3 - 2t^2 + 4t - 1$. Determine the velocity and acceleration when (a) $t = 0$, and (b) $t = 1.5$ s

$$\text{Distance} \quad x = 3t^3 - 2t^2 + 4t - 1 \text{ m.}$$

$$\text{Velocity} \quad v = \frac{dx}{dt} = 9t^2 - 4t + 4 \text{ m/s}$$

$$\text{Acceleration} \quad a = \frac{d^2x}{dx^2} = 18t - 4 \text{ m/s}^2$$

(a) When time $t = 0$,
velocity $\boldsymbol{v} = 9(0)^2 - 4(0) + 4 = \mathbf{4\ m/s}$
and acceleration $\boldsymbol{a} = 18(0) - 4 = \mathbf{-4\ m/s^2}$
(i.e. a deceleration)

(b) When time $t = 1.5$ s,
velocity $\boldsymbol{v} = 9(1.5)^2 - 4(1.5) + 4 = \mathbf{18.25\ m/s}$
and acceleration $\boldsymbol{a} = 18(1.5) - 4 = \mathbf{23\ m/s^2}$

> Problem 6. Supplies are dropped from a helicopter and the distance fallen in a time t seconds is given by: $x = \frac{1}{2}gt^2$, where $g = 9.8$ m/s^2. Determine the velocity and acceleration of the supplies after it has fallen for 2 seconds

$$\text{Distance} \quad x = \frac{1}{2}gt^2 = \frac{1}{2}(9.8)t^2 = 4.9t^2 \text{m}$$

$$\text{Velocity} \quad v = \frac{dv}{dt} = 9.8 \text{ t m/s}$$

$$\text{and acceleration} \quad a = \frac{d^2x}{dt^2} = 9.8 \text{ m/s}^2$$

When time $t = 2$ s,
velocity $\boldsymbol{v} = (9.8)(2) = \mathbf{19.6\ m/s}$
and **acceleration $\boldsymbol{a}$ = 9.8 m/s^2** (which is acceleration due to gravity).

> Problem 7. The distance x metres travelled by a vehicle in time t seconds after the brakes are applied is given by: $x = 20t - \dfrac{5}{3}t^2$. Determine (a) the speed of the vehicle (in km/h) at the instant the brakes are applied, and (b) the distance the car travels before it stops

(a) Distance, $x = 20t - \dfrac{5}{3}t^2$.

Hence velocity $v = \dfrac{dx}{dt} = 20 - \dfrac{10}{3}t$

At the instant the brakes are applied, time = 0

Hence

$$\textbf{velocity } v = 20 \text{ m/s} = \frac{20 \times 60 \times 60}{1000} \text{ km/h}$$

$$= \mathbf{72\ km/h}$$

Let the dimensions of the rectangle be x and y. Then the perimeter of the rectangle is $(2x + 2y)$. Hence $2x + 2y = 40$, or $x + y = 20$ (1)

Since the rectangle is to enclose the maximum possible area, a formula for area A must be obtained in terms of one variable only.

Area $A = xy$. From equation (1), $x = 20 - y$

Hence, area $A = (20 - y)y = 20y - y^2$

$\frac{dA}{dy} = 20 - 2y = 0$ for a turning point, from which, $y = 10$ cm.

$\frac{d^2A}{dy^2} = -2$, which is negative, giving a maximum point.

When $y = 10$ cm, $x = 10$ cm, from equation (1).

Hence the length and breadth of the rectangle are each 10 cm, i.e. a square gives the maximum possible area. When the perimeter of a rectangle is 40 cm, the maximum possible area is $10 \times 10 =$ **100 cm^2**.

Problem 16. A rectangular sheet of metal having dimensions 20 cm by 12 cm has squares removed from each of the four corners and the sides bent upwards to form an open box. Determine the maximum possible volume of the box

The squares to be removed from each corner are shown in Fig. 46.8, having sides x cm. When the sides are bent upwards the dimensions of the box will be: length $(20 - 2x)$ cm, breadth $(12 - 2x)$ cm and height, x cm.

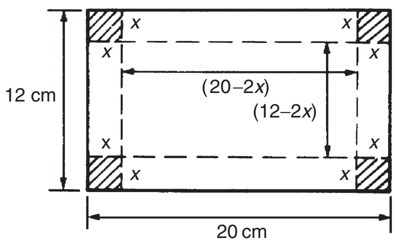


Figure 46.8

Volume of box, $V = (20 - 2x)(12 - 2x)(x)$
$= 240x - 64x^2 + 4x^3$

$\frac{dV}{dx} = 240 - 128x + 12x^2 = 0$ for a turning point.

Hence $4(60 - 32x + 3x^2) = 0$, i.e. $3x^2 - 32x + 60 = 0$

Using the quadratic formula,

$$x = \frac{32 \pm \sqrt{(-32)^2 - 4(3)(60)}}{2(3)} = 8.239 \text{ cm or } 2.427 \text{ cm.}$$

Since the breadth is $(12 - 2x)$ cm then $x = 8.239$ cm is not possible and is neglected. Hence $x = 2.427$ cm.

$$\frac{d^2V}{dx^2} = -128 + 24x.$$

When $x = 2.427$, $\frac{d^2V}{dx^2}$ is negative, giving a maximum value.

The dimensions of the box are:
length $= 20 - 2(2.427) = 15.146$ cm,
breadth $= 12 - 2(2.427) = 7.146$ cm, and
height $= 2.427$ cm.

Maximum volume $= (15.146)(7.146)(2.427)$
$=$ **262.7 cm^3**

Problem 17. Determine the height and radius of a cylinder of volume 200 cm^3 which has the least surface area

Let the cylinder have radius r and perpendicular height h.

Volume of cylinder, $V = \pi r^2 h = 200$ (1)

Surface area of cylinder, $A = 2\pi rh + 2\pi r^2$

Least surface area means minimum surface area and a formula for the surface area in terms of one variable only is required.

From equation (1), $h = \frac{200}{\pi r^2}$ (2)

Hence surface area,

$$A = 2\pi r\left(\frac{200}{\pi r^2}\right) + 2\pi r^2 = \frac{400}{r} + 2\pi r^2 = 400r^{-1} + 2\pi r^2$$

$\frac{dA}{dr} = \frac{-400}{r^2} + 4\pi r = 0$, for a turning point.

Hence $4\pi r = \frac{400}{r^2}$

and $r^3 = \frac{400}{4\pi}$,

from which, $r = \sqrt[3]{\frac{100}{\pi}} = 3.169$ cm.

$$\frac{d^2A}{dr^2} = \frac{800}{r^3} + 4\pi.$$

7. $\displaystyle\int_0^1 2\tan^2 2t\,dt$ $[-4.185]$

8. $\displaystyle\int_{\pi/6}^{\pi/3} \cot^2\theta\,d\theta$ $[0.3156]$

49.3 Worked problems on powers of sines and cosines

Problem 5. Determine: $\displaystyle\int \sin^5\theta\,d\theta$

Since $\cos^2\theta + \sin^2\theta = 1$ then $\sin^2\theta = (1 - \cos^2\theta)$.

Hence $\displaystyle\int \sin^5\theta\,d\theta$

$$= \int \sin\theta(\sin^2\theta)^2\,d\theta = \int \sin\theta(1-\cos^2\theta)^2\,d\theta$$

$$= \int \sin\theta(1 - 2\cos^2\theta + \cos^4\theta)\,d\theta$$

$$= \int (\sin\theta - 2\sin\theta\cos^2\theta + \sin\theta\cos^4\theta)\,d\theta$$

$$= \mathbf{-\cos\theta + \frac{2\cos^3\theta}{3} - \frac{\cos^5\theta}{5} + c}$$

[Whenever a power of a cosine is multiplied by a sine of power 1, or vice-versa, the integral may be determined by inspection as shown.

In general, $\displaystyle\int \cos^n\theta\sin\theta\,d\theta = \frac{-\cos^{n+1}\theta}{(n+1)} + c$

and $\displaystyle\int \sin^n\theta\cos\theta\,d\theta = \frac{\sin^{n+1}\theta}{(n+1)} + c$]

Alternatively, an algebraic substitution may be used as shown in Problem 6, chapter 50, page 415].

Problem 6. Evaluate: $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\,dx$

$$\int_0^{\frac{\pi}{2}} \sin^2 x\cos^3 x\,dx = \int_0^{\frac{\pi}{2}} \sin^2 x\cos^2 x\cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x)(1-\sin^2 x)(\cos x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x\cos x - \sin^4 x\cos x)\,dx$$

$$= \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\left(\sin\frac{\pi}{2}\right)^3}{3} - \frac{\left(\sin\frac{\pi}{2}\right)^5}{5}\right] - [0-0]$$

$$= \frac{1}{3} - \frac{1}{5} = \mathbf{\frac{2}{15}} \text{ or } \mathbf{0.1333}$$

Problem 7. Evaluate: $\displaystyle\int_0^{\frac{\pi}{4}} 4\cos^4\theta\,d\theta$, correct to 4 significant figures

$$\int_0^{\frac{\pi}{4}} 4\cos^4\theta\,d\theta = 4\int_0^{\frac{\pi}{4}} (\cos^2\theta)^2\,d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \left[\frac{1}{2}(1+\cos 2\theta)\right]^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} (1 + 2\cos 2\theta + \cos^2 2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta$$

$$= \left[\frac{3\theta}{2} + \sin 2\theta + \frac{\sin 4\theta}{8}\right]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{3}{2}\left(\frac{\pi}{4}\right) + \sin\frac{2\pi}{4} + \frac{\sin 4(\pi/4)}{8}\right] - [0]$$

$$= \frac{3\pi}{8} + 1$$

$= \mathbf{2.178}$, correct to 4 significant figures.

Problem 8. Find: $\displaystyle\int \sin^2 t\cos^4 t\,dt$

5. Evaluate: $\displaystyle\int_0^4 \frac{1}{\sqrt{16-x^2}}\,dx$

$\left[\dfrac{\pi}{2} \text{ or } 1.571\right]$

6. Evaluate: $\displaystyle\int_0^1 \sqrt{9-4x^2}\,dx$ [2.760]

49.6 Worked problems on integration using the tan θ substitution

Problem 17. Determine: $\displaystyle\int \frac{1}{(a^2+x^2)}\,dx$

Let $x = a\tan\theta$ then $\dfrac{dx}{d\theta} = a\sec^2\theta$ and $dx = a\sec^2\theta\, d\theta$

Hence $\displaystyle\int \frac{1}{(a^2+x^2)}\,dx$

$$= \int \frac{1}{(a^2+a^2\tan^2\theta)}(a\sec^2\theta\, d\theta)$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2(1+\tan^2\theta)}$$

$$= \int \frac{a\sec^2\theta\, d\theta}{a^2\sec^2\theta}, \quad \text{since } 1+\tan^2\theta = \sec^2\theta$$

$$= \int \frac{1}{a}\,d\theta = \frac{1}{a}(\theta) + c$$

Since $x = a\tan\theta$, $\theta = \tan^{-1}\dfrac{x}{a}$

Hence $\displaystyle\int \frac{\mathbf{1}}{(\boldsymbol{a^2+x^2})}\,\boldsymbol{dx} = \frac{\mathbf{1}}{\boldsymbol{a}}\,\mathbf{tan^{-1}}\,\frac{\boldsymbol{x}}{\boldsymbol{a}} + \boldsymbol{c}$

Problem 18. Evaluate: $\displaystyle\int_0^2 \frac{1}{(4+x^2)}\,dx$

From Problem 17, $\displaystyle\int_0^2 \frac{1}{(4+x^2)}\,dx$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2 \quad \text{since } a = 2$$

$$= \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0) = \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\boldsymbol{\pi}}{\mathbf{8}} \text{ or } \mathbf{0.3927}$$

Problem 19. Evaluate: $\displaystyle\int_0^1 \frac{5}{(3+2x^2)}\,dx$, correct to 4 decimal places

$$\int_0^1 \frac{5}{(3+2x^2)}\,dx = \int_0^1 \frac{5}{2[(3/2)+x^2]}\,dx$$

$$= \frac{5}{2}\int_0^1 \frac{1}{[\sqrt{3/2}]^2+x^2}\,dx$$

$$= \frac{5}{2}\left[\frac{1}{\sqrt{3/2}}\tan^{-1}\frac{x}{\sqrt{3/2}}\right]_0^1$$

$$= \frac{5}{2}\sqrt{\frac{2}{3}}\left[\tan^{-1}\sqrt{\frac{2}{3}} - \tan^{-1}0\right]$$

$$= (2.0412)[0.6847 - 0]$$

$$= \mathbf{1.3976}, \text{ correct to 4 decimal places.}$$

Now try the following exercise

Exercise 177 Further problems on integration using the tan θ substitution

1. Determine: $\displaystyle\int \frac{3}{4+t^2}\,dt$

$\left[\dfrac{3}{2}\tan^{-1}\dfrac{x}{2} + c\right]$

2. Determine: $\displaystyle\int \frac{5}{16+9\theta^2}\,d\theta$

$\left[\dfrac{5}{12}\tan^{-1}\dfrac{3\theta}{4} + c\right]$

3. Evaluate: $\displaystyle\int_0^1 \frac{3}{1+t^2}\,dt$ [2.356]

4. Evaluate: $\displaystyle\int_0^3 \frac{5}{4+x^2}\,dx$ [2.457]

1. $\int xe^{2x}\,dx$ $\quad\left[\frac{e^{2x}}{2}\left(x-\frac{1}{2}\right)+c\right]$

2. $\int \frac{4x}{e^{3x}}\,dx$ $\quad\left[-\frac{4}{3}e^{-3x}\left(x+\frac{1}{3}\right)+c\right]$

3. $\int x\sin x\,dx$ $\quad[-x\cos x+\sin x+c]$

4. $\int 5\theta\cos 2\theta\,d\theta$
$\quad\left[\frac{5}{2}\left(\theta\sin 2\theta+\frac{1}{2}\cos 2\theta\right)+c\right]$

5. $\int 3t^2e^{2t}\,dt$ $\quad\left[\frac{3}{2}e^{2t}\left(t^2-t+\frac{1}{2}\right)+c\right]$

Evaluate the integrals in Problems 6 to 9, correct to 4 significant figures.

6. $\int_0^2 2xe^x\,dx$ $\quad[16.78]$

7. $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx$ $\quad[0.2500]$

8. $\int_0^{\frac{\pi}{2}} t^2\cos t\,dt$ $\quad[0.4674]$

9. $\int_1^2 3x^2e^{\frac{x}{2}}\,dx$ $\quad[15.78]$

52.3 Further worked problems on integration by parts

Problem 6. Find: $\int x\ln x\,dx$

The logarithmic function is chosen as the 'u part'

Thus when $u=\ln x$, then $\frac{du}{dx}=\frac{1}{x}$, i.e. $du=\frac{dx}{x}$

Letting $dv=x\,dx$ gives $v=\int x\,dx=\frac{x^2}{2}$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int x\ln x\,dx=(\ln x)\left(\frac{x^2}{2}\right)-\int\left(\frac{x^2}{2}\right)\frac{dx}{x}$$
$$=\frac{x^2}{2}\ln x-\frac{1}{2}\int x\,dx$$
$$=\frac{x^2}{2}\ln x-\frac{1}{2}\left(\frac{x^2}{2}\right)+c$$

Hence $\quad\int x\ln x\,dx=\boldsymbol{\frac{x^2}{2}\left(\ln x-\frac{1}{2}\right)+c}$

or $\quad\boldsymbol{\frac{x^2}{4}(2\ln x-1)+c}$

Problem 7. Determine: $\int \ln x\,dx$

$\int\ln x\,dx$ is the same as $\int(1)\ln x\,dx$

Let $u=\ln x$, from which, $\frac{du}{dx}=\frac{1}{x}$, i.e. $du=\frac{dx}{x}$

and let $dv=1\,dx$, from which, $v=\int 1\,dx=x$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int\ln x\,dx=(\ln x)(x)-\int x\frac{dx}{x}$$
$$=x\ln x-\int dx=x\ln x-x+c$$

Hence $\quad\boldsymbol{\int\ln x\,dx=x(\ln x-1)+c}$

Problem 8. Evaluate: $\int_1^9\sqrt{x}\ln x\,dx$, correct to 3 significant figures

Let $u=\ln x$, from which $du=\frac{dx}{x}$

and let $dv=\sqrt{x}\,dx=x^{\frac{1}{2}}\,dx$, from which,

$$v=\int x^{\frac{1}{2}}\,dx=\frac{2}{3}x^{\frac{3}{2}}$$

Substituting into $\int u\,dv=uv-\int v\,du$ gives:

$$\int\sqrt{x}\ln x\,dx=(\ln x)\left(\frac{2}{3}x^{\frac{3}{2}}\right)-\int\left(\frac{2}{3}x^{\frac{3}{2}}\right)\left(\frac{dx}{x}\right)$$
$$=\frac{2}{3}\sqrt{x^3}\ln x-\frac{2}{3}\int x^{\frac{1}{2}}\,dx$$
$$=\frac{2}{3}\sqrt{x^3}\ln x-\frac{2}{3}\left(\frac{2}{3}x^{\frac{3}{2}}\right)+c$$
$$=\frac{2}{3}\sqrt{x^3}\left[\ln x-\frac{2}{3}\right]+c$$

Hence $\int_1^9\sqrt{x}\ln x\,dx=\left[\frac{2}{3}\sqrt{x^3}\left(\ln x-\frac{2}{3}\right)\right]_1^9$

Problem 4. Use the mid-ordinate rule with (a) 4 intervals, (b) 8 intervals, to evaluate $\int_1^3 \frac{2}{\sqrt{x}}\, dx$, correct to 3 decimal places

(a) With 4 intervals, each will have a width of $\frac{3-1}{4}$, i.e. 0.5 and the ordinates will occur at 1.0, 1.5, 2.0, 2.5 and 3.0. Hence the mid-ordinates y_1, y_2, y_3 and y_4 occur at 1.25, 1.75, 2.25 and 2.75

Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the following table:

x	$\frac{2}{\sqrt{x}}$
1.25	1.7889
1.75	1.5119
2.25	1.3333
2.75	1.2060

From equation (2):

$$\int_1^3 \frac{2}{\sqrt{x}}\, dx \approx (0.5)[1.7889 + 1.5119 + 1.3333 + 1.2060]$$

$$= \mathbf{2.920}, \text{ correct to 3 decimal places}$$

(b) With 8 intervals, each will have a width of 0.25 and the ordinates will occur at 1.00, 1.25, 1.50, 1.75, ... and thus mid-ordinates at 1.125, 1.375, 1.625, 1.875 .... Corresponding values of $\frac{2}{\sqrt{x}}$ are shown in the following table:

x	$\frac{2}{\sqrt{x}}$
1.125	1.8856
1.375	1.7056
1.625	1.5689
1.875	1.4606
2.125	1.3720
2.375	1.2978
2.625	1.2344
2.875	1.1795

From equation (2):

$$\int_1^3 \frac{2}{\sqrt{x}}\, dx \approx (0.25)[1.8856 + 1.7056 + 1.5689 + 1.4606 + 1.3720 + 1.2978 + 1.2344 + 1.1795]$$

$$= \mathbf{2.926}, \text{ correct to 3 decimal places}$$

As previously, the greater the number of intervals the nearer the result is to the true value of 2.928, correct to 3 decimal places.

Problem 5. Evaluate $\int_0^{2.4} e^{-x^2/3}\, dx$, correct to 4 significant figures, using the mid-ordinate rule with 6 intervals

With 6 intervals each will have a width of $\frac{2.4-0}{6}$, i.e. 0.40 and the ordinates will occur at 0, 0.40, 0.80, 1.20, 1.60, 2.00 and 2.40 and thus mid-ordinates at 0.20, 0.60, 1.00, 1.40, 1.80 and 2.20.

Corresponding values of $e^{-x^2/3}$ are shown in the following table:

x	$e^{\frac{-x^2}{3}}$
0.20	0.98676
0.60	0.88692
1.00	0.71653
1.40	0.52031
1.80	0.33960
2.20	0.19922

From equation (2):

$$\int_0^{2.4} e^{\frac{-x^2}{3}}\, dx \approx (0.40)[0.98676 + 0.88692 + 0.71653 + 0.52031 + 0.33960 + 0.19922]$$

$$= \mathbf{1.460}, \text{ correct to 4 significant figures.}$$

$0, \dfrac{\pi}{18}, \dfrac{\pi}{9}, \dfrac{\pi}{6}, \dfrac{2\pi}{9}, \dfrac{5\pi}{18}$ and $\dfrac{\pi}{3}$

Corresponding values of $\sqrt{1-\frac{1}{3}\sin^2\theta}$ are shown in the table below:

θ	0	$\frac{\pi}{18}$ (or 10°)	$\frac{\pi}{9}$ (or 20°)	$\frac{\pi}{6}$ (or 30°)
$\sqrt{1-\frac{1}{3}\sin^2\theta}$	1.0000	0.9950	0.9803	0.9574

θ	$\frac{2\pi}{9}$ (or 40°)	$\frac{5\pi}{18}$ (or 50°)	$\frac{\pi}{3}$ (or 60°)
$\sqrt{1-\frac{1}{3}\sin^2\theta}$	0.9286	0.8969	0.8660

From equation (5):

$$\int_0^{\frac{\pi}{3}} \sqrt{1-\frac{1}{3}\sin^2\theta}\, d\theta$$

$$\approx \frac{1}{3}\left(\frac{\pi}{18}\right)[(1.0000 + 0.8660) + 4(0.9950 + 0.9574 + 0.8969) + 2(0.9803 + 0.9286)]$$

$$= \frac{1}{3}\left(\frac{\pi}{18}\right)[1.8660 + 11.3972 + 3.8178]$$

$= \mathbf{0.994}$, correct to 3 decimal places.

Problem 8. An alternating current i has the following values at equal intervals of 2.0 milliseconds:

Time (ms)	0	2.0	4.0	6.0	8.0	10.0	12.0
Current i (A)	0	3.5	8.2	10.0	7.3	2.0	0

Charge, q, in millicoulombs, is given by $q = \int_0^{12.0} i\,dt$. Use Simpson's rule to determine the approximate charge in the 12 ms period

From equation (5):

$$\text{Charge, } q = \int_0^{12.0} i\,dt$$

$$\approx \frac{1}{3}(2.0)\,[(0+0) + 4(3.5 + 10.0 + 2.0) + 2(8.2 + 7.3)]$$

$= \mathbf{62\,mC}$

Now try the following exercise

Exercise 182 Further problems on Simpson's rule

In Problems 1 to 5, evaluate the definite integrals using **Simpson's rule**, giving the answers correct to 3 decimal places.

1. $\displaystyle\int_0^{\pi/2} \sqrt{\sin x}\, dx$ (Use 6 intervals) [1.187]

2. $\displaystyle\int_0^{1.6} \frac{1}{1+\theta^4}\, d\theta$ (Use 8 intervals) [1.034]

3. $\displaystyle\int_{0.2}^{1.0} \frac{\sin\theta}{\theta}\, d\theta$ (Use 8 intervals) [0.747]

4. $\displaystyle\int_0^{\pi/2} x\cos x\, dx$ (Use 6 intervals) [0.571]

5. $\displaystyle\int_0^{\pi/3} e^{x^2}\sin 2x\, dx$ (Use 10 intervals) [1.260]

In Problems 6 and 7 evaluate the definite integrals using (a) integration, (b) the trapezoidal rule, (c) the mid-ordinate rule, (d) Simpson's rule. Give answers correct to 3 decimal places.

6. $\displaystyle\int_1^4 \frac{4}{x^3}\, dx$ (Use 6 intervals) [(a) 1.875 (b) 2.107 (c) 1.765 (d) 1.916]

7. $\displaystyle\int_2^6 \frac{1}{\sqrt{2x-1}}\, dx$ (Use 8 intervals) [(a) 1.585 (b) 1.588 (c) 1.583 (d) 1.585]

In Problems 8 and 9 evaluate the definite integrals using (a) the trapezoidal rule, (b) the mid-ordinate rule, (c) Simpson's rule. Use 6 intervals in each case and give answers correct to 3 decimal places.

8. $\displaystyle\int_0^3 \sqrt{1+x^4}\,dx$

[(a) 10.194 (b) 10.007 (c) 10.070]

9. $\displaystyle\int_{0.1}^{0.7} \frac{1}{\sqrt{1-y^2}}\,dy$

[(a) 0.677 (b) 0.674 (c) 0.675]

10. A vehicle starts from rest and its velocity is measured every second for 8 seconds, with values as follows:

time t (s)	velocity v (ms^{-1})
0	0
1.0	0.4
2.0	1.0
3.0	1.7
4.0	2.9
5.0	4.1
6.0	6.2
7.0	8.0
8.0	9.4

The distance travelled in 8.0 seconds is given by $\int_0^{8.0} v\,dt$.

Estimate this distance using Simpson's rule, giving the answer correct to 3 significant figures. [28.8 m]

11. A pin moves along a straight guide so that its velocity v (m/s) when it is a distance x (m) from the beginning of the guide at time t (s) is given in the table below:

t (s)	v (m/s)
0	0
0.5	0.052
1.0	0.082
1.5	0.125
2.0	0.162
2.5	0.175
3.0	0.186
3.5	0.160
4.0	0

Use Simpson's rule with 8 intervals to determine the approximate total distance travelled by the pin in the 4.0 second period.

[0.485 m]

$$= \left(8 \sin \frac{\pi}{4}\right) - (8 \sin 0)$$

$$= \textbf{5.657 square units}$$

> Problem 8. Determine the area bounded by the curve $y = 3e^{t/4}$, the t-axis and ordinates $t = -1$ and $t = 4$, correct to 4 significant figures

A table of values is produced as shown.

t	-1	0	1	2	3	4
$y = 3e^{t/4}$	2.34	3.0	3.85	4.95	6.35	8.15

Since all the values of y are positive the area required is wholly above the t-axis.

$$\text{Hence area } = \int_1^4 y\, dt$$

$$= \int_1^4 3e^{t/4}\, dt = \left[\frac{3}{\left(\frac{1}{4}\right)} e^{t/4}\right]_{-1}^4$$

$$= 12[e^{t/4}]_{-1}^4 = 12(e^1 - e^{-1/4})$$

$$= 12(2.7183 - 0.7788)$$

$$= 12(1.9395) = \textbf{23.27 square units}$$

> Problem 9. Sketch the curve $y = x^2 + 5$ between $x = -1$ and $x = 4$. Find the area enclosed by the curve, the x-axis and the ordinates $x = 0$ and $x = 3$. Determine, also, by integration, the area enclosed by the curve and the y-axis, between the same limits

A table of values is produced and the curve $y = x^2 + 5$ plotted as shown in Fig. 54.9.

x	-1	0	1	2	3
y	6	5	6	9	14

$$\text{Shaded area} = \int_0^3 y\, dx = \int_0^3 (x^2 + 5)\, dx$$

$$= \left[\frac{x^3}{3} + 5x\right]_0^3$$

$$= \textbf{24 square units}$$

When $x = 3$, $y = 3^2 + 5 = 14$, and when $x = 0$, $y = 5$.

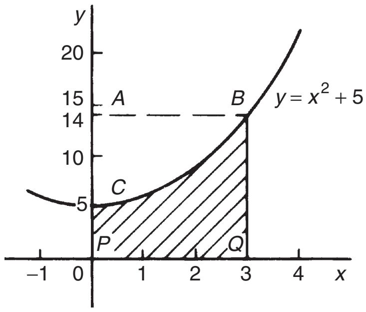


Figure 54.9

Since $y = x^2 + 5$ then $x^2 = y - 5$ and $x = \sqrt{y - 5}$

The area enclosed by the curve $y = x^2 + 5$ (i.e. $x = \sqrt{y - 5}$), the y-axis and the ordinates $y = 5$ and $y = 14$ (i.e. area ABC of Fig. 54.9) is given by:

$$\text{Area} = \int_{y=5}^{y=14} x\, dy = \int_5^{14} \sqrt{y - 5}\, dy$$

$$= \int_5^{14} (y - 5)^{1/2}\, dy$$

Let $u = y - 5$, then $\frac{du}{dy} = 1$ and $dy = du$

Hence $\int (y - 5)^{1/2}\, dy = \int u^{1/2}\, du = \frac{2}{3} u^{3/2}$ (for algebraic substitution, see Chapter 48)

Since $u = y - 5$ then

$$\int_5^{14} \sqrt{y - 5}\, dy = \frac{2}{3}[(y - 5)^{3/2}]_5^{14}$$

$$= \frac{2}{3}[\sqrt{9^3} - 0]$$

$$= \textbf{18 square units}$$

(Check: From Fig. 54.9, area BCPQ + area ABC = 24 + 18 = 42 square units, which is the area of rectangle $ABQP$.)

> Problem 10. Determine the area between the curve $y = x^3 - 2x^2 - 8x$ and the x-axis

$$y = x^3 - 2x^2 - 8x = x(x^2 - 2x - 8)$$

$$= x(x + 2)(x - 4)$$

When $y = 0$, then $x = 0$ or $(x + 2) = 0$ or $(x - 4) = 0$, i.e. when $y = 0$, $x = 0$ or -2 or 4, which means that the curve crosses the x-axis at 0, -2 and

3. The speed v of a vehicle is given by: $v = (4t + 3)$ m/s, where t is the time in seconds. Determine the average value of the speed from $t = 0$ to $t = 3$ s. [9 m/s]

4. Find the mean value of the curve $y = 6 + x - x^2$ which lies above the x-axis by using an approximate method. Check the result using integration. [4.17]

5. The vertical height h km of a missile varies with the horizontal distance d km, and is given by $h = 4d - d^2$. Determine the mean height of the missile from $d = 0$ to $d = 4$ km. [2.67 km]

6. The velocity v of a piston moving with simple harmonic motion at any time t is given by: $v = c \sin \omega t$, where c is a constant. Determine the mean velocity between $t = 0$ and $t = \dfrac{\pi}{\omega}$. $\left[\dfrac{2c}{\pi}\right]$

55.2 Root mean square values

The **root mean square value** of a quantity is 'the square root of the mean value of the squared values of the quantity' taken over an interval. With reference to Fig. 53.1, the r.m.s. value of $y = f(x)$ over the range $x = a$ to $x = b$ is given by:

$$\textbf{r.m.s. value} = \sqrt{\frac{1}{b-a}\int_a^b y^2\,dx}$$

One of the principal applications of r.m.s. values is with alternating currents and voltages. The r.m.s. value of an alternating current is defined as that current which will give the same heating effect as the equivalent direct current.

Problem 5. Determine the r.m.s. value of $y = 2x^2$ between $x = 1$ and $x = 4$

R.m.s. value

$$= \sqrt{\frac{1}{4-1}\int_1^4 y^2\,dx} = \sqrt{\frac{1}{3}\int_1^4 (2x^2)^2\,dx}$$

$$= \sqrt{\frac{1}{3}\int_1^4 4x^4\,dx} = \sqrt{\frac{4}{3}\left[\frac{x^5}{5}\right]_1^4}$$

$$= \sqrt{\frac{4}{15}(1024 - 1)} = \sqrt{272.8} = \mathbf{16.5}$$

Problem 6. A sinusoidal voltage has a maximum value of 100 V. Calculate its r.m.s. value

A sinusoidal voltage v having a maximum value of 10 V may be written as: $v = 10 \sin\theta$. Over the range $\theta = 0$ to $\theta = \pi$,

r.m.s. value

$$= \sqrt{\frac{1}{\pi - 0}\int_0^\pi v^2\,d\theta}$$

$$= \sqrt{\frac{1}{\pi}\int_0^\pi (100\sin\theta)^2\,d\theta}$$

$$= \sqrt{\frac{10\,000}{\pi}\int_0^\pi \sin^2\theta\,d\theta}$$

which is not a 'standard' integral. It is shown in Chapter 26 that $\cos 2A = 1 - 2\sin^2 A$ and this formula is used whenever $\sin^2 A$ needs to be integrated.

Rearranging $\cos 2A = 1 - 2\sin^2 A$ gives $\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

Hence $\sqrt{\dfrac{10\,000}{\pi}\displaystyle\int_0^\pi \sin^2\theta\,d\theta}$

$$= \sqrt{\frac{10\,000}{\pi}\int_0^\pi \frac{1}{2}(1 - \cos 2\theta)\,d\theta}$$

$$= \sqrt{\frac{10\,000}{\pi}\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi}$$

$$= \sqrt{\frac{10\,000}{\pi}\frac{1}{2}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right]}$$

$$= \sqrt{\frac{10\,000}{\pi}\frac{1}{2}[\pi]} = \sqrt{\frac{10\,000}{2}}$$

$$= \frac{100}{\sqrt{2}} = \mathbf{70.71\ volts}$$

$$= \frac{\int_0^5 (5x^2 - x^3)\,dx}{\int_0^5 (5x - x^2)\,dx} = \frac{\left[\frac{5x^3}{3} - \frac{x^4}{4}\right]_0^5}{\left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5}$$

$$= \frac{\frac{625}{3} - \frac{625}{4}}{\frac{125}{2} - \frac{125}{3}} = \frac{\frac{625}{12}}{\frac{125}{6}}$$

$$= \left(\frac{625}{12}\right)\left(\frac{6}{125}\right) = \frac{5}{2} = \mathbf{2.5}$$

$$\overline{y} = \frac{\frac{1}{2}\int_0^5 y^2\,dx}{\int_0^5 y\,dx} = \frac{\frac{1}{2}\int_0^5 (5x - x^2)^2\,dx}{\int_0^5 (5x - x^2)\,dx}$$

$$= \frac{\frac{1}{2}\int_0^5 (25x^2 - 10x^3 + x^4)\,dx}{\frac{125}{6}}$$

$$= \frac{\frac{1}{2}\left[\frac{25x^3}{3} - \frac{10x^4}{4} + \frac{x^5}{5}\right]_0^5}{\frac{125}{6}}$$

$$= \frac{\frac{1}{2}\left(\frac{25(125)}{3} - \frac{6250}{4} + 625\right)}{\frac{125}{6}} = \mathbf{2.5}$$

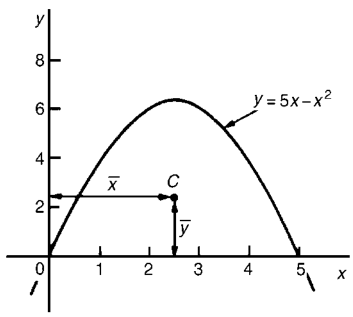


Figure 57.6

Hence the centroid of the area lies at (2.5, 2.5) (Note from Fig. 57.6 that the curve is symmetrical about $x = 2.5$ and thus $\overline{x}$ could have been determined 'on sight').

Problem 5. Locate the centroid of the area enclosed by the curve $y = 2x^2$, the y-axis and ordinates $y = 1$ and $y = 4$, correct to 3 decimal places

From Section 57.4,

$$\overline{x} = \frac{\frac{1}{2}\int_1^4 x^2\,dy}{\int_1^4 x\,dy} = \frac{\frac{1}{2}\int_1^4 \frac{y}{2}\,dy}{\int_1^4 \sqrt{\frac{y}{2}}\,dy}$$

$$= \frac{\frac{1}{2}\left[\frac{y^2}{4}\right]_1^4}{\left[\frac{2y^{3/2}}{3\sqrt{2}}\right]_1^4} = \frac{\frac{15}{8}}{\frac{14}{3\sqrt{2}}} = \mathbf{0.568}$$

and $$\overline{y} = \frac{\int_1^4 xy\,dy}{\int_1^4 x\,dy} = \frac{\int_1^4 \sqrt{\frac{y}{2}}(y)\,dy}{\frac{14}{3\sqrt{2}}}$$

$$= \frac{\int_1^4 \frac{y^{3/2}}{\sqrt{2}}\,dy}{\frac{14}{3\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}\left[\frac{y^{5/2}}{\frac{5}{2}}\right]_1^4}{\frac{14}{3\sqrt{2}}}$$

$$= \frac{\frac{2}{5\sqrt{2}}(31)}{\frac{14}{3\sqrt{2}}} = \mathbf{2.657}$$

Hence the position of the centroid is at (0.568, 2.657)

Problem 6. Locate the position of the centroid enclosed by the curves $y = x^2$ and $y^2 = 8x$

Figure 57.7 shows the two curves intersecting at (0, 0) and (2, 4). These are the same curves as used in Problem 12, Chapter 54, where the shaded area was calculated as $2\frac{2}{3}$ square units. Let the co-ordinates of centroid C be $\overline{x}$ and $\overline{y}$.

By integration, $\qquad \overline{x} = \dfrac{\displaystyle\int_0^2 xy\,dx}{\displaystyle\int_0^2 y\,dx}$

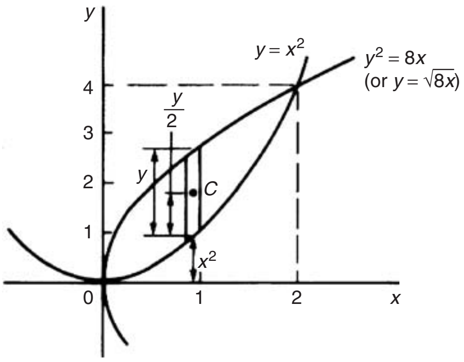


Figure 57.7

The value of y is given by the height of the typical strip shown in Fig. 55.7, i.e. $y = \sqrt{8x} - x^2$. Hence,

$$\overline{x} = \frac{\displaystyle\int_0^2 x\left(\sqrt{8x} - x^2\right)\,dx}{2\frac{2}{3}} = \frac{\displaystyle\int_0^2 \left(\sqrt{8}\,x^{3/2} - x^3\right)}{2\frac{2}{3}}$$

$$= \frac{\left[\sqrt{8}\dfrac{x^{5/2}}{\frac{5}{2}} - \dfrac{x^4}{4}\right]_0^2}{2\frac{2}{3}} = \frac{\left(\sqrt{8}\dfrac{\sqrt{2^5}}{\frac{5}{2}} - 4\right)}{2\frac{2}{3}}$$

$$= \frac{2\frac{2}{5}}{2\frac{2}{3}} = \mathbf{0.9}$$

Care needs to be taken when finding $\overline{y}$ in such examples as this. From Fig. 57.7, $y = \sqrt{8x} - x^2$ and $\dfrac{y}{2} = \dfrac{1}{2}(\sqrt{8x} - x^2)$. The perpendicular distance from centroid C of the strip to Ox is $\dfrac{1}{2}(\sqrt{8x} - x^2) + x^2$.

Taking moments about Ox gives:

(total area) $(\overline{y}) = \sum_{x=0}^{x=2}$ (area of strip) (perpendicular distance of centroid of strip to Ox)

Hence (area) $(\overline{y})$

$$= \int \left[\sqrt{8x} - x^2\right]\left[\frac{1}{2}(\sqrt{8x} - x^2) + x^2\right]dx$$

i.e. $$\left(2\frac{2}{3}\right)(\overline{y}) = \int_0^2 \left[\sqrt{8x} - x^2\right]\left(\frac{\sqrt{8x}}{2} + \frac{x^2}{2}\right)dx$$

$$= \int_0^2 \left(\frac{8x}{2} - \frac{x^4}{2}\right)dx = \left[\frac{8x^2}{4} - \frac{x^5}{10}\right]_0^2$$

$$= \left(8 - 3\frac{1}{5}\right) - (0) = 4\frac{4}{5}$$

Hence $\qquad \overline{y} = \dfrac{4\frac{4}{5}}{2\frac{2}{3}} = \mathbf{1.8}$

Thus the position of the centroid of the shaded area in Fig. 55.7 is at (0.9, 1.8)

Now try the following exercise

Exercise 191 Further problems on centroids of simple shapes

1. Determine the position of the centroid of a sheet of metal formed by the curve $y = 4x - x^2$ which lies above the x-axis. [(2, 1.6)]

2. Find the coordinates of the centroid of the area that lies between the curve $\dfrac{y}{x} = x - 2$ and the x-axis. [(1, −0.4)]

3. Determine the coordinates of the centroid of the area formed between the curve $y = 9 - x^2$ and the x-axis. [(0, 3.6)]

4. Determine the centroid of the area lying between $y = 4x^2$, the y-axis and the ordinates $y = 0$ and $y = 4$. [(0.375, 2.40]

5. Find the position of the centroid of the area enclosed by the curve $y = \sqrt{5x}$, the x-axis and the ordinate $x = 5$. [(3.0, 1.875)]

6. Sketch the curve $y^2 = 9x$ between the limits $x = 0$ and $x = 4$. Determine the position of the centroid of this area. [(2.4, 0)]

Problem 7. Determine the second moment of area and radius of gyration of a rectangular lamina of length 40 mm and width 15 mm about an axis through one corner, perpendicular to the plane of the lamina

The lamina is shown in Fig. 58.11.

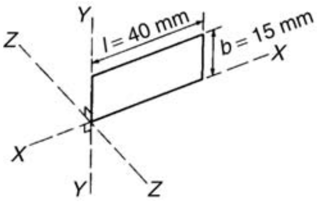


Figure 58.11

From the perpendicular axis theorem:

$$I_{ZZ} = I_{XX} + I_{YY}$$

$$I_{XX} = \frac{lb^3}{3} = \frac{(40)(15)^3}{3} = 45\,000 \text{ mm}^4$$

and $$I_{YY} = \frac{bl^3}{3} = \frac{(15)(40)^3}{3} = 320\,000 \text{ mm}^4$$

Hence $$\boldsymbol{I_{ZZ}} = 45\,000 + 320\,000$$

$$= \mathbf{365\,000\ mm^4} \text{ or } \mathbf{36.5\ cm^4}$$

Radius of gyration,

$$k_{ZZ} = \sqrt{\frac{I_{ZZ}}{\text{area}}} = \sqrt{\frac{365\,000}{(40)(15)}}$$

$$= \mathbf{24.7\ mm} \text{ or } \mathbf{2.47\ cm}$$

Now try the following exercise

Exercise 193 Further problems on second moments of area of regular sections

1. Determine the second moment of area and radius of gyration for the rectangle shown in Fig. 58.12 about (a) axis AA (b) axis BB, and (c) axis CC.

[(a) 72 cm^4, 1.73 cm
(b) 128 cm^4, 2.31 cm
(c) 512 cm^4, 4.62 cm]

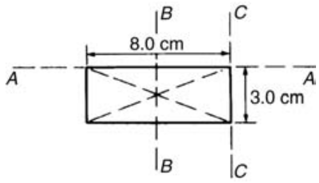


Figure 58.12

2. Determine the second moment of area and radius of gyration for the triangle shown in Fig. 58.13 about (a) axis DD (b) axis EE, and (c) an axis through the centroid of the triangle parallel to axis DD.

[(a) 729 mm^4, 3.67 mm
(b) 2187 mm^4, 6.36 mm
(c) 243 mm^4, 2.12 mm]

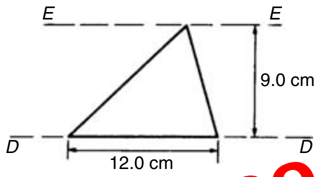


Figure 58.13

3. For the circle shown in Fig. 58.14, find the second moment of area and radius of gyration about (a) axis FF, and (b) axis HH.

[(a) 201 cm^4, 2.0 cm
(b) 1005 cm^4, 4.47 cm]

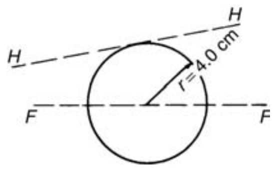


Figure 58.14

4. For the semicircle shown in Fig. 58.15, find the second moment of area and radius of gyration about axis JJ.

[3927 mm^4, 5.0 mm]

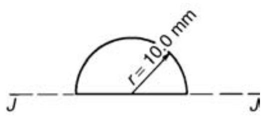


Figure 58.15

5. For each of the areas shown in Fig. 58.16 determine the second moment of area and radius of gyration about axis LL, by using the parallel axis theorem.

$$\left[\begin{array}{l}\text{(a) } 335\ \text{cm}^4,\ 4.73\ \text{cm}\\ \text{(b) } 22\,030\ \text{cm}^4,\ 14.3\ \text{cm}\\ \text{(c) } 628\ \text{cm}^4,\ 7.07\ \text{cm}\end{array}\right]$$

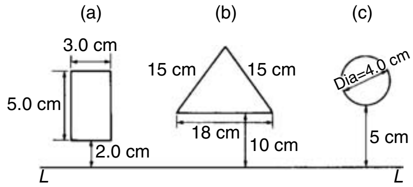


Figure 58.16

6. Calculate the radius of gyration of a rectangular door 2.0 m high by 1.5 m wide about a vertical axis through its hinge.

[0.866 m]

7. A circular door of a boiler is hinged so that it turns about a tangent. If its diameter is 1.0 m, determine its second moment of area and radius of gyration about the hinge.

[0.0245 m^4, 0.559 m]

8. A circular cover, centre 0, has a radius of 12.0 cm. A hole of radius 4.0 cm and centre X, where $OX = 6.0$ cm, is cut in the cover. Determine the second moment of area and the radius of gyration of the remainder about a diameter through 0 perpendicular to OX.

[14 280 cm^4, 5.96 cm]

58.7 Worked problems on second moments of areas of composite areas

Problem 8. Determine correct to 3 significant figures, the second moment of area about axis XX for the composite area shown in Fig. 58.17

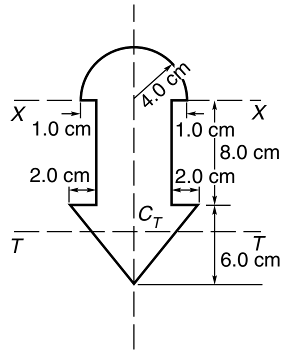


Figure 58.17

For the semicircle, $I_{XX} = \dfrac{\pi r^4}{8} = \dfrac{\pi(4.0)^4}{8}$

$= 100.5\ \text{cm}^4$

For the rectangle, $I_{XX} = \dfrac{bl^3}{3} = \dfrac{(6.0)(8.0)^3}{3}$

$= 1024\ \text{cm}^4$

For the triangle, about axis TT through centroid C_T,

$$I_{TT} = \frac{bh^3}{36} = \frac{(10)(6.0)^3}{36} = 60\ \text{cm}^4$$

By the parallel axis theorem, the second moment of area of the triangle about axis XX

$= 60 + \left[\frac{1}{2}(10)(6.0)\right]\left[8.0 + \frac{1}{3}(6.0)\right]^2 = 3060\ \text{cm}^4$.

Total second moment of area about XX

$= 100.5 + 1024 + 3060 = 4184.5 =$ **4180 cm^4**, correct to 3 significant figures

Problem 9. Determine the second moment of area and the radius of gyration about axis XX for the I-section shown in Fig. 58.18

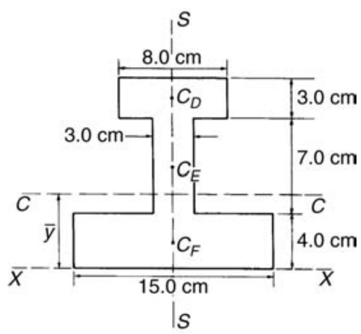


Figure 58.18

Part 10 Further Number and Algebra

59

Boolean algebra and logic circuits

59.1 Boolean algebra and switching circuits

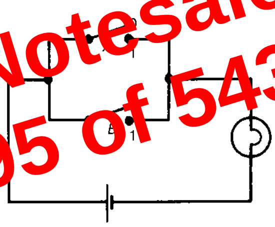
A **two-state device** is one whose basic elements can only have one of two conditions. Thus, two-way switches, which can either be on or off, and the binary numbering system, having the digits 0 and 1 only, are two-state devices. In Boolean algebra, if A represents one state, then $\overline{A}$, called 'not-A', represents the second state.

The or-function

In Boolean algebra, the **or**-function for two elements A and B is written as $A + B$, and is defined as 'A, or B, or both A and B'. The equivalent electrical circuit for a two-input **or**-function is given by two switches connected in parallel. With reference to Fig. 59.1(a), the lamp will be on when A is on, when B is on, or when both A and B are on. In the table shown in Fig. 59.1(b), all the possible switch combinations are shown in columns 1 and 2, in which a 0 represents a switch being off and a 1 represents the switch being on, these columns being called the inputs. Column 3 is called the output and a 0 represents the lamp being off and a 1 represents the lamp being on. Such a table is called a **truth table**.

The and-function

In Boolean algebra, the **and**-function for two elements A and B is written as $A \cdot B$ and is defined as 'both A and B'. The equivalent electrical circuit for a two-input **and**-function is given by two switches connected in series. With reference to Fig. 59.2(a) the lamp will be on only when both A and B are on. The truth table for a two-input **and**-function is shown in Fig. 59.2(b).

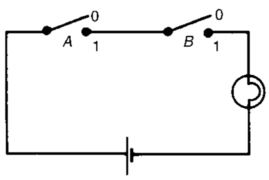


(a) Switching circuit for or - function

1	2	3
Input (switches)		Output (lamp)
A	B	$Z = A + B$
0	0	0
0	1	1
1	0	1
1	1	1

(b) Truth table for or - function

Figure 59.1



(a) Switching circuit for and - function

Input (switches)		Output (lamp)
A	B	$Z = A . B$
0	0	0
0	1	0
1	0	0
1	1	1

(b) Truth table for and - function

Figure 59.2

Hence the required switching circuit is as shown in Fig. 59.7. The corresponding truth table is shown in Table 59.4.

Table 59.4

1	2	3	4	5	6	7	8	9
A	B	C	$\overline{C}$	$A\cdot\overline{C}$	$\overline{A}$	$\overline{A}\cdot B$	$\overline{A}\cdot B\cdot\overline{C}$	$Z=A\cdot\overline{C}+\overline{A}\cdot B+\overline{A}\cdot B\cdot\overline{C}$
0	0	0	1	0	1	0	0	0
0	0	1	0	0	1	0	0	0
0	1	0	1	0	1	1	1	1
0	1	1	0	0	1	1	0	1
1	0	0	1	1	0	0	0	1
1	0	1	0	0	0	0	0	0
1	1	0	1	1	0	0	0	1
1	1	1	0	0	0	0	0	0

Column 4 is $\overline{C}$, i.e. the opposite to column 3

Column 5 is $A\cdot\overline{C}$, obtained by applying the **and**-function to columns 1 and 4

Column 6 is $\overline{A}$, the opposite to column 1

Column 7 is $\overline{A}\cdot B$, obtained by applying the **and**-function to columns 2 and 6

Column 8 is $\overline{A}\cdot B\cdot\overline{C}$, obtained by applying the **and**-function to columns 4 and 7

Column 9 is the output, obtained by applying the **or**-function to columns 5, 7 and 8

Problem 4. Derive the Boolean expression and construct the switching circuit for the truth table given in Table 59.5.

Table 59.5

	A	B	C	Z
1	0	0	0	1
2	0	0	1	0
3	0	1	0	1
4	0	1	1	1
5	1	0	0	0
6	1	0	1	1
7	1	1	0	0
8	1	1	1	0

Examination of the truth table shown in Table 59.5 shows that there is a 1 output in the Z-column in rows 1, 3, 4 and 6. Thus, the Boolean expression and switching circuit should be such that a 1 output is obtained for row 1 **or** row 3 **or** row 4 **or** row 6. In row 1, A is 0 **and** B is 0 **and** C is 0 and this corresponds to the Boolean expression $\overline{A}\cdot\overline{B}\cdot\overline{C}$. In row 3, A is 0 **and** B is 1 **and** C is 0, i.e. the Boolean expression in $\overline{A}\cdot B\cdot\overline{C}$. Similarly in rows 4 and 6, the Boolean expressions are $\overline{A}\cdot B\cdot C$ and $A\cdot\overline{B}\cdot C$ respectively. Hence the Boolean expression is:

$$\boldsymbol{Z=\overline{A}\cdot\overline{B}\cdot\overline{C}+\overline{A}\cdot B\cdot\overline{C}}$$
$$\boldsymbol{+\overline{A}\cdot B\cdot C+A\cdot\overline{B}\cdot C}$$

The corresponding switching circuit is shown in Fig. 59.8. The four terms are joined by **or**-functions, (+), and are represented by four parallel circuits. Each term has three elements joined by an **and**-function, and is represented by three elements connected in series.

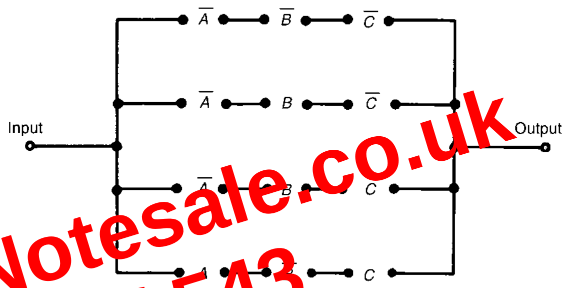


Figure 59.8

Now try the following exercise

Exercise 195 Further problems on Boolean algebra and switching circuits

In Problems 1 to 4, determine the Boolean expressions and construct truth tables for the switching circuits given.

1. The circuit shown in Fig. 59.9

$$\left[\begin{array}{l} C\cdot(A\cdot B+\overline{A}\cdot B); \\ \text{see Table 59.6, col. 4} \end{array}\right]$$

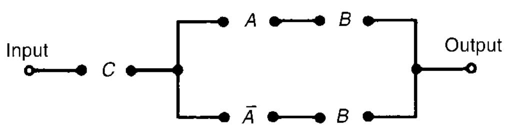


Figure 59.9

9. $\overline{F}\cdot\overline{G}\cdot\overline{H}+\overline{F}\cdot\overline{G}\cdot H+F\cdot\overline{G}\cdot\overline{H}+F\cdot\overline{G}\cdot H$ [$\overline{G}$]

10. $F\cdot\overline{G}\cdot H+F\cdot G\cdot H+F\cdot G\cdot\overline{H}+\overline{F}\cdot G\cdot\overline{H}$ [$F\cdot H+G\cdot\overline{H}$]

11. $R\cdot(P\cdot Q+P\cdot\overline{Q})+\overline{R}\cdot(\overline{P}\cdot\overline{Q}+\overline{P}\cdot Q)$ [$P\cdot R+\overline{P}\cdot\overline{R}$]

12. $\overline{R}\cdot(\overline{P}\cdot\overline{Q}+P\cdot Q+P\cdot\overline{Q})+P\cdot(Q\cdot R+\overline{Q}\cdot R)$ [$P+\overline{Q}\cdot\overline{R}$]

59.4 De Morgan's laws

De Morgan's laws may be used to simplify **not**-functions having two or more elements. The laws state that:

$$\boxed{\overline{A+B}=\overline{A}\cdot\overline{B}} \quad \textbf{and} \quad \boxed{\overline{\boldsymbol{A\cdot B}}=\overline{\boldsymbol{A}}+\overline{\boldsymbol{B}}}$$

and may be verified by using a truth table (see Problem 11). The application of de Morgan's laws in simplifying Boolean expressions is shown in Problems 12 and 13.

Problem 11. Verify that $\overline{A+B}=\overline{A}\cdot\overline{B}$

A Boolean expression may be verified by using a truth table. In Table 59.9, columns 1 and 2 give all the possible arrangements of the inputs A and B. Column 3 is the **or**-function applied to columns 1 and 2 and column 4 is the **not**-function applied to column 3. Columns 5 and 6 are the **not**-function applied to columns 1 and 2 respectively and column 7 is the **and**-function applied to columns 5 and 6.

Table 59.9

1	2	3	4	5	6	7
A	B	$A+B$	$\overline{A+B}$	$\overline{A}$	$\overline{B}$	$\overline{A}\cdot\overline{B}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Since columns 4 and 7 have the same pattern of 0's and 1's this verifies that $\overline{A+B}=\overline{A}\cdot\overline{B}$.

Problem 12. Simplify the Boolean expression $(\overline{\overline{A}\cdot B})+(\overline{\overline{A}+B})$ by using de Morgan's laws and the rules of Boolean algebra.

Applying de Morgan's law to the first term gives:

$$\overline{\overline{A}\cdot B}=\overline{\overline{A}}+\overline{B}=A+\overline{B} \quad \text{since } \overline{\overline{A}}=A$$

Applying de Morgan's law to the second term gives:

$$\overline{\overline{A}+B}=\overline{\overline{A}}\cdot\overline{B}=A\cdot\overline{B}$$

Thus, $(\overline{\overline{A}\cdot B})+(\overline{\overline{A}+B})=(A+\overline{B})+A\cdot\overline{B}$

Removing the bracket and reordering gives:
$A+A\cdot\overline{B}+\overline{B}$
But, by rule 15, Table 59.8, $A+A\cdot B=A$. It follows that: $A+A\cdot\overline{B}=A$

Thus: $\boldsymbol{(\overline{\overline{A}\cdot B})+(\overline{\overline{A}+B})=A+\overline{B}}$

Problem 13. Simplify the Boolean expression $(\overline{A\cdot\overline{B}+C})\cdot(\overline{A}+\overline{B\cdot\overline{C}})$ by using de Morgan's laws and the rules of Boolean algebra.

Applying de Morgan's laws to the first term gives:

$$\overline{A\cdot\overline{B}+C}=\overline{A\cdot\overline{B}}\cdot\overline{C}=(\overline{A}+\overline{\overline{B}})\cdot\overline{C}$$
$$=(\overline{A}+B)\cdot\overline{C}=\overline{A}\cdot\overline{C}+B\cdot\overline{C}$$

Applying de Morgan's law to the second term gives:

$$\overline{A}+\overline{B\cdot\overline{C}}=\overline{A}+(\overline{B}+\overline{\overline{C}})=\overline{A}+(\overline{B}+C)$$

Thus $(\overline{A\cdot\overline{B}+C})\cdot(\overline{A}+\overline{B\cdot\overline{C}})$

$$=(\overline{A}\cdot\overline{C}+B\cdot\overline{C})\cdot(\overline{A}+\overline{B}+C)$$
$$=\overline{A}\cdot\overline{A}\cdot\overline{C}+\overline{A}\cdot\overline{B}\cdot\overline{C}+\overline{A}\cdot\overline{C}\cdot C$$
$$+\overline{A}\cdot B\cdot\overline{C}+B\cdot\overline{B}\cdot\overline{C}+B\cdot\overline{C}\cdot C$$

But from Table 59.8, $\overline{A}\cdot\overline{A}=\overline{A}$ and $\overline{C}\cdot C=B\cdot\overline{B}=0$

Hence the Boolean expression becomes:

$$\overline{A}\cdot\overline{C}+\overline{A}\cdot\overline{B}\cdot\overline{C}+\overline{A}\cdot B\cdot\overline{C}$$
$$=\overline{A}\cdot\overline{C}(1+\overline{B}+B)$$
$$=\overline{A}\cdot\overline{C}(1+B)$$
$$=\overline{A}\cdot\overline{C}$$

Thus: $\boldsymbol{(\overline{A\cdot\overline{B}+C})\cdot(\overline{A}+\overline{B\cdot\overline{C}})=\overline{A}\cdot\overline{C}}$

the techniques introduced in Sections 59.3 to 59.5, resulting in the cost of the circuit being reduced. Any of the techniques can be used, and in this case, the rules of Boolean algebra (see Table 59.8) are used.

$$\begin{aligned} Z &= A\cdot\overline{B}\cdot C + A\cdot B\cdot\overline{C} + A\cdot B\cdot C \\ &= A\cdot[\overline{B}\cdot C + B\cdot\overline{C} + B\cdot C] \\ &= A\cdot[\overline{B}\cdot C + B(\overline{C}+C)] = A\cdot[\overline{B}\cdot C + B] \\ &= A\cdot[B + \overline{B}\cdot C] = \boldsymbol{A\cdot[B+C]} \end{aligned}$$

The logic circuit to give this simplified expression is shown in Fig. 59.26.

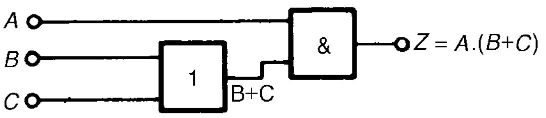


Figure 59.26

Problem 23. Simplify the expression:

$$Z = \overline{P}\cdot\overline{Q}\cdot\overline{R}\cdot\overline{S} + \overline{P}\cdot\overline{Q}\cdot\overline{R}\cdot S + \overline{P}\cdot Q\cdot\overline{R}\cdot\overline{S} + \overline{P}\cdot Q\cdot\overline{R}\cdot S + P\cdot\overline{Q}\cdot\overline{R}\cdot\overline{S}$$

and devise a logic circuit to give this output.

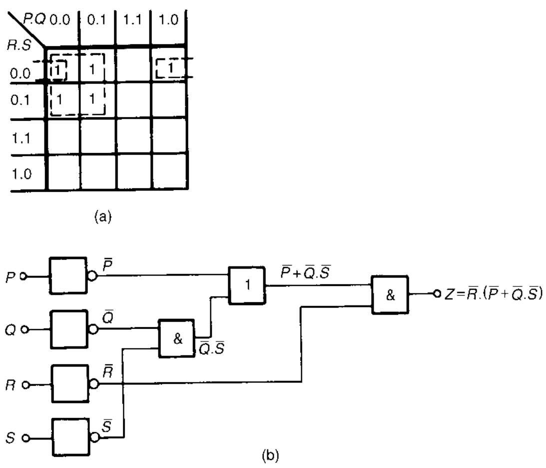


Figure 59.27

The given expression is simplified using the Karnaugh map techniques introduced in Section 59.5. Two couples are formed as shown in Fig. 59.27(a) and the simplified expression becomes:

$$Z = \overline{Q}\cdot\overline{R}\cdot\overline{S} + \overline{P}\cdot\overline{R}$$

i.e $\quad \boldsymbol{Z = \overline{R}\cdot(\overline{P} + \overline{Q}\cdot\overline{S})}$

The logic circuit to produce this expression is shown in Fig. 59.27(b).

Now try the following exercise

Exercise 199 Further problems on logic circuits

In Problems 1 to 4, devise logic systems to meet the requirements of the Boolean expressions given.

1. $Z = \overline{A} + B\cdot C$ [See Fig. 59.28(a)]

2. $Z = A\cdot\overline{B} + B\cdot\overline{C}$ [See Fig. 59.28(b)]

3. $Z = A\cdot B\cdot\overline{C} + \overline{A}\cdot\overline{B}\cdot C$ [See Fig. 59.28(c)]

4. $Z = (\overline{A} + B)\cdot(\overline{C} + D)$ [See Fig. 59.28(d)]

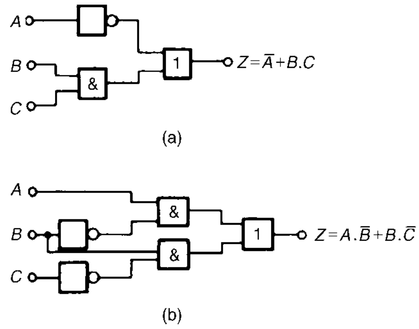


Figure 59.28

In Problems 5 to 7, simplify the expression given in the truth table and devise a logic circuit to meet the requirements stated.

Following the above procedure:

(i) $3x - 4y - 12 = 0$

$7x + 5y - 6.5 = 0$

(ii) $$\frac{x}{\begin{vmatrix} -4 & -12 \\ 5 & -6.5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & -12 \\ 7 & -6.5 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & -4 \\ 7 & 5 \end{vmatrix}}$$

i.e. $$\frac{x}{(-4)(-6.5) - (-12)(5)} = \frac{-y}{(3)(-6.5) - (-12)(7)} = \frac{1}{(3)(5) - (-4)(7)}$$

i.e. $$\frac{x}{26 + 60} = \frac{-y}{-19.5 + 84} = \frac{1}{15 + 28}$$

i.e. $$\frac{x}{86} = \frac{-y}{64.5} = \frac{1}{43}$$

Since $\frac{x}{86} = \frac{1}{43}$ then $\boldsymbol{x} = \frac{86}{43} = \mathbf{2}$

and since

$\frac{-y}{64.5} = \frac{1}{43}$ then $\boldsymbol{y} = -\frac{64.5}{43} = \mathbf{-1.5}$

Problem 4. The velocity of a car, accelerating at uniform acceleration a between two points, is given by $v = u + at$, where u is its velocity when passing the first point and t is the time taken to pass between the two points. If $v = 21$ m/s when $t = 3.5$ s and $v = 33$ m/s when $t = 6.1$ s, use determinants to find the values of u and a, each correct to 4 significant figures.

Substituting the given values in $v = u + at$ gives:

$$21 = u + 3.5a \qquad (1)$$

$$33 = u + 6.1a \qquad (2)$$

(i) The equations are written in the form $a_1x + b_1y + c_1 = 0$,

i.e. $u + 3.5a - 21 = 0$

and $u + 6.1a - 33 = 0$

(ii) The solution is given by

$$\frac{u}{D_u} = \frac{-a}{D_a} = \frac{1}{D},$$

where D_u is the determinant of coefficients left when the u column is covered up,

i.e. $$D_u = \begin{vmatrix} 3.5 & -21 \\ 6.1 & -33 \end{vmatrix}$$
$$= (3.5)(-33) - (-21)(6.1)$$
$$= 12.6$$

Similarly, $$D_a = \begin{vmatrix} 1 & -21 \\ 1 & -33 \end{vmatrix}$$
$$= (1)(-33) - (-21)(1)$$
$$= -12$$

and $$D = \begin{vmatrix} 1 & 3.5 \\ 1 & 6.1 \end{vmatrix}$$
$$= (1)(6.1) - (3.5)(1) = 2.6$$

Thus $$\frac{u}{12.6} = \frac{-a}{-12} = \frac{1}{2.6}$$

i.e. $$\boldsymbol{u} = \frac{12.6}{2.6} = \mathbf{4.846\ m/s}$$

and $$\boldsymbol{a} = \frac{12}{2.6} = \mathbf{4.615\ m/s^2},$$

each correct to 4 significant figures

Problem 5. Applying Kirchhoff's laws to an electric circuit results in the following equations:

$$(9 + j12)I_1 - (6 + j8)I_2 = 5$$

$$-(6 + j8)I_1 + (8 + j3)I_2 = (2 + j4)$$

Solve the equations for I_1 and I_2.

Following the procedure:

(i) $(9 + j12)I_1 - (6 + j8)I_2 - 5 = 0$

$-(6 + j8)I_1 + (8 + j3)I_2 - (2 + j4) = 0$

(ii) $$\frac{I_1}{\begin{vmatrix} -(6 + j8) & -5 \\ (8 + j3) & -(2 + j4) \end{vmatrix}} = \frac{-I_2}{\begin{vmatrix} (9 + j12) & -5 \\ -(6 + j8) & -(2 + j4) \end{vmatrix}} = \frac{1}{\begin{vmatrix} (9 + j12) & -(6 + j8) \\ -(6 + j8) & (8 + j3) \end{vmatrix}}$$