Part (c) What do you observe?

Answer

2. The Convolution Theorem

Let $f(t)$ and $g(t)$ be causal functions with Laplace transforms $F(s)$ and $G(s)$ respectively, i.e. $\mathcal{L}\{f(t)\} = F(s)$ adn $\mathcal{L}\{g(t)\} = G(s)$. Then it can be shown that

Key Point

$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$ or equivalently $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$

Example Use the Convolution Theorem to find the inverse transform of $\dfrac{6}{s(s^2+9)}$.

Solution

In this case we can, of course, find the inverse transform by using partial fractions and then using the table of transforms. That is

$$\frac{6}{s(s^2+9)} = \frac{(2/3)}{s} - \frac{(2/3)s}{s^2+9}$$

and so

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{6}{s(s^2+9)}\right\} &= \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \tfrac{2}{3}\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} \\ &= \tfrac{2}{3}u(t) - \tfrac{2}{3}\cos 3t.u(t) \end{aligned}$$

However, we can alternatively use the Convolution Theorem. Let us choose

$$F(s) = \frac{2}{s} \qquad \text{and} \qquad G(s) = \frac{3}{s^2+9}$$

then

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = 2u(t) \quad \text{and} \quad g(t) = \mathcal{L}^{-1}\{G(s)\} = \sin 3t.u(t)$$

So

$$\begin{aligned} \mathcal{L}^{-1}\{F(s)G(s)\} &= (f * g)(t) \qquad \text{by the Convolution Theorem} \\ &= \int_0^t 2u(t-x)\sin 3x.u(x)\,\mathrm{d}x \end{aligned}$$

From the example above $(f * g)(t) = t - \sin t$ and so

$$\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{t - \sin t\} = \frac{1}{s^2} - \frac{1}{s^2 + 1}$$

Back to the theory