Contents

1 Board work

1.1 Tuesday, 29 September 2015

1. For a cluster point every deleted neighbourhood actually has infinitely many points of A.

   `Example.` Let 0 be a cluster point of A. Then there must be a point $a \in A$ such that $a \in D_1(0)$. Since $D_{|a|}(0)$ is also a deleted neighbourhood of 0, there must be a point $b \in A$ such that $b \in D_{|a|}(0)$. Next there must be a point $c \in A$ with $c \in D_{|b|}(0)$. And so on.

2. `Sequence`. Think of a sequence as a collection of a first term, a second term,..., an n^{th} term,..., ad infinitum.

3. `Tail`. A tail of a sequence is new sequence obtained by removing finitely many initial terms of the given sequence. Thus $x_N, x_{N+1}, x_{N+2}, \ldots$ is a tail of the sequence $x_1, x_2, x_3, \ldots$.

4. Finite vs Infinite. Inside vs Outside.

   x_n converges to x if and only if for every $\epsilon > 0$ the number of terms of (x_n) outside $B_\epsilon(x)$ is finite. (In which case the number of terms of (x_n) inside $B_\epsilon(x)$ will automatically be infinity).

5. Note that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) where

(i) (x_n) has finitely many terms outside $B_\epsilon(x)$,
(ii) (x_n) has a tail with zero terms outside $B_\epsilon(x)$, and
(iii) (x_n) has a tail with all terms inside $B_\epsilon(x)$.

It follows that (x_n) converges to a limit l if and only if given any $\epsilon > 0$ there is a tail of (x_n) with all terms of the tail inside $B_\epsilon(l)$.

6. It follows that if for some $\epsilon > 0$ the number of terms of (x_n) outside $B_\epsilon(x)$ is infinity then x_n cannot converge to x.

7. Every convergent sequence is bounded. Assume $x_n \to l$. Then a tail of (x_n) is entirely contained in $B_1(l)$. So every element of the tail is bounded by $|l|+1$. What about the terms of the sequence that precede the tail? Are they bounded also? There are only finitely many terms in the sequence (x_n) which do not belong to the tail. Since the maximum of finitely many finite numbers is also finite, we get a finite bound for the entire sequence.

1.2 Wednesday, 30 September 2015

1. Sandwich theorem. $a_n \leq b_n \leq c_n$ with $a_n \to l$ and $c_n \to l$. Then $b_n \to l$.

   Since $a_n \to l$, there exists n_a such that $l - \epsilon < a_n$ for all $n > n_a$. Since $c_n \to l$, there exists n_c such that $c_n < l + \epsilon$ for all $n > n_c$. Therefore $b_n \in B_\epsilon(l)$ whenever n greater than both n_a and n_b. (In other words, every $B_\epsilon(l)$ contains a tail of (b_n).)

2. If $a_n \to a$ the $ca_n \to ca$ for every $c \in \mathbb{R}$.

   Trivial for $c = 0$.
   For $c \neq 0$ there exists n_0 such that $a_n \in B_{\epsilon/|c|}(a)$ for all $n > n_0$. So $ca_n \in B_\epsilon(a)$ for all $n > n_0$. (In other words, an arbitrary neighbourhood of ca and contains a tail of the sequence (ca_n).)

3. $a_n \to a$ and $b_n \to b$ implies that $a_n + b_n \to a + b$.

   Given arbitrary $\epsilon > 0$ there exists n_a such that

   $$|a - a_n| < \epsilon/2 \text{ for all } n > n_a.$$