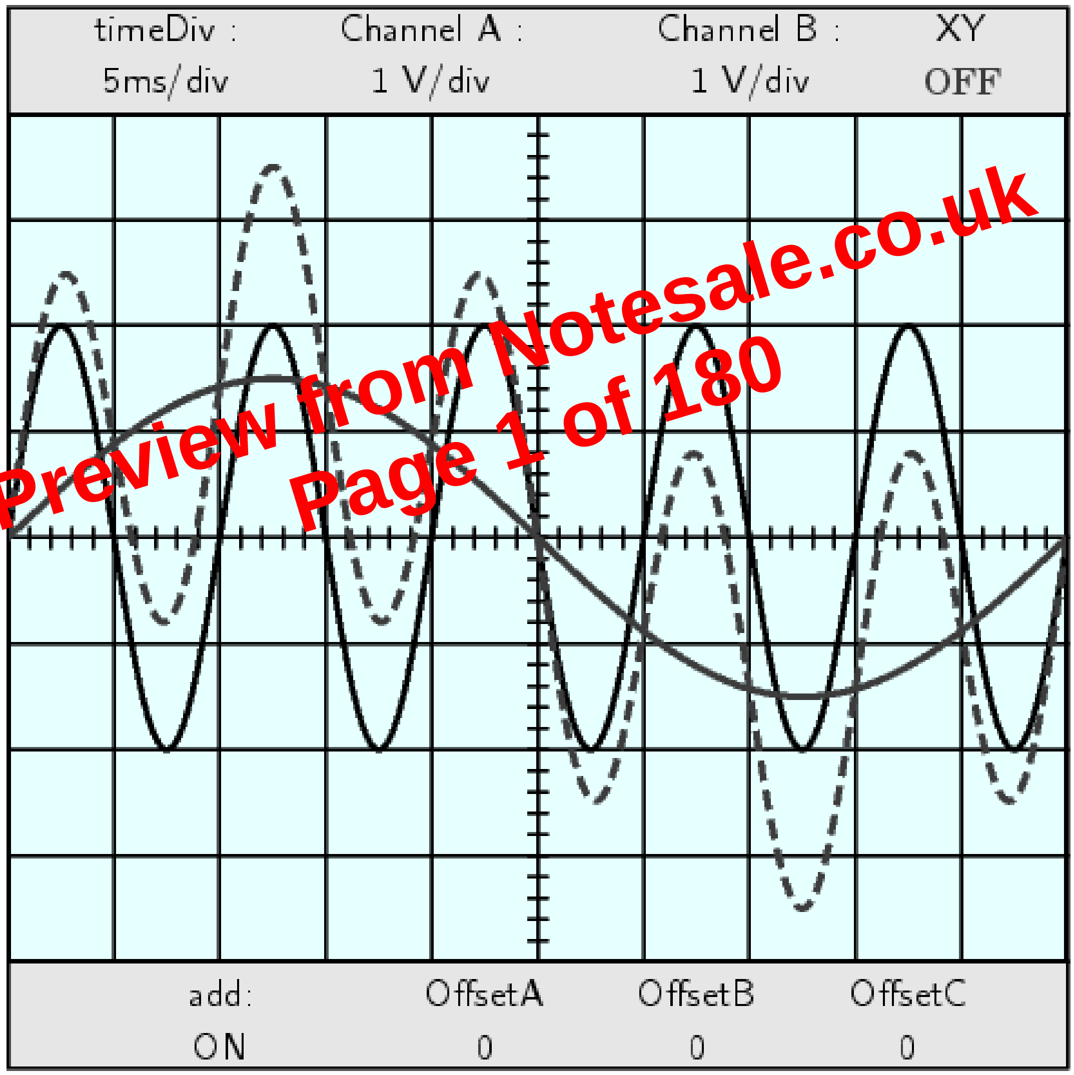
TRIGONOMETRY

MICHAEL CORRAL



1 Right Triangle Trigonometry

Trigonometry is the study of the relations between the sides and angles of triangles. The word "trigonometry" is derived from the Greek words *trigono* (τρίγωνο), meaning "triangle", and *metro* (μετρώ), meaning "measure". Though the ancient Greeks, such as Hipparchus and Ptolemy, used trigonometry in their study of astronomy between roughly 150 B.C. - A.D. 200, its history is much older. For example, the Egyptian scribe Ahmes recorded some rudimentary trigonometric calculations (concerning ratios of sides of pyramids) in the famous Rhind Papyrus sometime around 1650 B.C.[1]

Trigonometry is distinguished from elementary geometry in part by its extensive use of certain functions of angles, known as the *trigonometric functions*. Before discussing those functions, we will review some basic terminology about angles.

1.1 Angles

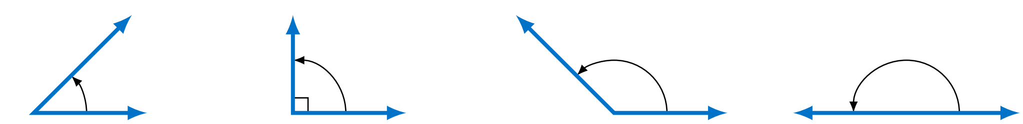
Recall the following definitions from elementary geometry:

(a) An angle is **acute** if it is between 0° and 90°.

(b) An angle is a **right angle** if it equals 90°.

(c) An angle is **obtuse** if it is between 90° and 180°.

(d) An angle is a **straight angle** if it equals 180°.



(a) acute angle **(b)** right angle **(c)** obtuse angle **(d)** straight angle

Figure 1.1.1 Types of angles

In elementary geometry, angles are always considered to be positive and not larger than 360°. For now we will only consider such angles.[2] The following definitions will be used throughout the text:

[1]Ahmes claimed that he copied the papyrus from a work that may date as far back as 3000 B.C.

[2]Later in the text we will discuss negative angles and angles larger than 360°.

In a right triangle, the side opposite the right angle is called the **hypotenuse**, and the other two sides are called its **legs**. For example, in Figure 1.1.4 the right angle is C, the hypotenuse is the line segment $\overline{AB}$, which has length c, and $\overline{BC}$ and $\overline{AC}$ are the legs, with lengths a and b, respectively. The hypotenuse is always the longest side of a right triangle (see Exercise 11).

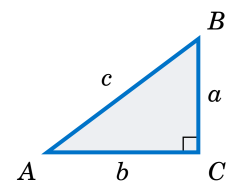


Figure 1.1.4

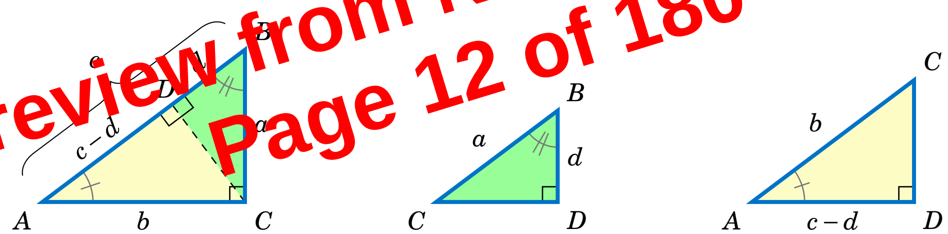
By knowing the lengths of two sides of a right triangle, the length of the third side can be determined by using the **Pythagorean Theorem**:

Theorem 1.1. Pythagorean Theorem: The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.

Thus, if a right triangle has a hypotenuse of length c and legs of lengths a and b, as in Figure 1.1.4, then the Pythagorean Theorem says:

$$\boxed{a^2 + b^2 = c^2} \tag{1.1}$$

Let us prove this. In the right triangle $\triangle ABC$ in Figure 1.1.5(a) below, if we draw a line segment from the vertex C to the point D on the hypotenuse such that $\overline{CD}$ is **perpendicular** to $\overline{AB}$ (that is, $\overline{CD}$ forms a right angle with $\overline{AB}$), then this divides $\triangle ABC$ into two smaller triangles $\triangle CBD$ and $\triangle ACD$, which are both similar to $\triangle ABC$.



(a) $\triangle ABC$ **(b)** $\triangle CBD$ **(c)** $\triangle ACD$

Figure 1.1.5 Similar triangles $\triangle ABC$, $\triangle CBD$, $\triangle ACD$

Recall that triangles are **similar** if their corresponding angles are equal, and that similarity implies that corresponding sides are proportional. Thus, since $\triangle ABC$ is similar to $\triangle CBD$, by proportionality of corresponding sides we see that

$$\overline{AB} \text{ is to } \overline{CB} \text{ (hypotenuses) as } \overline{BC} \text{ is to } \overline{BD} \text{ (vertical legs)} \quad \Rightarrow \quad \frac{c}{a} = \frac{a}{d} \quad \Rightarrow \quad cd = a^2 .$$

Since $\triangle ABC$ is similar to $\triangle ACD$, comparing horizontal legs and hypotenuses gives

$$\frac{b}{c-d} = \frac{c}{b} \quad \Rightarrow \quad b^2 = c^2 - cd = c^2 - a^2 \quad \Rightarrow \quad a^2 + b^2 = c^2 . \quad \textbf{QED}$$

Note: The symbols $\perp$ and $\sim$ denote perpendicularity and similarity, respectively. For example, in the above proof we had $\overline{CD} \perp \overline{AB}$ and $\triangle ABC \sim \triangle CBD \sim \triangle ACD$.

We now know the lengths of all sides of the triangle $\triangle ABC$, so we have:

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{5}}{3} \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{\sqrt{5}}$$

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{3}{2} \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{3}{\sqrt{5}} \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{\sqrt{5}}{2}$$

You may have noticed the connections between the sine and cosine, secant and cosecant, and tangent and cotangent of the complementary angles in Examples 1.5 and 1.7. Generalizing those examples gives us the following theorem:

Theorem 1.2. Cofunction Theorem: If A and B are the complementary acute angles in a right triangle $\triangle ABC$, then the following relations hold:

$$\sin A = \cos B \qquad \sec A = \csc B \qquad \tan A = \cot B$$

$$\sin B = \cos A \qquad \sec B = \csc A \qquad \tan B = \cot A$$

We say that the pairs of functions $\{\sin, \cos\}$, $\{\sec, \csc\}$, and $\{\tan, \cot\}$ are **cofunctions**.

So sine and cosine are cofunctions, secant and cosecant are cofunctions, and tangent and cotangent are cofunctions. That is how the functions cosine, cosecant, and cotangent got the "co" in their names. The Cofunction Theorem says that any trigonometric function of an acute angle is equal to its cofunction of the complementary angle.

Example 1.9

Write each of the following numbers as trigonometric functions of an angle less than 45°: **(a)** $\sin 65°$; **(b)** $\cos 78°$; **(c)** $\tan 59°$.

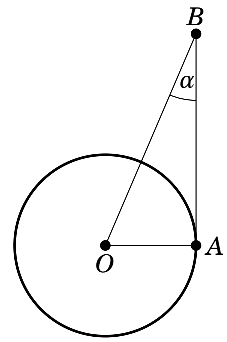
Solution: **(a)** The complement of 65° is $90° - 65° = 25°$ and the cofunction of sin is cos, so by the Cofunction Theorem we know that $\sin 65° = \cos 25°$.
(b) The complement of 78° is $90° - 78° = 12°$ and the cofunction of cos is sin, so $\cos 78° = \sin 12°$.
(c) The complement of 59° is $90° - 59° = 31°$ and the cofunction of tan is cot, so $\tan 59° = \cot 31°$.

(a) 45 – 45 – 90 **(b)** 30 – 60 – 90

Figure 1.2.2 Two general right triangles (any $a > 0$)

The angles 30°, 45°, and 60° arise often in applications. We can use the Pythagorean Theorem to generalize the right triangles in Examples 1.6 and 1.7 and see what *any* 45 – 45 – 90 and 30 – 60 – 90 right triangles look like, as in Figure 1.2.2 above.

Example 1.15



As another application of trigonometry to astronomy, we will find the distance from the earth to the sun. Let O be the center of the earth, let A be a point on the equator, and let B represent an object (e.g. a star) in space, as in the picture on the right. If the earth is positioned in such a way that the angle $\angle OAB = 90°$, then we say that the angle $\alpha = \angle OBA$ is the *equatorial parallax* of the object. The equatorial parallax of the sun has been observed to be approximately $\alpha = 0.00244°$. Use this to estimate the distance from the center of the earth to the sun.

Solution: Let B be the position of the sun. We want to find the length of $\overline{OB}$. We will use the actual radius of the earth, mentioned at the end of Example 1.14, to get $OA = 3956.6$ miles. Since $\angle OAB = 90°$, we have

$$\frac{OA}{OB} = \sin\alpha \quad\Rightarrow\quad OB = \frac{OA}{\sin\alpha} = \frac{3956.6}{\sin 0.00244°} = 92908394\,,$$

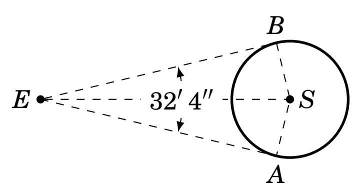
so the distance from the center of the earth to the sun is approximately $\boxed{\text{93 million miles}}$.
Note: The earth's orbit around the sun is an ellipse, so the actual distance to the sun varies.

In the above example we used a very small angle (0.00244°). A degree can be divided into smaller units: a **minute** is one-sixtieth of a degree, and a **second** is one-sixtieth of a minute. The symbol for a minute is $'$ and the symbol for a second is $''$. For example, $4.5° = 4°\,30'$. And $4.505° = 4°\,30'\,18''$:

$$4°\,30'\,18'' = 4 + \frac{30}{60} + \frac{18}{3600} \text{ degrees} = 4.505°$$

In Example 1.15 we used $\alpha = 0.00244° \approx 8.8''$, which we mention only because some angle measurement devices do use minutes and seconds.

Example 1.16



An observer on earth measures an angle of $32'\,4''$ from one visible edge of the sun to the other (opposite) edge, as in the picture on the right. Use this to estimate the radius of the sun.

Solution: Let the point E be the earth and let S be the center of the sun. The observer's lines of sight to the visible edges of the sun are tangent lines to the sun's surface at the points A and B. Thus, $\angle EAS = \angle EBS = 90°$. The radius of the sun equals AS. Clearly $AS = BS$. So since $EB = EA$ (why?), the triangles $\triangle EAS$ and $\triangle EBS$ are similar. Thus, $\angle AES = \angle BES = \frac{1}{2}\angle AEB = \frac{1}{2}(32'\,4'') = 16'\,2'' = (16/60) + (2/3600) = 0.26722°$.

Now, ES is the distance from the *surface* of the earth (where the observer stands) to the center of the sun. In Example 1.15 we found the distance from the *center* of the earth to the sun to be 92,908,394 miles. Since we treated the sun in that example as a point, then we are justified in treating that distance as the distance between the centers of the earth and sun. So $ES = 92908394 -$ radius of earth $= 92908394 - 3956.6 = 92904437.4$ miles. Hence,

$$\sin(\angle AES) = \frac{AS}{ES} \quad\Rightarrow\quad AS = ES \sin 0.26722° = (92904437.4)\sin 0.26722° = \boxed{433{,}293 \text{ miles}}\,.$$

Note: This answer is close to the sun's actual (mean) radius of 432,200 miles.

Example 1.18

A *slider-crank mechanism* is shown in Figure 1.3.2 below. As the piston moves downward the connecting rod rotates the crank in the clockwise direction, as indicated.

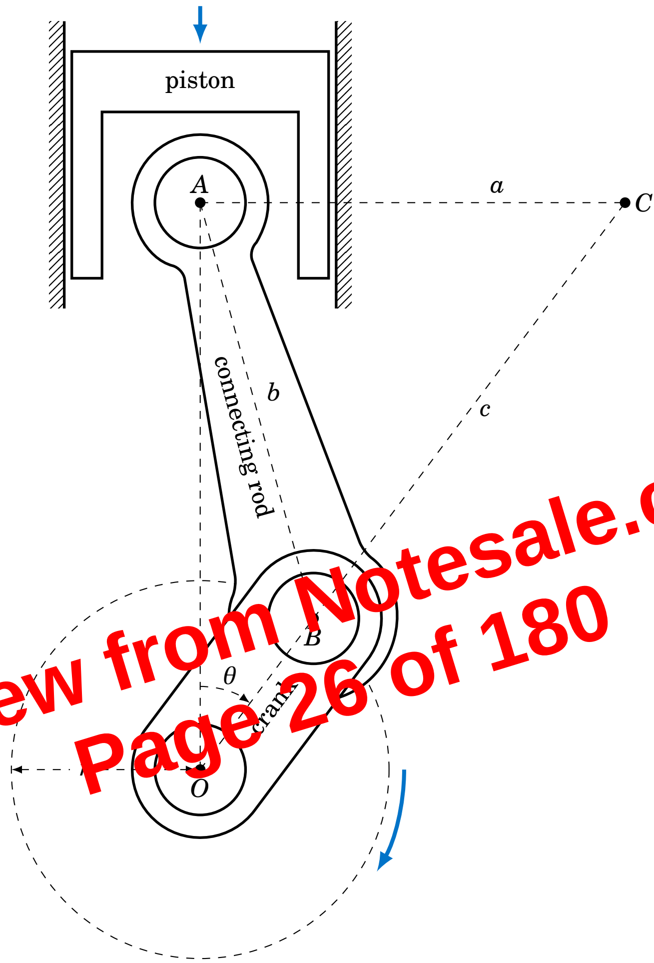
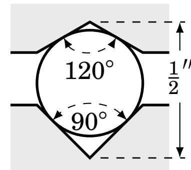


Figure 1.3.2 Slider-crank mechanism

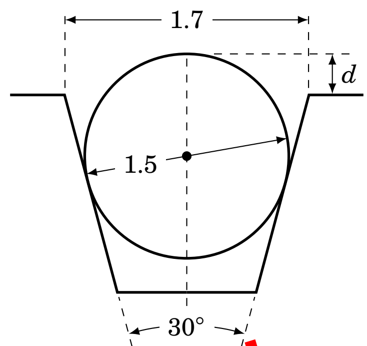
The point A is the center of the connecting rod's *wrist pin* and only moves vertically. The point B is the center of the *crank pin* and moves around a circle of radius r centered at the point O, which is directly below A and does not move. As the crank rotates it makes an angle θ with the line $\overline{OA}$. The *instantaneous center of rotation* of the connecting rod at a given time is the point C where the horizontal line through A intersects the extended line through O and B. From Figure 1.3.2 we see that $\angle OAC = 90°$, and we let $a = AC$, $b = AB$, and $c = BC$. In the exercises you will show that for $0° < \theta < 90°$,

$$c = \frac{\sqrt{b^2 - r^2(\sin\theta)^2}}{\cos\theta} \qquad \text{and} \qquad a = r\sin\theta + \sqrt{b^2 - r^2(\sin\theta)^2}\,\tan\theta\,.$$

8. A ball bearing sits between two metal grooves, with the top groove having an angle of 120° and the bottom groove having an angle of 90°, as in the picture on the right. What must the diameter of the ball bearing be for the distance between the vertexes of the grooves to be half an inch? You may assume that the top vertex is directly above the bottom vertex.



9. The machine tool diagram on the right shows a symmetric *worm thread*, in which a circular roller of diameter 1.5 inches sits. Find the amount d that the top of the roller rises above the top of the thread, given the information in the diagram. (*Hint: Extend the slanted sides of the thread until they meet at a point.*)



10. Repeat Exercise 9 using 1.8 inches as the distance across the top of the worm thread.

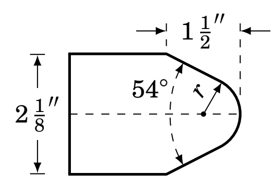
11. In Exercise 9, what would the distance across the top of the worm thread have to be to make d equal to 0 inches?

12. For $0° < \theta < 90°$ in the slider-crank mechanism in Example 1.18, show that

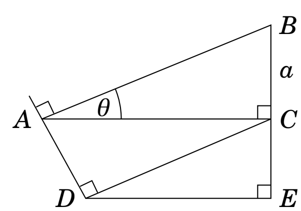
$$c = \frac{\sqrt{b^2 - r^2(\sin\theta)^2}}{\cos\theta} \qquad \text{and} \qquad a = r\sin\theta + \sqrt{b^2 - r^2(\sin\theta)^2}\,\tan\theta\,.$$

(*Hint: In Figure 1.3.2 draw line segments from B perpendicular to $\overline{OA}$ and $\overline{AC}$.*)

13. The machine tool diagram on the right shows a symmetric *die punch*. In this view, the rounded tip is part of a circle of radius r, and the slanted sides are tangent to that circle and form an angle of 54°. The top and bottom sides of the die punch are horizontal. Use the information in the diagram to find the radius r.



14. In the figure on the right, $\angle BAC = \theta$ and $BC = a$. Use this to find AB, AC, AD, DC, CE, and DE in terms of θ and a.
(*Hint: What is the angle $\angle ACD$?*)



For Exercises 15-23, solve the right triangle in Figure 1.3.4 using the given information.

15. $a = 5,\ b = 12$

16. $c = 6,\ B = 35°$

17. $b = 2,\ A = 8°$

18. $a = 2,\ c = 7$

19. $a = 3,\ A = 26°$

20. $b = 1,\ c = 2$

21. $b = 3,\ B = 26°$

22. $a = 2,\ B = 8°$

23. $c = 2,\ A = 45°$

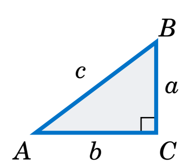
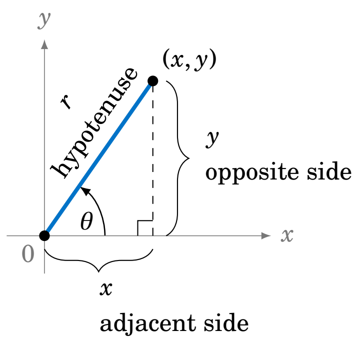
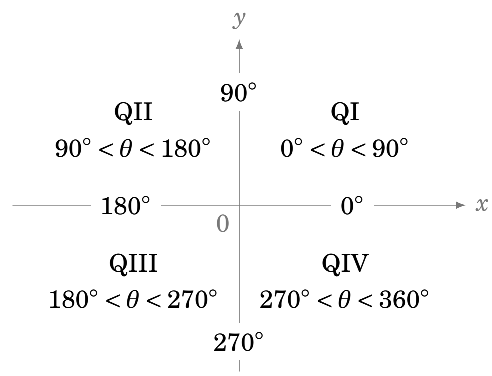


Figure 1.3.4

Notice that in the case of an acute angle these definitions are equivalent to our earlier definitions in terms of right triangles: draw a right triangle with angle θ such that $x =$ adjacent side, $y =$ opposite side, and $r =$ hypotenuse. For example, this would give us $\sin\theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\cos\theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$, just as before (see Figure 1.4.4(a)).



(a) Acute angle θ



(b) Angles by quadrant

Figure 1.4.4

In Figure 1.4.4(b) we see in which quadrants or on which axes the terminal side of an angle $0° \le \theta < 360°$ may fall. From Figure 1.4.3(a) and formulas (1.2) and (1.3), we see that we can get negative values for a trigonometric function. For example, $\sin\theta < 0$ when $y < 0$. Figure 1.4.5 summarizes the signs (positive or negative) for the trigonometric functions based on the angle's quadrant.

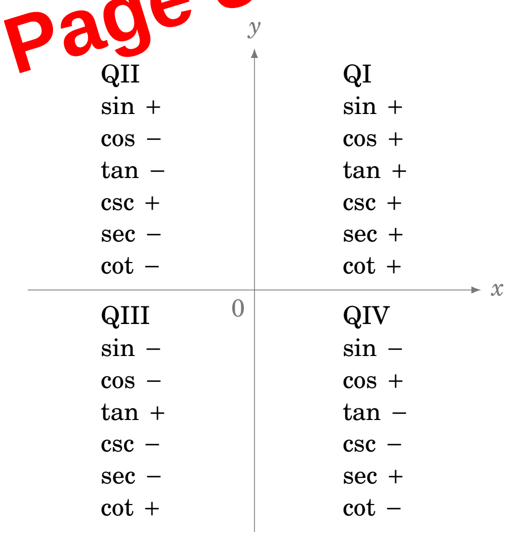


Figure 1.4.5 Signs of the trigonometric functions by quadrant

Notice that reflection around the y-axis is equivalent to reflection around the x-axis ($\theta \mapsto -\theta$) followed by a rotation of 180° ($-\theta \mapsto -\theta + 180° = 180° - \theta$), as in Figure 1.5.7.

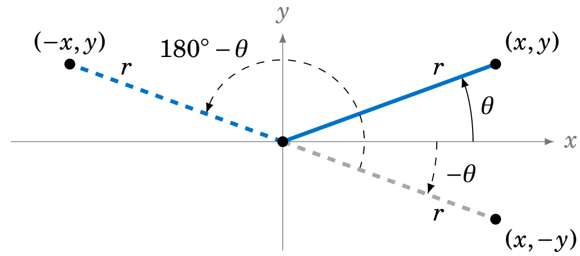


Figure 1.5.7 Reflection of θ around the y-axis $= 180° - \theta$

It may seem that these geometrical operations and formulas are not necessary for evaluating the trigonometric functions, since we could just use a calculator. However, there are two reasons for why they are useful. First, the formulas work for any angles, so they are often used to prove general formulas in mathematics and other fields, as we will see later in the text. Second, they can help in determining which angles have a given trigonometric function value.

Example 1.27

Find all angles $0° \le \theta < 360°$ such that $\sin\theta = -0.682$.

Solution: Using the $\boxed{\sin^{-1}}$ button on a calculator with -0.682 as the input, we get $\theta = -43°$, which is not between 0° and 360°.[7] Since $\theta = -43°$ is in QIV, its reflection $180° - \theta$ around the y-axis will be in QIII and have the same sine value. But $180° - \theta = 180° - (-43°) = 223°$ (see Figure 1.5.8). Also, we know that $-43°$ and $-43° + 360° = 317°$ have the same trigonometric function values. So since angles in QI and QII have positive sine values, we see that the only angles between 0° and 360° with a sine of -0.682 are $\boxed{\theta = 223° \text{ and } 317°}$.

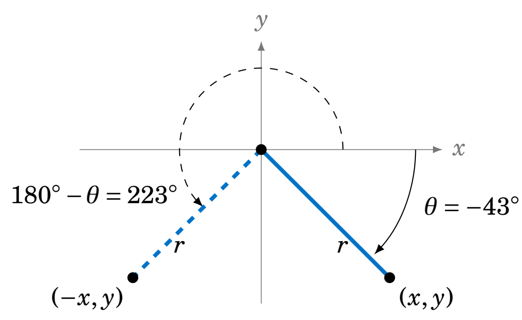
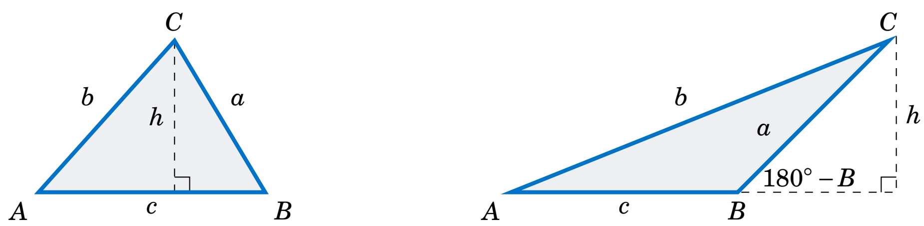


Figure 1.5.8 Reflection around the y-axis: $-43°$ and $223°$

[7] In Chapter 5 we will discuss why the $\boxed{\sin^{-1}}$ button returns that value.

Another way of stating the Law of Sines is: *The sides of a triangle are proportional to the sines of their opposite angles.*

To prove the Law of Sines, let $\triangle ABC$ be an oblique triangle. Then $\triangle ABC$ can be acute, as in Figure 2.1.1(a), or it can be obtuse, as in Figure 2.1.1(b). In each case, draw the *altitude*[1] from the vertex at C to the side $\overline{AB}$. In Figure 2.1.1(a) the altitude lies inside the triangle, while in Figure 2.1.1(b) the altitude lies outside the triangle.



(a) Acute triangle **(b)** Obtuse triangle

Figure 2.1.1 Proof of the Law of Sines for an oblique triangle $\triangle ABC$

Let h be the height of the altitude. For each triangle in Figure 2.1.1, we see that

$$\frac{h}{b} = \sin A \tag{2.4}$$

and

$$\frac{h}{a} = \sin B \tag{2.5}$$

(in Figure 2.1.1(b), $\frac{h}{a} = \sin(180° - B) = \sin B$ by formula (1.19) in Section 1.5). Thus, solving for h in equation (2.5) and substituting that into equation (2.4) gives

$$\frac{a \sin B}{b} = \sin A \,, \tag{2.6}$$

and so putting a and A on the left side and b and B on the right side, we get

$$\frac{a}{\sin A} = \frac{b}{\sin B} \,. \tag{2.7}$$

By a similar argument, drawing the altitude from A to $\overline{BC}$ gives

$$\frac{b}{\sin B} = \frac{c}{\sin C} \,, \tag{2.8}$$

so putting the last two equations together proves the theorem. **QED**

Note that we did not prove the Law of Sines for right triangles, since it turns out (see Exercise 12) to be trivially true for that case.

[1] Recall from geometry that an altitude of a triangle is a perpendicular line segment from any vertex to the line containing the side opposite the vertex.

2.3 The Law of Tangents

We have shown how to solve a triangle in all four cases discussed at the beginning of this chapter. An alternative to the Law of Cosines for Case 3 (two sides and the included angle) is the *Law of Tangents*:

Theorem 2.3. Law of Tangents: If a triangle has sides of lengths a, b, and c opposite the angles A, B, and C, respectively, then

$$\frac{a-b}{a+b} = \frac{\tan\,\frac{1}{2}(A-B)}{\tan\,\frac{1}{2}(A+B)}\,, \tag{2.17}$$

$$\frac{b-c}{b+c} = \frac{\tan\,\frac{1}{2}(B-C)}{\tan\,\frac{1}{2}(B+C)}\,, \tag{2.18}$$

$$\frac{c-a}{c+a} = \frac{\tan\,\frac{1}{2}(C-A)}{\tan\,\frac{1}{2}(C+A)}\,. \tag{2.19}$$

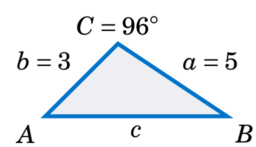
Note that since $\tan(-\theta) = -\tan\theta$ for any angle θ, we can switch the order of the letters in each of the above formulas. For example, we can rewrite formula (2.17) as

$$\frac{b-a}{b+a} = \frac{\tan\,\frac{1}{2}(B-A)}{\tan\,\frac{1}{2}(B+A)}\,, \tag{2.20}$$

and similarly for the other formulas. If $a > b$, then it is usually more convenient to use formula (2.17), while formula (2.20) is more convenient when $b > a$.

Example 2.10

Case 3: Two sides and the included angle.
Solve the triangle $\triangle ABC$ given $a = 5$, $b = 3$, and $C = 96°$.



Solution: $A + B + C = 180°$, so $A + B = 180° - C = 180° - 96° = 84°$. Thus, by the Law of Tangents,

$$\frac{a-b}{a+b} = \frac{\tan\,\frac{1}{2}(A-B)}{\tan\,\frac{1}{2}(A+B)} \quad\Rightarrow\quad \frac{5-3}{5+3} = \frac{\tan\,\frac{1}{2}(A-B)}{\tan\,\frac{1}{2}(84°)}$$
$$\Rightarrow\quad \tan\,\tfrac{1}{2}(A-B) = \tfrac{2}{8}\tan 42° = 0.2251$$
$$\Rightarrow\quad \tfrac{1}{2}(A-B) = 12.7° \quad\Rightarrow\quad A-B = 25.4°\,.$$

We now have two equations involving A and B, which we can solve by adding the equations:

$$\begin{aligned} A-B &= 25.4° \\ A+B &= 84° \\ \hline 2A &= 109.4° \quad\Rightarrow\quad \boxed{A = 54.7°} \quad\Rightarrow\quad B = 84° - 54.7° \quad\Rightarrow\quad \boxed{B = 29.3°} \end{aligned}$$

We can find the remaining side c by using the Law of Sines:

$$c = \frac{a\,\sin C}{\sin A} = \frac{5\,\sin 96°}{\sin 54.7°} \quad\Rightarrow\quad \boxed{c = 6.09}$$

Case 2: Three angles and any side.
Suppose that we have a triangle $\triangle ABC$ in which one side, say, a, and all three angles are known.[7] By the Law of Sines we know that

$$c = \frac{a \sin C}{\sin A} ,$$

so substituting this into formula (2.24) we get:

$$\boxed{\text{Area} = K = \frac{a^2 \sin B \sin C}{2 \sin A}} \tag{2.26}$$

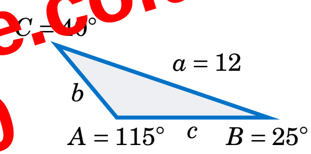
Similar arguments for the sides b and c give us:

$$\text{Area} = K = \frac{b^2 \sin A \sin C}{2 \sin B} \tag{2.27}$$

$$\text{Area} = K = \frac{c^2 \sin A \sin B}{2 \sin C} \tag{2.28}$$

Example 2.14

Find the area of the triangle $\triangle ABC$ given $A = 115°$, $B = 25°$, $C = 40°$, and $a = 12$.



Solution: Using formula (2.26), the area K is given by:

$$\begin{aligned} K &= \frac{a^2 \sin B \sin C}{2 \sin A} \\ &= \frac{12^2 \sin 25° \sin 40°}{2 \sin 115°} \\ K &= 21.58 \end{aligned}$$

Case 3: Three sides.
Suppose that we have a triangle $\triangle ABC$ in which all three sides are known. Then *Heron's formula*[8] gives us the area:

Heron's formula: For a triangle $\triangle ABC$ with sides a, b, and c, let $s = \frac{1}{2}(a+b+c)$ (i.e. $2s = a+b+c$ is the perimeter of the triangle). Then the area K of the triangle is

$$\text{Area} = K = \sqrt{s(s-a)(s-b)(s-c)} . \tag{2.29}$$

To prove this, first remember that the area K is one-half the base times the height. Using c as the base and the altitude h as the height, as before in Figure 2.4.1, we have $K = \frac{1}{2}hc$. Squaring both sides gives us

$$K^2 = \tfrac{1}{4} h^2 c^2 . \tag{2.30}$$

[7] Note that this is equivalent to knowing just *two* angles and a side (why?).
[8] Due to the ancient Greek engineer and mathematician Heron of Alexandria (c. 10-70 A.D.).

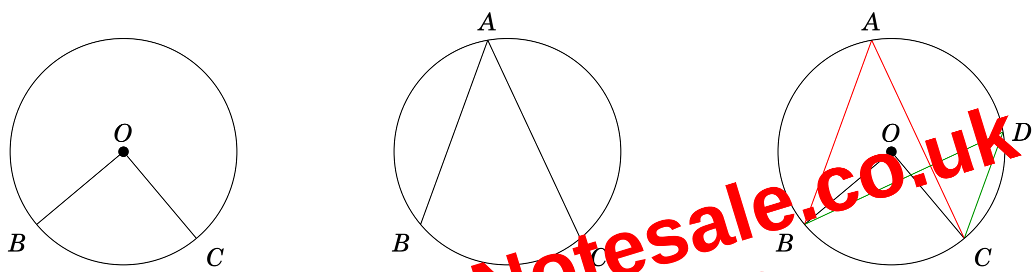
2.5 Circumscribed and Inscribed Circles

Recall from the Law of Sines that any triangle $\triangle ABC$ has a common ratio of sides to sines of opposite angles, namely

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\,.$$

This common ratio has a geometric meaning: it is the diameter (i.e. twice the radius) of the unique circle in which $\triangle ABC$ can be inscribed, called the **circumscribed circle** of the triangle. Before proving this, we need to review some elementary geometry.

A **central angle** of a circle is an angle whose vertex is the center O of the circle and whose sides (called **radii**) are line segments from O to two points on the circle. In Figure 2.5.1(a), $\angle O$ is a central angle and we say that it *intercepts the arc* $\widehat{BC}$.



(a) Central angle $\angle O$ **(b)** Inscribed angle $\angle A$ **(c)** $\angle A = \angle D = \frac{1}{2}\angle O$

Figure 2.5.1 Types of angles in a circle

An **inscribed angle** of a circle is an angle whose vertex is a point A on the circle and whose sides are line segments (called **chords**) from A to two other points on the circle. In Figure 2.5.1(b), $\angle A$ is an inscribed angle that intercepts the arc $\widehat{BC}$. We state here without proof[12] a useful relation between inscribed and central angles:

Theorem 2.4. If an inscribed angle $\angle A$ and a central angle $\angle O$ intercept the same arc, then $\angle A = \frac{1}{2}\angle O$. Thus, inscribed angles which intercept the same arc are equal.

Figure 2.5.1(c) shows two inscribed angles, $\angle A$ and $\angle D$, which intercept the same arc $\widehat{BC}$ as the central angle $\angle O$, and hence $\angle A = \angle D = \frac{1}{2}\angle O$ (so $\angle O = 2\angle A = 2\angle D$).

We will now prove our assertion about the common ratio in the Law of Sines:

Theorem 2.5. For any triangle $\triangle ABC$, the radius R of its circumscribed circle is given by:

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \tag{2.35}$$

(Note: For a circle of diameter 1, this means $a = \sin A$, $b = \sin B$, and $c = \sin C$.)

[12]For a proof, see pp. 210-211 in R.A. AVERY, *Plane Geometry*, Boston: Allyn & Bacon, 1950.

We will use Figure 2.5.6 to find the radius r of the inscribed circle. Since $\overline{OA}$ bisects A, we see that $\tan \frac{1}{2}A = \frac{r}{AD}$, and so $r = AD \cdot \tan \frac{1}{2}A$. Now, $\triangle OAD$ and $\triangle OAF$ are equivalent triangles, so $AD = AF$. Similarly, $DB = EB$ and $FC = CE$. Thus, if we let $s = \frac{1}{2}(a+b+c)$, we see that

$$\begin{aligned}
2s \;&=\; a \;+\; b \;+\; c \;=\; (AD+DB) \;+\; (CE+EB) \;+\; (AF+FC)\\
&=\; AD \;+\; EB \;+\; CE \;+\; EB \;+\; AD \;+\; CE \;=\; 2(AD+EB+CE)\\
s \;&=\; AD \;+\; EB \;+\; CE \;=\; AD \;+\; a\\
AD \;&=\; s-a\;.
\end{aligned}$$

Hence, $r = (s-a)\tan \frac{1}{2}A$. Similar arguments for the angles B and C give us:

Theorem 2.10. For any triangle $\triangle ABC$, let $s = \frac{1}{2}(a+b+c)$. Then the radius r of its inscribed circle is

$$r \;=\; (s-a)\tan \tfrac{1}{2}A \;=\; (s-b)\tan \tfrac{1}{2}B \;=\; (s-c)\tan \tfrac{1}{2}C\;. \tag{2.38}$$

We also see from Figure 2.5.6 that the area of the triangle $\triangle AOB$ is

$$\text{Area}(\triangle AOB) \;=\; \tfrac{1}{2}\,\text{base} \times \text{height} \;=\; \tfrac{1}{2}\,c\,r\;.$$

Similarly, $\text{Area}(\triangle BOC) = \frac{1}{2}\,a\,r$ and $\text{Area}(\triangle AOC) = \frac{1}{2}\,b\,r$. Thus, the area K of $\triangle ABC$ is

$$\begin{aligned}
K \;&=\; \text{Area}(\triangle AOB) \;+\; \text{Area}(\triangle BOC) \;+\; \text{Area}(\triangle AOC) \;=\; \tfrac{1}{2}\,c\,r \;+\; \tfrac{1}{2}\,a\,r \;+\; \tfrac{1}{2}\,b\,r\\
&=\; \tfrac{1}{2}(a+b+c)\,r \;=\; s\,r\;, \text{ so by Heron's formula we get}
\end{aligned}$$

$$r \;=\; \frac{K}{s} \;=\; \frac{\sqrt{s\,(s-a)(s-b)(s-c)}}{s} \;=\; \sqrt{\frac{s\,(s-a)(s-b)(s-c)}{s^2}} \;=\; \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\;.$$

We have thus proved the following theorem:

Theorem 2.11. For any triangle $\triangle ABC$, let $s = \frac{1}{2}(a+b+c)$. Then the radius r of its inscribed circle is

$$r \;=\; \frac{K}{s} \;=\; \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\;. \tag{2.39}$$

Recall from geometry how to bisect an angle: use a compass centered at the vertex to draw an arc that intersects the sides of the angle at two points. At those two points use a compass to draw an arc with the same radius, large enough so that the two arcs intersect at a point, as in Figure 2.5.7. The line through that point and the vertex is the bisector of the angle. For the inscribed circle of a triangle, you need only *two* angle bisectors; their intersection will be the center of the circle.

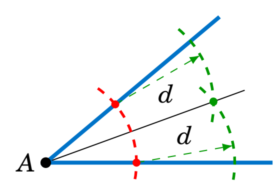


Figure 2.5.7

Note how we proved the identity by expanding one of its sides ($\frac{\sin\theta}{\cos\theta}$) until we got an expression that was equal to the other side ($\tan\theta$). This is probably the most common technique for proving identities. Taking reciprocals in the above identity gives:

$$\cot\theta \;=\; \frac{\cos\theta}{\sin\theta} \qquad \text{when } \sin\theta \neq 0 \tag{3.2}$$

We will now derive one of the most important trigonometric identities. Let θ be any angle with a point (x, y) on its terminal side a distance $r > 0$ from the origin. By the Pythagorean Theorem, $r^2 = x^2 + y^2$ (and hence $r = \sqrt{x^2 + y^2}$). For example, if θ is in QIII as in Figure 3.1.1, then the legs of the right triangle formed by the reference angle have lengths $|x|$ and $|y|$ (we use absolute values because x and y are negative in QIII). The same argument holds if θ is in the other quadrants or on either axis. Thus,

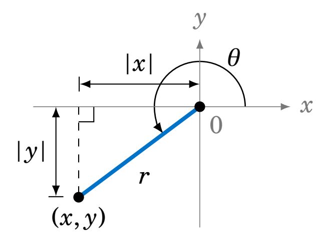


Figure 3.1.1

$$r^2 \;=\; |x|^2 \;+\; |y|^2 \;=\; x^2 \;+\; y^2\,,$$

so dividing both sides of the equation by r^2 (which we can do since $r > 0$) gives

$$\frac{r^2}{r^2} \;=\; \frac{x^2 \;+\; y^2}{r^2} \;=\; \frac{x^2}{r^2} \;+\; \frac{y^2}{r^2} \;=\; \left(\frac{x}{r}\right)^2 \;+\; \left(\frac{y}{r}\right)^2 .$$

Since $\frac{r^2}{r^2} = 1$, $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$, we can rewrite this as:

$$\cos^2\theta \;+\; \sin^2\theta \;=\; 1 \tag{3.3}$$

You can think of this as sort of a trigonometric variant of the Pythagorean Theorem. Note that we use the notation $\sin^2\theta$ to mean $(\sin\theta)^2$, likewise for cosine and the other trigonometric functions. We will use the same notation for other powers besides 2.

From the above identity we can derive more identities. For example:

$$\sin^2\theta \;=\; 1 \;-\; \cos^2\theta \tag{3.4}$$

$$\cos^2\theta \;=\; 1 \;-\; \sin^2\theta \tag{3.5}$$

from which we get (after taking square roots):

$$\sin\theta \;=\; \pm\sqrt{1 \;-\; \cos^2\theta} \tag{3.6}$$

$$\cos\theta \;=\; \pm\sqrt{1 \;-\; \sin^2\theta} \tag{3.7}$$

Example 3.5

Prove that $\dfrac{\tan^2\theta + 2}{1 + \tan^2\theta} = 1 + \cos^2\theta$.

Solution: Expand the left side:

$$\begin{aligned}
\frac{\tan^2\theta + 2}{1 + \tan^2\theta} &= \frac{\left(\tan^2\theta + 1\right) + 1}{1 + \tan^2\theta} \\
&= \frac{\sec^2\theta + 1}{\sec^2\theta} \qquad \text{(by (3.10))} \\
&= \frac{\sec^2\theta}{\sec^2\theta} + \frac{1}{\sec^2\theta} \\
&= 1 + \cos^2\theta
\end{aligned}$$

When trying to prove an identity where at least one side is a ratio of expressions, *cross-multiplying* can be an effective technique:

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc$$

Example 3.6

Prove that $\dfrac{1 + \sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1 - \sin\theta}$.

Solution: Cross-multiply and reduce both sides until it is clear that they are equal:

$$\begin{aligned}
(1 + \sin\theta)(1 - \sin\theta) &= \cos\theta \cdot \cos\theta \\
1 - \sin^2\theta &= \cos^2\theta
\end{aligned}$$

By (3.5) both sides of the last equation are indeed equal. Thus, the original identity holds.

Example 3.7

Suppose that $a\cos\theta = b$ and $c\sin\theta = d$ for some angle θ and some constants a, b, c, and d. Show that $a^2c^2 = b^2c^2 + a^2d^2$.

Solution: Multiply both sides of the first equation by c and the second equation by a:

$$\begin{aligned}
ac\cos\theta &= bc \\
ac\sin\theta &= ad
\end{aligned}$$

Now square each of the above equations then add them together to get:

$$\begin{aligned}
(ac\cos\theta)^2 + (ac\sin\theta)^2 &= (bc)^2 + (ad)^2 \\
(ac)^2\left(\cos^2\theta + \sin^2\theta\right) &= b^2c^2 + a^2d^2 \\
a^2c^2 &= b^2c^2 + a^2d^2 \qquad \text{(by (3.3))}
\end{aligned}$$

Notice how θ does not appear in our final result. The trick was to get a common coefficient (ac) for $\cos\theta$ and $\sin\theta$ so that we could use $\cos^2\theta + \sin^2\theta = 1$. This is a common technique for eliminating trigonometric functions from systems of equations.

Exercises

1. We showed that $\sin\theta = \pm\sqrt{1-\cos^2\theta}$ for all θ. Give an example of an angle θ such that $\sin\theta = -\sqrt{1-\cos^2\theta}$.

2. We showed that $\cos\theta = \pm\sqrt{1-\sin^2\theta}$ for all θ. Give an example of an angle θ such that $\cos\theta = -\sqrt{1-\sin^2\theta}$.

3. Suppose that you are given a system of two equations of the following form:[1]

$$\begin{aligned} A\cos\phi &= Bv_1 - Bv_2\cos\theta \\ A\sin\phi &= Bv_2\sin\theta\,. \end{aligned}$$

Show that $A^2 = B^2\left(v_1^2 + v_2^2 - 2v_1v_2\cos\theta\right)$.

For Exercises 4-16, prove the given identity.

4. $\cos\theta\,\tan\theta = \sin\theta$

5. $\sin\theta\,\cot\theta = \cos\theta$

6. $\dfrac{\tan\theta}{\cot\theta} = \tan^2\theta$

7. $\dfrac{\csc\theta}{\sin\theta} = \csc^2\theta$

8. $\dfrac{\cos^2\theta}{1+\sin\theta} = 1 - \sin\theta$

9. $\dfrac{1-2\cos^2\theta}{\sin\theta\,\cos\theta} = \tan\theta - \cot\theta$

10. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

11. $\cos^4\theta - \sin^4\theta = 1 - 2\sin^2\theta$

12. $\dfrac{1-\tan\theta}{1+\tan\theta} = \dfrac{\cot\theta-1}{\cot\theta+1}$

13. $\dfrac{\tan\theta+\tan\phi}{\cot\theta+\cot\phi} = \tan\theta\,\tan\phi$

14. $\dfrac{\sin^2\theta}{1-\sin^2\theta} = \tan^2\theta$

15. $\dfrac{1-\tan^2\theta}{1-\cot^2\theta} = 1 - \sec^2\theta$

16. $\sin\theta = \pm\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$ *(Hint: Solve for* $\sin^2\theta$ *in Exercise 14.)*

17. Sometimes identities can be proved by geometrical methods. For example, to prove the identity in Exercise 16, draw an acute angle θ in QI and pick the point $(1, y)$ on its terminal side, as in Figure 3.1.2. What must y equal? Use that to prove the identity for acute θ. Explain the adjustment(s) you would need to make in Figure 3.1.2 to prove the identity for θ in the other quadrants. Does the identity hold if θ is on either axis?

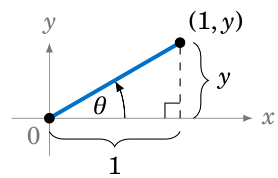


Figure 3.1.2

18. Similar to Exercise 16 , find an expression for $\cos\theta$ solely in terms of $\tan\theta$.

19. Find an expression for $\tan\theta$ solely in terms of $\sin\theta$, and one solely in terms of $\cos\theta$.

20. Suppose that a point with coordinates $(x, y) = (a(\cos\psi - \epsilon), a\sqrt{1-\epsilon^2}\,\sin\psi)$ is a distance $r > 0$ from the origin, where $a > 0$ and $0 < \epsilon < 1$. Use $r^2 = x^2 + y^2$ to show that $r = a(1 - \epsilon\cos\psi)$. (Note: These coordinates arise in the study of elliptical orbits of planets.)

21. Show that each trigonometric function can be put in terms of the sine function.

[1]These types of equations arise in physics, e.g. in the study of photon-electron collisions. See pp. 95-97 in W. Rindler, *Special Relativity*, Edinburgh: Oliver and Boyd, LTD., 1960.

and

$$\begin{aligned}
\cos(A+B) &= \frac{OM}{OP} = \frac{ON-MN}{OP} = \frac{ON-RQ}{OP} = \frac{ON}{OP} - \frac{RQ}{OP} \\
&= \frac{ON}{OQ}\cdot\frac{OQ}{OP} + \frac{RQ}{PQ}\cdot\frac{PQ}{OP} \\
&= \cos A\,\cos B \,-\, \sin A\,\sin B\,. \qquad (3.15)
\end{aligned}$$

So we have proved the identities for acute angles A and B. It is simple to verify that they hold in the special case of $A = B = 0°$. For general angles, we will need to use the relations we derived in Section 1.5 which involve adding or subtracting $90°$:

$$\begin{aligned}
\sin(\theta+90°) &= \cos\theta & \qquad \sin(\theta-90°) &= -\cos\theta \\
\cos(\theta+90°) &= -\sin\theta & \qquad \cos(\theta-90°) &= \sin\theta
\end{aligned}$$

These will be useful because *any angle can be written as the sum of an acute angle (or $0°$) and integer multiples of* $\pm 90°$. For example, $155° = 65° + 90°$, $222° = 42° + 2(90°)$, $-77° = 13° - 90°$, etc. So if we can prove that the identities hold when adding or subtracting $90°$ to or from either A or B, respectively, where A and B are acute or $0°$, then the identities will also hold when repeatedly adding or subtracting $90°$, and hence will hold for all angles. Replacing A by $A + 90°$ and using the relations for adding $90°$ gives

$$\begin{aligned}
\sin((A+90°)+B) &= \sin((A+B)+90°) = \cos(A+B)\,, \\
&= \cos A\,\cos B \,-\, \sin A\,\sin B \quad \text{(by equation (3.15))} \\
&= \sin(A+90°)\,\cos B \,+\, \cos(A+90°)\,\sin B\,,
\end{aligned}$$

so the identity holds for $A + 90°$ and B (and, similarly, for A and $B + 90°$). Likewise,

$$\begin{aligned}
\sin((A-90°)+B) &= \sin((A+B)-90°) = -\cos(A+B)\,, \\
&= -(\cos A\,\cos B \,-\, \sin A\,\sin B) \\
&= (-\cos A)\,\cos B \,+\, \sin A\,\sin B \\
&= \sin(A-90°)\,\cos B \,+\, \cos(A-90°)\,\sin B\,,
\end{aligned}$$

so the identity holds for $A - 90°$ and B (and, similarly, for A and $B + 90°$). Thus, the addition formula (3.12) for sine holds for *all* A and B. A similar argument shows that the addition formula (3.13) for cosine is true for all A and B. **QED**

Replacing B by $-B$ in the addition formulas and using the relations $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$ from Section 1.5 gives us the *subtraction formulas*:

$$\sin(A-B) = \sin A\,\cos B \,-\, \cos A\,\sin B \qquad (3.16)$$

$$\cos(A-B) = \cos A\,\cos B \,+\, \sin A\,\sin B \qquad (3.17)$$

Using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and the addition formulas for sine and cosine, we can derive the addition formula for tangent:

$$\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A\,\cos B \,+\, \cos A\,\sin B}{\cos A\,\cos B \,-\, \sin A\,\sin B} \\
&= \frac{\dfrac{\sin A\,\cos B}{\cos A\,\cos B} + \dfrac{\cos A\,\sin B}{\cos A\,\cos B}}{\dfrac{\cos A\,\cos B}{\cos A\,\cos B} - \dfrac{\sin A\,\sin B}{\cos A\,\cos B}} \quad \text{(divide top and bottom by } \cos A\,\cos B\text{)} \\
&= \frac{\dfrac{\sin A}{\cos A}\cdot\dfrac{\cancel{\cos B}}{\cancel{\cos B}} + \dfrac{\cancel{\cos A}}{\cancel{\cos A}}\cdot\dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}} = \frac{\tan A \,+\, \tan B}{1 \,-\, \tan A\,\tan B}
\end{aligned}$$

This, combined with replacing B by $-B$ and using the relation $\tan(-\theta) = -\tan\theta$, gives us the addition and subtraction formulas for tangent:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\,\tan B} \tag{3.18}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\,\tan B} \tag{3.19}$$

Example 3.8

Given angles A and B such that $\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\sin B = \frac{12}{13}$, and $\cos B = \frac{5}{13}$, find the exact values of $\sin(A+B)$, $\cos(A+B)$, and $\tan(A+B)$.

Solution: Using the addition formula for sine, we get:

$$\begin{aligned}
\sin(A+B) &= \sin A\,\cos B \,+\, \cos A\,\sin B \\
&= \frac{4}{5}\cdot\frac{5}{13} + \frac{3}{5}\cdot\frac{12}{13} \quad\Rightarrow\quad \boxed{\sin(A+B) = \frac{56}{65}}
\end{aligned}$$

Using the addition formula for cosine, we get:

$$\begin{aligned}
\cos(A+B) &= \cos A\,\cos B \,-\, \sin A\,\sin B \\
&= \frac{3}{5}\cdot\frac{5}{13} - \frac{4}{5}\cdot\frac{12}{13} \quad\Rightarrow\quad \boxed{\cos(A+B) = -\frac{33}{65}}
\end{aligned}$$

Instead of using the addition formula for tangent, we can use the results above:

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\frac{56}{65}}{-\frac{33}{65}} \quad\Rightarrow\quad \boxed{\tan(A+B) = -\frac{56}{33}}$$

11. $\tan(\theta + 45°) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$

12. $\dfrac{\cos(A+B)}{\sin A \cos B} = \cot A - \tan B$

13. $\cot A + \cot B = \dfrac{\sin(A+B)}{\sin A \sin B}$

14. $\dfrac{\sin(A-B)}{\sin(A+B)} = \dfrac{\cot B - \cot A}{\cot B + \cot A}$

15. Generalize Exercise 6: For any a and b, $-\sqrt{a^2+b^2} \le a \sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$ for all θ.

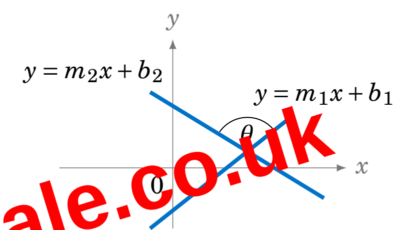
16. Continuing Example 3.12, use Snell's law to show that the *s-polarization transmission Fresnel coefficient*

$$t_{12s} = \frac{2\,n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2} \tag{3.22}$$

can be written as:

$$t_{12s} = \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_2 + \theta_1)}$$

17. Suppose that two lines with slopes m_1 and m_2, respectively, intersect at an angle θ and are not perpendicular (i.e. $\theta \ne 90°$), as in the figure on the right. Show that



$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| .$$

(*Hint: Use Example 1.26 from Section 1.5.*)

18. Use Exercise 17 to find the angle between the lines $y = 2x+3$ and $y = -5x-4$.

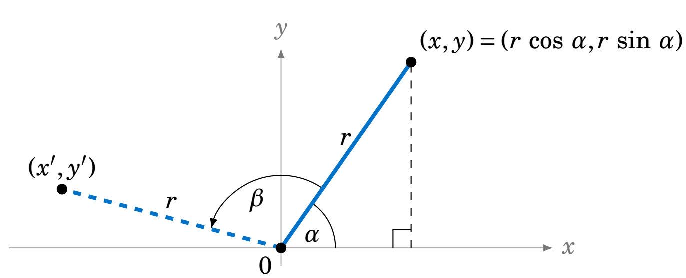
19. For any triangle $\triangle ABC$, show that $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$.
(*Hint: Use Exercise 9 and* $C = 180° - (A+B)$.)

20. For any positive angles A, B, and C such that $A+B+C = 90°$, show that

$$\tan A \tan B + \tan B \tan C + \tan C \tan A = 1 .$$

21. Prove the identity $\sin(A+B)\cos B - \cos(A+B)\sin B = \sin A$.
Note that the right side depends only on A, while the left side depends on both A and B.

22. A line segment of length $r > 0$ from the origin to the point (x,y) makes an angle α with the positive x-axis, so that $(x,y) = (r\cos\alpha, r\sin\alpha)$, as in the figure below. What are the endpoint's new coordinates (x', y') after a counterclockwise rotation by an angle β? Your answer should be in terms of r, α, and β.



Example 4.5

A central angle in a circle of radius 5 m cuts off an arc of length 2 m. What is the measure of the angle in radians? What is the measure in degrees?

Solution: Letting $r = 5$ and $s = 2$ in formula (4.4), we get:

$$\theta = \frac{s}{r} = \frac{2}{5} = \boxed{0.4 \text{ rad}}$$

In degrees, the angle is:

$$\theta = 0.4 \text{ rad} = \frac{180}{\pi} \cdot 0.4 = \boxed{22.92^\circ}$$

For central angles $\theta > 2\pi$ rad, i.e. $\theta > 360^\circ$, it may not be clear what is meant by the intercepted arc, since the angle is larger than one revolution and hence "wraps around" the circle more than once. We will take the approach that such an arc consists of the full circumference plus any additional arc length determined by the angle. In other words, formula (4.4) is still valid for angles $\theta > 2\pi$ rad.

What about negative angles? In this case using $s = r\theta$ would mean that the arc length is negative, which violates the usual concept of length. So we will adopt the convention of only using nonnegative central angles when discussing arc length.

Example 4.6

A rope is fastened to a wall in two places 8 ft apart at the same height. A cylindrical container with a radius of 2 ft is pushed away from the wall as far as it can go while being held in by the rope, as in Figure 4.2.3 which shows the top view. If the center of the container is 3 feet away from the point on the wall midway between the ends of the rope, what is the length L of the rope?

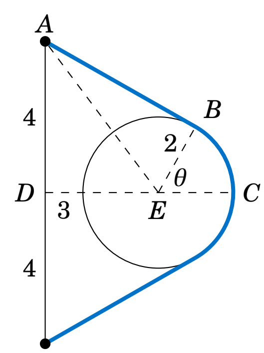


Figure 4.2.3

Solution: We see that, by symmetry, the total length of the rope is $L = 2(AB + \widehat{BC})$. Also, notice that $\triangle ADE$ is a right triangle, so the hypotenuse has length $AE = \sqrt{DE^2 + DA^2} = \sqrt{3^2 + 4^2} = 5$ ft, by the Pythagorean Theorem. Now since $\overline{AB}$ is tangent to the circular container, we know that $\angle ABE$ is a right angle. So by the Pythagorean Theorem we have

$$AB = \sqrt{AE^2 - BE^2} = \sqrt{5^2 - 2^2} = \sqrt{21} \text{ ft.}$$

By formula (4.4) the arc $\widehat{BC}$ has length $BE \cdot \theta$, where $\theta = \angle BEC$ is the supplement of $\angle AED + \angle AEB$. So since

$$\tan \angle AED = \frac{4}{3} \Rightarrow \angle AED = 53.1^\circ \quad \text{and} \quad \cos \angle AEB = \frac{BE}{AE} = \frac{2}{5} \Rightarrow \angle AEB = 66.4^\circ,$$

we have

$$\theta = \angle BEC = 180^\circ - (\angle AED + \angle AEB) = 180^\circ - (53.1^\circ + 66.4^\circ) = 60.5^\circ .$$

Converting to radians, we get $\theta = \frac{\pi}{180} \cdot 60.5 = 1.06$ rad. Thus,

$$L = 2(AB + \widehat{BC}) = 2(\sqrt{21} + BE \cdot \theta) = 2(\sqrt{21} + (2)(1.06)) = \boxed{13.4 \text{ ft}} .$$

The graph of $y = \cot x$ can also be determined by using $\cot x = \frac{1}{\cot x}$. Alternatively, we can use the relation $\cot x = -\tan(x + 90°)$ from Section 1.5, so that the graph of the cotangent function is just the graph of the tangent function shifted to the left by $\frac{\pi}{2}$ radians and then reflected about the x-axis, as in Figure 5.1.9:

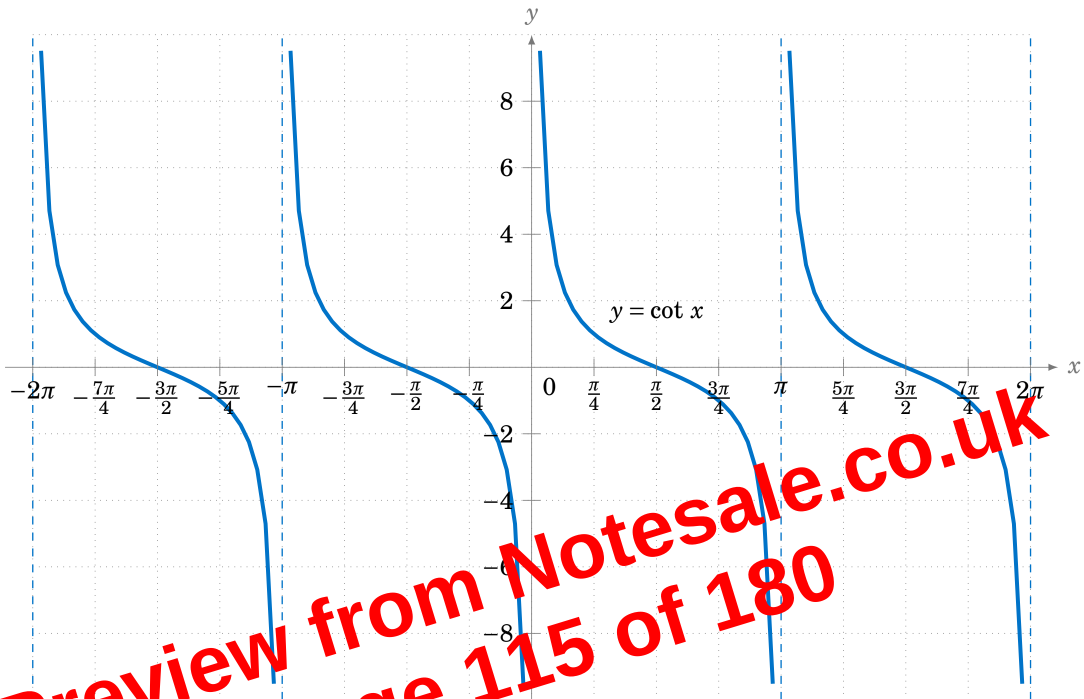
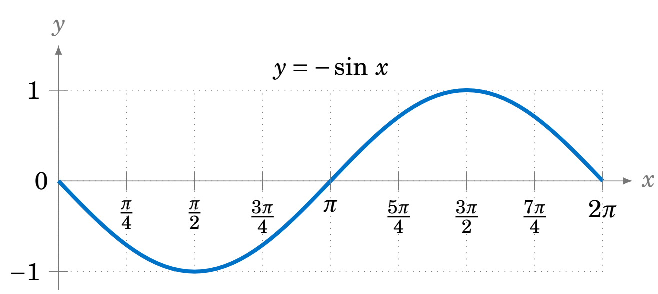


Figure 5.1.9 Graph of $y = \cot x$

Example 5.1

Draw the graph of $y = -\sin x$ for $0 \le x \le 2\pi$.

Solution: Multiplying a function by -1 just reflects its graph around the x-axis. So reflecting the graph of $y = \sin x$ around the x-axis gives us the graph of $y = -\sin x$:



Note that this graph is the same as the graphs of $y = \sin(x \pm \pi)$ and $y = \cos(x + \frac{\pi}{2})$.

5.3 Inverse Trigonometric Functions

We have briefly mentioned the inverse trigonometric functions before, for example in Section 1.3 when we discussed how to use the $\boxed{\sin^{-1}}$, $\boxed{\cos^{-1}}$, and $\boxed{\tan^{-1}}$ buttons on a calculator to find an angle that has a certain trigonometric function value. We will now define those inverse functions and determine their graphs.

Recall that a **function** is a rule that assigns a single object y from one set (the **range**) to each object x from another set (the **domain**). We can write that rule as $y = f(x)$, where f is the function (see Figure 5.3.1). There is a simple *vertical rule* for determining whether a rule $y = f(x)$ is a function: f is a function if and only if every vertical line intersects the graph of $y = f(x)$ in the xy-coordinate plane at most once (see Figure 5.3.2).

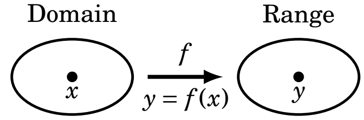
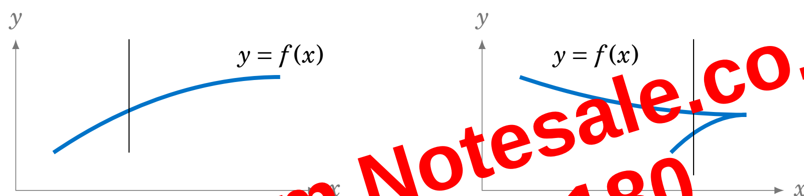


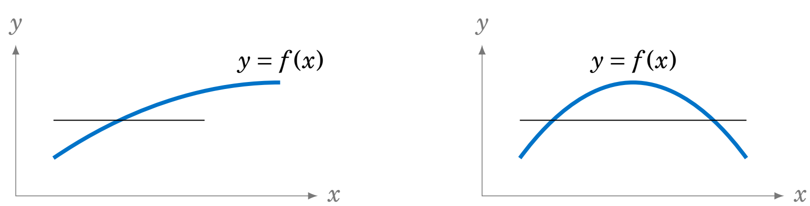
Figure 5.3.1



(a) f is a function **(b)** f is not a function

Figure 5.3.2 Vertical rule for functions

Recall that a function f is **one-to-one** (often written as $1-1$) if it assigns distinct values of y to distinct values of x. In other words, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. Equivalently, f is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. There is a simple *horizontal rule* for determining whether a function $y = f(x)$ is one-to-one: f is one-to-one if and only if every horizontal line intersects the graph of $y = f(x)$ in the xy-coordinate plane at most once (see Figure 5.3.3).



(a) f is one-to-one **(b)** f is not one-to-one

Figure 5.3.3 Horizontal rule for one-to-one functions

If a function f is one-to-one on its domain, then f has an **inverse function**, denoted by f^{-1}, such that $y = f(x)$ if and only if $f^{-1}(y) = x$. The domain of f^{-1} is the range of f.

The basic idea is that f^{-1} "undoes" what f does, and vice versa. In other words,

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for all } x \text{ in the domain of } f \text{, and} \\ f(f^{-1}(y)) &= y \quad \text{for all } y \text{ in the range of } f. \end{aligned}$$

We know from their graphs that none of the trigonometric functions are one-to-one over their entire domains. However, we can restrict those functions to *subsets* of their domains where they *are* one-to-one. For example, $y = \sin x$ is one-to-one over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as we see in the graph below:

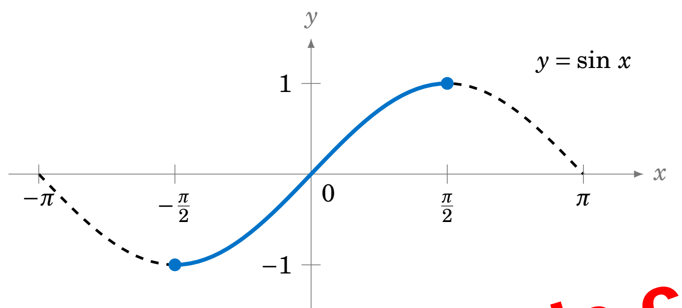


Figure 5.3.4 $y = \sin x$ with x restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

For $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ we have $-1 \le \sin x \le 1$, so we can define the **inverse sine** function $y = \sin^{-1} x$ (sometimes called the arcsine and denoted by $y = \arcsin x$) whose domain is the interval $[-1, 1]$ and whose range is the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In other words:

$$\sin^{-1}(\sin y) = y \quad \text{for } -\tfrac{\pi}{2} \le y \le \tfrac{\pi}{2} \tag{5.2}$$

$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \le x \le 1 \tag{5.3}$$

Example 5.13

Find $\sin^{-1}\left(\sin \frac{\pi}{4}\right)$.

Solution: Since $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$, we know that $\sin^{-1}\left(\sin \frac{\pi}{4}\right) = \boxed{\dfrac{\pi}{4}}$, by formula (5.2).

Example 5.14

Find $\sin^{-1}\left(\sin \frac{5\pi}{4}\right)$.

Solution: Since $\frac{5\pi}{4} > \frac{\pi}{2}$, we can not use formula (5.2). But we know that $\sin \frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$. Thus, $\sin^{-1}\left(\sin \frac{5\pi}{4}\right) = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is, by definition, the angle y such that $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $\sin y = -\frac{1}{\sqrt{2}}$. That angle is $y = -\frac{\pi}{4}$, since

$$\sin\left(-\tfrac{\pi}{4}\right) = -\sin\left(\tfrac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\,.$$

Thus, $\sin^{-1}\left(\sin \frac{5\pi}{4}\right) = \boxed{-\frac{\pi}{4}}$.

The inverse functions for cotangent, cosecant, and secant can be determined by looking at their graphs. For example, the function $y = \cot x$ is one-to-one in the interval $(0,\pi)$, where it has a range equal to the set of all real numbers. Thus, the **inverse cotangent** $y = \cot^{-1} x$ is a function whose domain is the set of all real numbers and whose range is the interval $(0,\pi)$. In other words:

$$\cot^{-1}(\cot y) = y \quad \text{for } 0 < y < \pi \tag{5.8}$$

$$\cot(\cot^{-1} x) = x \quad \text{for all real } x \tag{5.9}$$

The graph of $y = \cot^{-1} x$ is shown below in Figure 5.3.11.

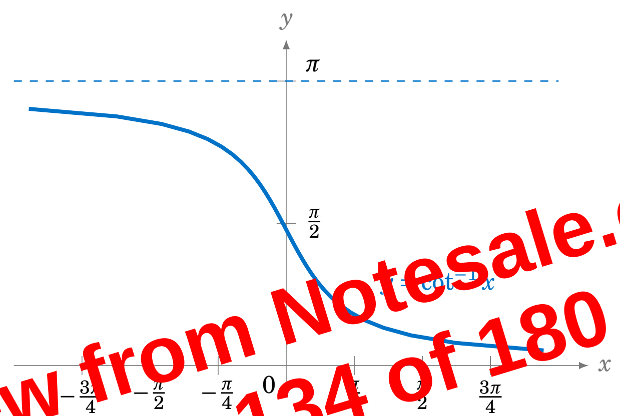


Figure 5.3.11 Graph of $y = \cot^{-1} x$

Similarly, it can be shown that the **inverse cosecant** $y = \csc^{-1} x$ is a function whose domain is $|x| \ge 1$ and whose range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$. Likewise, the **inverse secant** $y = \sec^{-1} x$ is a function whose domain is $|x| \ge 1$ and whose range is $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$.

$$\csc^{-1}(\csc y) = y \quad \text{for } -\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0 \tag{5.10}$$

$$\csc(\csc^{-1} x) = x \quad \text{for } |x| \ge 1 \tag{5.11}$$

$$\sec^{-1}(\sec y) = y \quad \text{for } 0 \le y \le \pi,\ y \ne \frac{\pi}{2} \tag{5.12}$$

$$\sec(\sec^{-1} x) = x \quad \text{for } |x| \ge 1 \tag{5.13}$$

It is also common to call $\cot^{-1} x$, $\csc^{-1} x$, and $\sec^{-1} x$ the **arc cotangent**, **arc cosecant**, and **arc secant**, respectively, of x. The graphs of $y = \csc^{-1} x$ and $y = \sec^{-1} x$ are shown in Figure 5.3.12:

Exercises

For Exercises 1-25, find the exact value of the given expression in radians.

1. $\tan^{-1} 1$ **2.** $\tan^{-1}(-1)$ **3.** $\tan^{-1} 0$ **4.** $\cos^{-1} 1$ **5.** $\cos^{-1}(-1)$

6. $\cos^{-1} 0$ **7.** $\sin^{-1} 1$ **8.** $\sin^{-1}(-1)$ **9.** $\sin^{-1} 0$ **10.** $\sin^{-1}\left(\sin \frac{\pi}{3}\right)$

11. $\sin^{-1}\left(\sin \frac{4\pi}{3}\right)$ **12.** $\sin^{-1}\left(\sin\left(-\frac{5\pi}{6}\right)\right)$ **13.** $\cos^{-1}\left(\cos \frac{\pi}{7}\right)$ **14.** $\cos^{-1}\left(\cos\left(-\frac{\pi}{10}\right)\right)$

15. $\cos^{-1}\left(\cos \frac{6\pi}{5}\right)$ **16.** $\tan^{-1}\left(\tan \frac{4\pi}{3}\right)$ **17.** $\tan^{-1}\left(\tan\left(-\frac{5\pi}{6}\right)\right)$ **18.** $\cot^{-1}\left(\cot \frac{4\pi}{3}\right)$

19. $\csc^{-1}\left(\csc\left(-\frac{\pi}{9}\right)\right)$ **20.** $\sec^{-1}\left(\sec \frac{6\pi}{5}\right)$ **21.** $\cos\left(\sin^{-1}\left(\frac{5}{13}\right)\right)$ **22.** $\cos\left(\sin^{-1}\left(-\frac{4}{5}\right)\right)$

23. $\sin^{-1} \frac{3}{5} + \sin^{-1} \frac{4}{5}$ **24.** $\sin^{-1} \frac{5}{13} + \cos^{-1} \frac{5}{13}$ **25.** $\tan^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{5}$

For Exercises 26-33, prove the given identity.

26. $\cos(\sin^{-1} x) = \sqrt{1-x^2}$ **27.** $\sin(\cos^{-1} x) = \sqrt{1-x^2}$

28. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ **29.** $\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$

30. $\sin^{-1}(-x) = -\sin^{-1} x$ **31.** $\cos^{-1}(-x) + \cos^{-1} x = \pi$

32. $\cot^{-1} x = \tan^{-1} \frac{1}{x}$ for $x > 0$ **33.** $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$ for $x > 0$

34. In Example 5.22 we showed that the formula $\tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$ does not always hold. Does the formula $\tan(\tan^{-1} a + \tan^{-1} b) = \frac{a+b}{1-ab}$, which was part of that example, always hold? Explain your answer.

35. Show that $\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{4}{7}$.

36. Show that $\tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \tan^{-1} \frac{1}{2}$.

37. Figure 5.3.13 shows three equal squares lined up against each other. For the angles α, β, and γ in the picture, show that $\alpha = \beta + \gamma$. (*Hint: Consider the tangents of the angles.*)

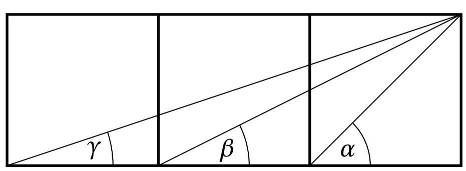


Figure 5.3.13 Exercise 37

38. Sketch the graph of $y = \sin^{-1} 2x$.

39. Write a computer program to solve a triangle in the case where you are given three sides. Your program should read in the three sides as input parameters and print the three angles in degrees as output if a solution exists. Note that since most computer languages use radians for their inverse trigonometric functions, you will likely have to do the conversion from radians to degrees yourself in the program.

Below is the result of compiling and running the program using $x_0 = 0$ and $x_1 = 1$:

```
javac secant.java
java secant 0 1
x2 = 0.6850733573260451
x3 = 0.736298997613654
x4 = 0.7391193619116293
x5 = 0.7390851121274639
x6 = 0.7390851332150012
x7 = 0.7390851332151607
x8 = 0.7390851332151607
x = 0.73908513321516067229310920083662495017051696777344
```

Notice that the program only got up to x_8, not x_{10}. The reason is that the difference between x_8 and x_7 was small enough (less than $\epsilon_{error} = 1.0 \times 10^{-50}$) to stop at x_8 and call that our solution. The last line shows that solution to 50 decimal places.

Does that number look familiar? It should, since it is the answer to Exercise 11 in Section 4.1. That is, when taking repeated cosines starting with any number (in radians), you eventually start getting the above number repeatedly after enough iterations. This turns out not to be a coincidence. Figure 6.2.2 gives an idea of why.

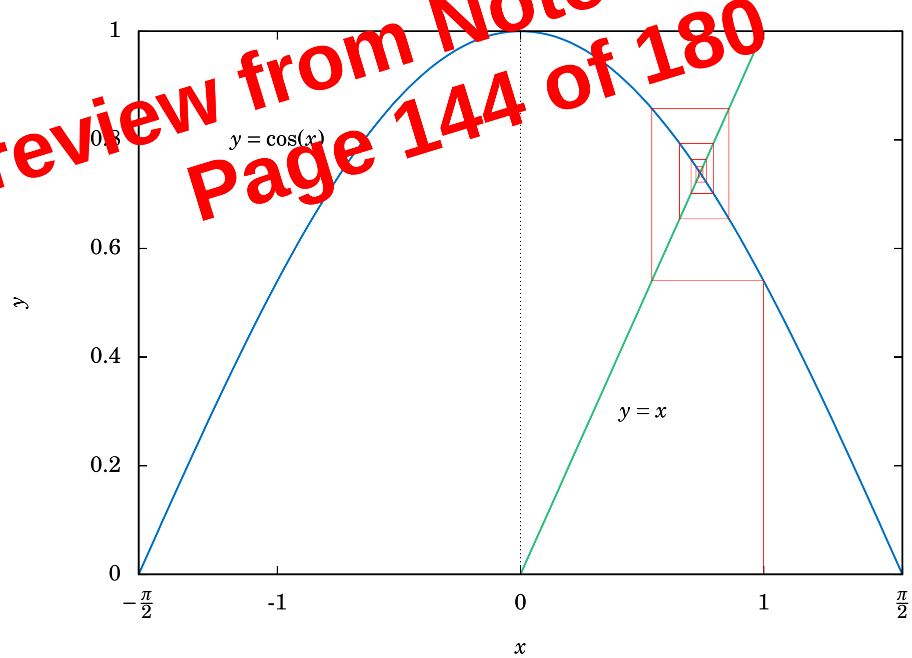


Figure 6.2.2 Attractive fixed point for $\cos x$

Since $x = 0.73908513321516...$ is the solution of $\cos x = x$, you would get $\cos(\cos x) = \cos x = x$, so $\cos(\cos(\cos x)) = \cos x = x$, and so on. This number x is called an *attractive fixed*

Example 6.10

Represent the complex number $-2-i$ in trigonometric form.

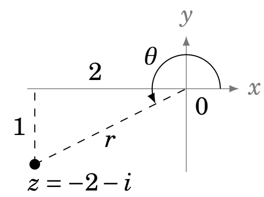


Figure 6.3.2

Solution: Let $z = -2-i = x+yi$, so that $x = -2$ and $y = -1$. Then θ is in QIII, as we see in Figure 6.3.2. So since $\tan\theta = \frac{y}{x} = \frac{-1}{-2} = \frac{1}{2}$, we have $\theta = 206.6°$. Also,

$$r = \sqrt{x^2+y^2} = \sqrt{(-2)^2+(-1)^2} = \sqrt{5}\,.$$

Thus, $\boxed{-2-i = \sqrt{5}\,(\cos 206.6° + i\sin 206.6°)}$, or $\sqrt{5}\,\mathrm{cis}\,206.6°$.

For complex numbers in trigonometric form, we have the following formulas for multiplication and division:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be complex numbers. Then

$$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))\,, \text{ and} \tag{6.5}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \quad \text{if } z_2 \neq 0. \tag{6.6}$$

The proofs of these formulas are straightforward:

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\,\cos\theta_2 - \sin\theta_1\,\sin\theta_2) + i(\sin\theta_1\,\cos\theta_2 + \cos\theta_1\,\sin\theta_2)] \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

by the addition formulas for sine and cosine. And

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\ &= \frac{r_1}{r_2}\cdot\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}\cdot\frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2} \\ &= \frac{r_1}{r_2}\cdot\frac{(\cos\theta_1\,\cos\theta_2 + \sin\theta_1\,\sin\theta_2) + i(\sin\theta_1\,\cos\theta_2 - \cos\theta_1\,\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2} \\ &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \end{aligned}$$

by the subtraction formulas for sine and cosine, and since $\cos^2\theta_2 + \sin^2\theta_2 = 1$. **QED**

Note that formulas (6.5) and (6.6) say that when multiplying complex numbers the moduli are multiplied and the arguments are added, while when dividing complex numbers the moduli are divided and the arguments are subtracted. This makes working with complex numbers in trigonometric form fairly simple.

Example 6.20

Prove that the distance d between two points (r_1,θ_1) and (r_2,θ_2) in polar coordinates is

$$d \;=\; \sqrt{r_1^2 \,+\, r_2^2 \,-\, 2r_1r_2\cos(\theta_1-\theta_2)}\;. \tag{6.11}$$

Solution: The idea here is to use the distance formula in Cartesian coordinates, then convert that to polar coordinates. So write

$$\begin{aligned} x_1 &= r_1\cos\theta_1 & y_1 &= r_1\sin\theta_1 \\ x_2 &= r_2\cos\theta_2 & y_2 &= r_2\sin\theta_2\;. \end{aligned}$$

Then (x_1,y_1) and (x_2,y_2) are the Cartesian equivalents of (r_1,θ_1) and (r_2,θ_2), respectively. Thus, by the Cartesian coordinate distance formula,

$$\begin{aligned} d^2 &= (x_1-x_2)^2 + (y_1-y_2)^2 \\ &= (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2 \\ &= r_1^2\cos^2\theta_1 - 2r_1r_2\cos\theta_1\cos\theta_2 + r_2^2\cos^2\theta_2 + r_1^2\sin^2\theta_1 - 2r_1r_2\sin\theta_1\sin\theta_2 + r_2^2\sin^2\theta_2 \\ &= r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2) - 2r_1r_2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) \\ d^2 &= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1-\theta_2)\,, \end{aligned}$$

so the result follows by taking square roots of both sides.

In Example 6.17 we saw that the equation $x^2 + y^2 = 9$ in Cartesian coordinates could be expressed as $r = 3$ in polar coordinates. This equation describes a circle centered at the origin, so the circle is symmetric about the origin. In general, polar coordinates are useful in situations when there is symmetry about the origin (though there are other situations), which arise in many physical applications.

Exercises

For Exercises 1-5, convert the given point from polar coordinates to Cartesian coordinates.

1. $(6,210°)$ **2.** $(-4,3\pi)$ **3.** $(2,11\pi/6)$ **4.** $(6,90°)$ **5.** $(-1,405°)$

For Exercises 6-10, convert the given point from Cartesian coordinates to polar coordinates.

6. $(3,1)$ **7.** $(-1,-3)$ **8.** $(0,2)$ **9.** $(4,-2)$ **10.** $(-2,0)$

For Exercises 11-18, write the given equation in polar coordinates.

11. $(x-3)^2 + y^2 = 9$ **12.** $y = -x$ **13.** $x^2 - y^2 = 1$ **14.** $3x^2 + 4y^2 - 6x = 9$

15. Graph the function $r = 1 + 2\cos\theta$ in polar coordinates.

In Linux you would do this:

```
set terminal wxt enhanced
```

You can then (provided the Symbol font is installed, which it usually is) set the x-axis to have multiples of $\pi/2$ from 0 to 2π as labels with this command (all on one line):

```
set xtics ('0' 0,'{/Symbol p}/2' pi/2,'{/Symbol p}' pi,'3{/Symbol p}/2' 3*pi/2,
'2{/Symbol p}' 2*pi)
```

In the above example, to also plot the function $y = \cos 2x + \sin 3x$ on the same graph, put a comma after the first function then append the new function:

```
plot [0:2*pi] sin(x), cos(2*x) + sin(3*x)
```

By default, the x-axis is not shown in the graph. To display it, use this command *before* the **plot** command:

```
set zeroaxis
```

Also, to label the axes, use these commands:

```
set xlabel "x"
set ylabel "y"
```

The default sample size for plots is 100 units, which can result in jagged edges if the curve is complicated. To get a smoother curve, increase the sample size (to, say, 1000) like this:

```
set samples 1000
```

Putting all this together, we get the following graph: