Unit 2 - 1.1 Polynomials

Polynomial:

An expression of the form:

$a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots a_3x^3 + a_2x^2 + a_1x + a_0$

with $a_0, \ldots. a_n$ constants and $a_n \neq 0$
is called a polynomial of degree n
(the highest power of x is the degree)

Division – some terms:

Divisor – what you are dividing by
Dividend – the number you are dividing into
Quotient – how many times the divisor goes into the dividend
Remainder – What is left over.

$$\begin{array}{r|l} & \text{Quotient} \quad \text{r. Remainder} \\ \hline \text{Divisor} & \text{Dividend} \end{array}$$

Division of polynomials
– Nested or synthetic division

Dividing f(x) by x – h

Note: the divisor **MUST** be in the form x – h

Example:

Find the quotient and remainder
when $x^3 + 6x^2 + 3x – 15$ is divided by x - 3

Follow the working opposite.

The quotient is $x^2 + 9x + 30$ and the **remainder is 75**

It should also be noted that the remainder when the divisor is x – 3 is f(3)

To illustrate this $f(3) = 3^3 + 6(3^2) + 3(3) – 15$
$= 27 + 54 + 9 – 15 = 75$

If we divided the quadratic f(x) by x – h then the remainder would be f(h)

Example of synthetic (nested) division:

Find the quotient and remainder when $x^3 + 6x^2 + 3x – 15$ is divided by x - 3

Write down the coefficients of the polynomial
– taking care to put a 0 where a power of x is missing

	A	**B**	**C**	**D**	
3	1	6	3	-15	1
	↓	3	27	90	2
	1	9	30	75	3

The shaded row and column are used there only to refer to the table for explanation.

Step 1. Put the contents of A1 straight down to A3
Step 2. Multiply A3 by the divisor and put result in B2
Step 3. Add B1 to B2 and put result in B3

Step 4. Multiply B3 by the divisor and put result in C2
Step 5. Add C1 to C2 and put result into C3

Step 6. Multiply C3 by divisor and put result into D2
Step 7. Add D1 and D2 and put result into D3

A3, B3, C3 are the coefficients of the quotient and D3 is the remainder.

Example:

Find the quotient and remainder when
$x^3 + 6x^2 + 3x – 15$ is divided by x + 3

Note we **must** make x + 3 into **x – (-3)**

The **quotient is $x^2 + 3x - 6$** and the **remainder is 3**

-3	1	6	3	-15
	↓	-3	-9	18
	1	3	-6	3

Example:

Find the quotient and remainder when
$2x^3 + 3x^2 - 5x + 3$ is divided by 2x + 1

Again we have to arrange the divisor into the form **x – h**

2x + 1 is the same as 2(x + ½) or **2(x – (-½))**

ignore the factor of 2 at the moment – we will deal with that separately.

-½	2	3	-5	3
	↓	-1	-1	3
	2	2	-6	6

The **quotient is $2x^2 + 2x - 6$** and the **remainder is 6**

However we now have to **divide the quotient by the factor of 2** that we took out.

So, the **quotient becomes: $x^2 + x - 3$** and the **remainder is 6**

NB we do NOT divide the remainder as well.

Unit 2 - 1.1 Polynomials

The Remainder Theorem

When any polynomial f(x) is divided by x – h the remainder is given by f(h)

We can find f(h) directly to obtain the remainder,

or we can use synthetic division.

The Factor Theorem

If the remainder when dividing a polynomial f(x) by x – h is 0 then x – h is a factor of f(x)

i.e. if f(h) = 0 then x – h is a factor.

This allows us to find factors of polynomials of any degree. Once we have a factor, we can divide by the factor using synthetic division, and obtain another polynomial of degree one less.

We can then repeat the process to obtain another factor, if one exists.

Using the Factor Theorem.

The easiest way to use the factor Theorem is as follows:

1. Look at the factors of the constant term in the polynomial f(x) these are the only possible values for h
2. Evaluate f(h) until you find f(h) = 0 and then you have a factor.
3. Once you have a factor, divide the polynomial by it using synthetic division and obtain the polynomial quotient which is of degree one less.
4. Repeat the process until you can find no more factors.

This is a follow on from the Remainder Theorem and is perhaps more important and certainly useful.

Example: Find the factors of : $f(x) = 2x^3 - 11x^2 + 17x - 6$

possible values for h are ±1, ±2, ±3, ±6,

Try h = 1

f(h) = f(1) = 2 – 11 + 17 – 6 = 2 this is not zero so (x – 1) is not a factor

Try h = -1

f(h) = f(-1) = -2 -11 -17 – 6 = -36 this is not zero so (x + 1) is not a factor

Try h = 2

f(h) = f(2) = 16 – 44 +34 – 6 = 0 so (x – 2) is a factor

Now obtain the quotient:

2	**2**	**-11**	**17**	**-6**
	↓	**4**	**-14**	**6**
	2	**-7**	**3**	**0**

Quotient is $2x^2 - 7x + 3$ so polynomial is **$(x - 2)(2x^2 - 7x + 3)$**

Now factorise the quadratic factor using two brackets:

$2x^2 - 7x + 3 \Rightarrow (2x - 1)(x - 3)$

Hence: $\mathbf{f(x) = 2x^3 - 11x^2 + 17x - 6}$ factorises to $\mathbf{f(x) = (x - 2)(2x - 1)(x - 3)}$

Solving Polynomial Equations

If h is a root of the equation f(x) = 0 then (x – h) is a factor of f(x) and so f(h) = 0

Recall the graph of f(x) a root is where f(x) crosses the x axis – in other words f(x) = 0

Consequently the value of x, at which f(x) = 0, is a root of the equation f(x) = 0

So if we can find h such that f(h) = 0 then we have a root of the equation f(x) = 0

Example: solve the equation $x^3 - 2x^2 - x + 2 = 0$

First find a factor – try possible values: ±1, ±2

f(1) = 1 – 2 – 1 + 2 ⇒ 0 so (x – 1) is a factor

Use synthetic division to divide f(x) by the factor

1	**1**	**-2**	**-1**	**2**
	↓	**1**	**-1**	**-2**
	1	**-1**	**-2**	**0**

hence: $x^3 - 2x^2 - x + 2 = 0$ factorises to $(x - 1)(x^2 - x - 2) = 0$

now factorise the quadratic part to get: $(x - 1)(x - 2)(x + 1) = 0$

Hence solutions of the equation: $x^3 - 2x^2 - x + 2 = 0$

are: x = 1, x = 2 and x = -1

Unit 2 - 3.1 Calculations in 2 and 3 dimensions

Three Dimensions

We live in a 3 dimensional world – a world of length, breadth and height.

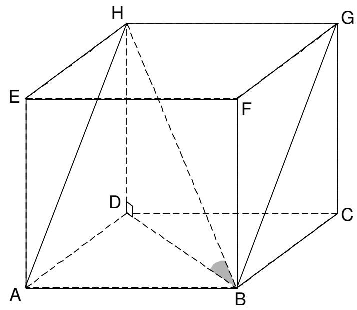
We can use the rules listed above, by applying them to 2-dimensional planes within the 3 dimensional solid.

(i) Angle between a line and a plane.

To find the angle between HB and the plane ABCDuse the perpendicular HD and form a right angled triangle Δ HDB

$\angle$ HBD is the required angle

Calculations may involve Pythagoras and SOH-CAH-TOA



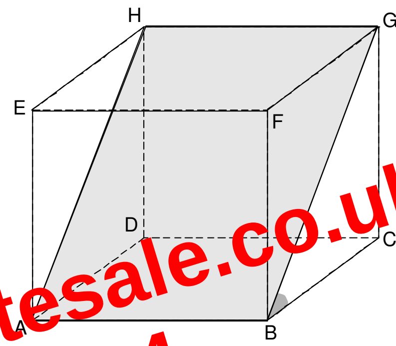
(ii) Angle between two planes.

To find the angle between planes ABGH and ABCD find their line of intersection AB.

Then a line in each plane perpendicular to AB, in this diagram, (BC and BG).

$\angle$ CBG is the required angle.

$\angle$ DAH would also do



Some terminology:

Face diagonal – this is a diagonal across a face. e.g. AH, ED, EC, FH etc.

Space diagonal – this is a diagonal linking two vertices which are not in the same face.

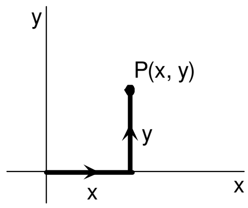
e.g. BH, AG, EC, DF

To find lengths of diagonals, calculations may involve Pythagoras and SOH-CAH-TOA.

Co-ordinates in 2 and 3 dimensions

To fix the position of a point on a plane (2 dimensions), you need two axes OX and OY.

P is the point (x, y)

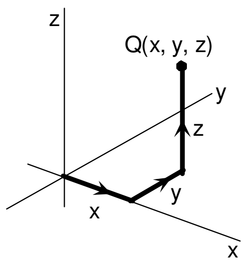


To fix the position of a point in space (3 dimensions), you need 3 axes – OX, OY and OZ.

and three co-ordinates – x, y and z

Q is the point (x, y, z)

In 3 dimensions, we usually show the **z direction vertically.**



Unit 2 - 4 The Circle

The circle – centre O(0, 0) and radius r

$\mathbf{x^2 + y^2 = r^2}$

The equation of a circle is given by the locus of Point P

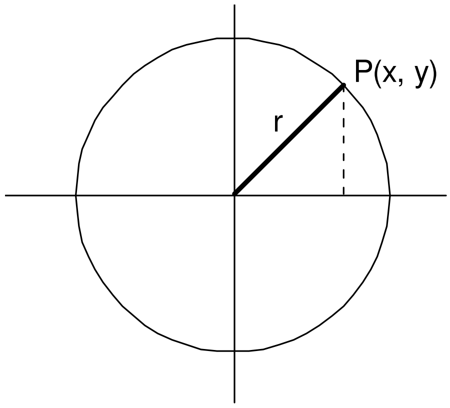
which describes a path at a constant distance r from the origin.

We need to find a relationship between x and y that satisfies this condition.

By Pythagoras: $x^2 + y^2 = r^2$

Hence the equation of the circle is:

$$x^2 + y^2 = r^2$$



Application:

Given the equation of a circle in the form

$x^2 + y^2 = r^2$

we can write down the radius.

Example: the radius of the circle: $x^2 + y^2 = 64$ is $\mathbf{r = 8}$

Example: the radius of the circle: $3x^2 + 3y^2 = 48$

first divide by 3 to get the form $x^2 + y^2 = r^2$

$x^2 + y^2 = 16$ so $\mathbf{r = 4}$

Application:

If we know that the circle is centred on the origin and passes through a given point, we can find its equation:

Example:

Find the equation of the circle centre O passing through P(3, 4)

using the distance formula, we can calculate OP as 5

This is the radius of the circle.

Hence $x^2 + y^2 = 25$

Application:

We can check that a point lies on a circle – if it does then it will satisfy the equation of the circle:

Example: Does the point R(12, -9) lie on the circle $x^2 + y^2 = 225$

LHS	RHS
$x^2 + y^2$	225
144 + 81	
225	

Since LHS = RHS, point R satisfies the equation, so R lies on the circle.

Alternative method:

If the point R(12, -9) lies on the circle, then OR will be equal to the radius of the circle (which is 15).

Using the distance formula we find that OR = 15, so R lies on the circle.

Example:

Find p if (p, 3) lies on the circle $x^2 + y^2 = 13$

(p, 3) must satisfy the equation of the circle, so:

$$p^2 + 3^2 = 13 \Rightarrow p^2 = 13 - 9 \Rightarrow p^2 = 4$$

so $p = \pm 2$

Example:

Does the point Q(7, -4) lie on the circle $x^2 + y^2 = 64$

The distance OQ (by the distance formula = $\sqrt{65}$)

This is larger than the radius of the circle,

so Q does NOT lie on the circle