Unit 3 - 3 The Exponential and Logarithmic Functions

The exponential function

An exponential function is of the form

$$a^x$$

where a is a constant.

If $a > 0$, the function is increasing (growth)
If $a < 0$, the function is decreasing (decay)

a may take any positive value
depends on situation function is modelling.

Note: In general an exponential function will take the form:

$$A(x) = ab^x$$

where both a and b are constants.

a will represent an initial value
b will represent the multiplier
x will represent the variable

A special exponential function ~ e^x

$$e^x$$

e is a special constant
– a never ending decimal like π.

e = 2.718 282 828 ….

The number e crops up on many occasions in the natural world.

It is: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

You can find this by pressing the e^x key on your calculator followed by ‘1 =’

This effectively is evaluating e^1.

Linking the exponential and logarithmic functions.

$y = a^x \Leftrightarrow \log_a y = x$

$1 = a^0 \Leftrightarrow \log_a 1 = 0$

$a = a^1 \Leftrightarrow \log_a a = 1$

Use this relationship to switch between log and exponential forms.

Use these two relationships to simplify and evaluate logarithmic and exponential functions and expressions.

Examples:	Solutions:
1. Write in log form: $81 = 3^4$	1. $\log_3 81 = 4$
2. Write in log form: $y^4 = 20$	2. $\log_y 20 = 4$
3. Write in log form: $^1/_9 = 3^{-2}$	3. $\log_3 {}^1/_9 = -2$
4. Write in log form: $z^{½} = 10$	4. $\log_z 10 = ½$
5. Write in exp. form: $\log_2 4 = 2$	5. $2^2 = 4$
6. Write in exp. form: $\log_{10} 100 = 2$	6. $10^2 = 100$
7. Write in exp. form: $\log_9 3 = ½$	7. $9^{½} = 3$
8. Write in exp. form: $\log_8 4 = {}^2/_3$	8. $8^{2/3} = 4$ i.e. $(\sqrt[3]{8})^2 = 4$
9. Write in exp. form: $\log_a c = b$	9. $a^b = c$
10. Solve: $\log_x 9 = 2$	10. $x^2 = 9$ so $x = 3$
11. Solve: $\log_4 x = 0.5$	11. $4^{0.5} = x$ so $4^{½} = x$ $\sqrt{4} = x$ $x = 2$
12. Solve: $\log_3 81 = x$	12. $3^x = 81$ so $x = 4$
13. Solve: $\log_x 7 = 1$	13. $x^1 = 7$ $x = 7$
14. Solve: $\log_{10} x = 0.5$	14. $10^{0.5} = x$ Use calculator 10 y^x 0.5 = 3.162… x = 3.16 (2 d.p)

Example

Six spherical sponges were dipped in water and weighed to see how much water each could absorb.
The diameter (x millimetres) and gain in weight (y grams) were measured and recorded for each sponge.

It is thought that x and y are connected by a relationship of the form $y = ax^b$

By taking logarithms of the values of x and y, this table was constructed.

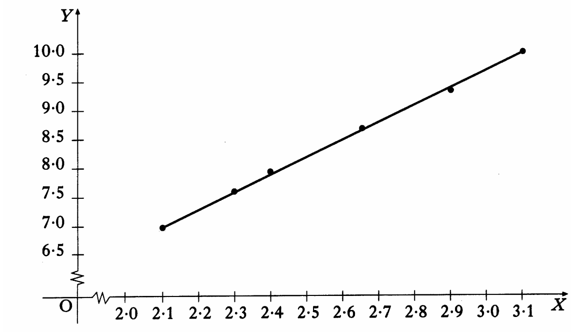
$X(=\log_e x)$	2.10	2.31	2.40	2.65	2.90	3.10
$Y(=\log_e y)$	7.00	7.60	7.92	8.70	9.38	10.00

A graph was drawn and is shown here.

a) Find the equation of the line in the form $Y = mX + c$

b) Hence find the values of the constants a and b

in the relationship $y = ax^b$



Solution:

a)

$y = ax^b$

$\log_e y = \log_e ax^b$

$\log_e y = \log_e a + \log_e x^b$

$\log_e y = \log_e a + b\log_e x$

This is of the form Y = mX + c where m = b and $\log_e a = c$

b) Choose two points on the line of best fit. (2.1, 7.0) and (3.1, 10.0)

Substitute into $\log_e y = \log_e a + b\log_e x$

giving: $7.0 = \log_e a + 2.1\,b$ ….. (1)

$10.0 = \log_e a + 3.1\,b$ ….. (2)

subtracting: (2) – (1) $\Rightarrow$ 3.0 = b substituting $\Rightarrow$ $\log_e a = 0.7$ so $a = e^{0.7}$ a = 2.01…

Hence relationship is: $y = 2x^3$ i.e. a = 2.0 and b = 3.0 (1 d.p.)

Note: You should be confident in applying the method in part (b) rather than relying on the gradient and y-intercept, as in this case, you cannot determine the y-intercept.

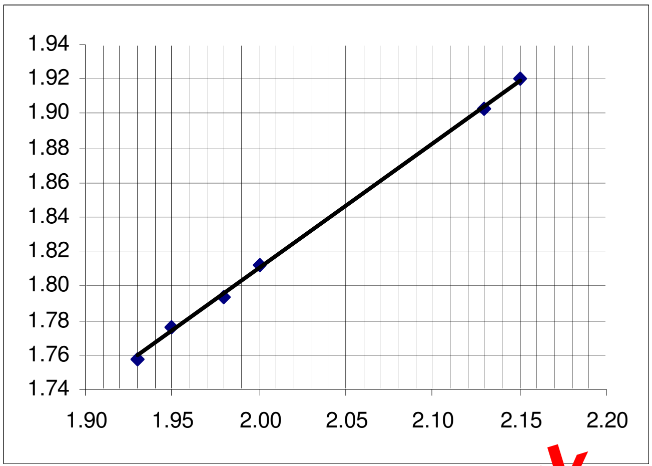
Example

Find the relation $y = ab^x$ for this data

x	2.15	2.13	2.00	1.98	1.95	1.93
y	83.33	79.93	64.89	62.24	59.70	57.26

Solution:

$$y = ab^x$$
$$\log_{10} y = \log_{10} ab^x$$
$$\log_{10} y = \log_{10} a + \log_{10} b^x$$
$$\log_{10} y = \log_{10} a + x \log_{10} b$$



Add a row to the table showing $\log_{10}$ y

Plot data **$\log_{10}$ y** against **x**
(because relationship is exponential)
to determine **line of best fit** which will indicate which points to use.

x	**2.15**	**2.13**	**2.00**	**1.98**	**1.95**	**1.93**
y	83.33	79.93	64.89	62.24	59.70	57.26
$\log_{10}$ y	**1.92**	**1.90**	**1.81**	**1.79**	**1.78**	**1.76**

From graph, choose points (1.93, 1.76) and (2.15, 1.92) corresponding to (x, $\log_{10}$ y)

Substituting into $\log_{10} y = \log_{10} a + x \log_{10} b$

gives: $1.92 = \log_{10} a + 2.15 \log_{10} b$ ….. (1)

and: $1.76 = \log_{10} a + 1.93 \log_{10} b$ ….. (2)

Subtracting: (1) – (2) $0.16 = 2.15 \log_{10} b - 1.93 \log_{10} b$

$0.16 = 0.22 \log_{10} b$

$\log_{10} b = 0.727$

$b = 10^{0.727} = 5.3$ (1 d.p.)

Substituting into (1) $\Rightarrow$ $\log_{10} a = 1.92 - 2.15 \log_{10} 5.3$

$\log_{10} a = 1.92 - 1.56$

$\log_{10} a = 0.36$

$a = 10^{0.36} = 2.29 = 2.3$ (1 d.p.)

Hence relationship is: $y = 2.3(5.3)^x$

Unit 3 - 4	The Wave Function a cos x + b sin x

Examples:

1. Solve for $0 \leq x \leq 180$ $\quad 6\cos(3x + 60) - 3 = 0$

$6\cos(3x + 60) = 3$

$\cos(3x + 60) = 0.5$ so, acute $(3x + 60) = 60°$

The range for x is: $0 \leq x \leq 180$ so the range for 3x is: $0 \leq x \leq 540$

The cosine is positive, so the required quadrants are 1st, 4th and 5th (1st quadrant – second time around)

$\therefore\ 3x + 60 = 60 \qquad 3x + 60 = 360 - 60 \qquad 3x + 60 = 360 + 60$

$\therefore$ **x = 0°, 80° or 120°**

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2. i) Express $\sqrt{3}\cos x - \sin x$ in the form $k\sin(x - \alpha)$
ii) and hence solve the equation $\sqrt{3}\cos x - \sin x = 0$ for $0 \leq x \leq 360$

i) $k\sin(x - \alpha) = k\sin x\cos\alpha - k\cos x\sin\alpha$

comparing coefficients:	$-k\sin\alpha = \sqrt{3}$	$k\sin\alpha = -\sqrt{3}$	… (1)
	$k\cos\alpha = -1$	$k\cos\alpha = -1$	… (2)
squaring and adding:	$k^2 = (\sqrt{3})^2 + 1^2$	$k^2 = 3 + 1 = 4$	$k = 2$
dividing:	$\tan\alpha = \sqrt{3}$	acute $\alpha = 60°$	
from (1) and (2)	$\sin\alpha$ and $\cos\alpha$ both negative, so α lies in 3rd quadrant		
	$\therefore\ \alpha = 180 + 60° = 240°$		

Hence: $\mathbf{\sqrt{3}\cos x - \sin x = 2\sin(x - 240)}$

ii) Using $2\sin(x - 240) = 0$ $\quad \sin(x - 240) = 0$ $\quad (x - 240) = -180°,\ 0°,\ 180°$, or $360°$

$\therefore$ **x = 60° or x = 240°**

(because we are adding 240°, we need to make sure we cover all the range, so we need to consider the solution –180° as well, we do not need to go any further back, since we would be then out of the range)

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3. Using $R\cos(2x - \alpha)$, find the maximum and minimum values of : $4\cos 2x + 3\sin 2x + 5$
and the corresponding values for x in $0 \leq x \leq 2\pi$.

$R\cos(2x - \alpha) = R\cos 2x\cos\alpha + R\sin 2x\sin\alpha$

compare coefficients:	$R\sin\alpha = 3$		
	$R\cos\alpha = 4$		
squaring and adding:	$R^2 = 3^2 + 4^2$	$R^2 = 25$	$R = 5$
dividing:	$\tan\alpha = ¾$	acute $\alpha = 0.643$ rad	

$\sin\alpha$ and $\cos\alpha$ both positive, so α is in first quadrant,

Hence: $4\cos 2x + 3\sin 2x + 5$ can be expressed as: $5\cos(2x - 0.643) + 5$

Maximum value is: **10** when $(2x - 0.643) = 0,\ 2\pi$, or 4π (since we have 2x and not x)
when x = 0.32 rad, 3.46 rad (6.60 rad – discard – out of range)

Minimum value is: **0** when $(2x - 0.643) = \pi$ or 3π (since we have 2x and not x)
when x = 1.89 rad or 5.03 rad.