

$$\text{Sol 1) } \nabla \times \nabla \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

= mixed partial derivatives  
= 0

$\therefore \nabla \times \nabla \phi = 0$   $\therefore$  irrotational.

$$\text{Sol 2) } \vec{v} = (xyz^2)\mathbf{i} + (3xz^2y)\mathbf{j} + (xz^2 - y^2z)\mathbf{k}$$

$$\text{div } v = \nabla \cdot v$$

$$= yz + 3xz^2 + (2xz - y^2)$$

$$\text{Sub } (2, -1, 1) \rightarrow$$

b) find grad of  $(x^3 + y^3 + z^3 - 3xyz)$   $\frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z}$  at (2)

2b) find  $\nabla \cdot \vec{F}$  and  $\text{Curl } \vec{F}$  of  $F = (x^3 + y^3 + z^3 - 3xyz)$

$$\vec{F} = \mathbf{i}(3x^2 + y^3 + z^3 - 3yz) + \mathbf{j}(3xy^2 + z^3 - 3xz) + \mathbf{k}(3y^2 - 3xz)$$

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 6x + 6y + 6z$$

$$\nabla \cdot \vec{F} \text{ at } (2, -1, 1)$$

$$\therefore \nabla \cdot \vec{F} = 12 - 6 + 6$$

$$= 12 \leftarrow$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \nabla \times \nabla \phi$$

$$= 0 \quad \therefore \vec{F} \text{ is irrotational.}$$

Check: For  $\nabla \cdot \vec{F} =$

$$\begin{vmatrix} + & - & + \\ i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$$= i \left[ \frac{d}{dy} (3z^2 - 3xy) - \frac{d}{dz} (3y^2 - 3xz) \right] - j \left[ \frac{d}{dx} (3z^2 - 3xy) - \frac{d}{dz} (3x^2 - 3yz) \right] + k \left[ \frac{d}{dx} (3y^2 - 3xz) - \frac{d}{dy} (3x^2 - 3yz) \right]$$

$$= i[-3x + 3z] - j(-3y + 3y) + k(-3z + 3z)$$

$$= 0j + 0j + 0k \leftarrow$$

$\nabla \cdot \vec{F} = 0 \therefore F$  is irrotational vector.

3a)  $(-3x^2z + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$   
 is Solenoidal  $\therefore \nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = -6xz + 6xz = 0 \text{ solenoidal.}$$

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### Integration of vector functions

$$\frac{d}{dt} [\vec{F}(t)] = \vec{f}(t)$$

int on both sides:

$$\int \frac{d}{dt} [\vec{F}(t)] dt = \int \vec{f}(t) dt$$

$$\vec{F}(t) + c = \int \vec{f}(t) dt$$

If  $t$  between  $t_1$  &  $t_2$  then

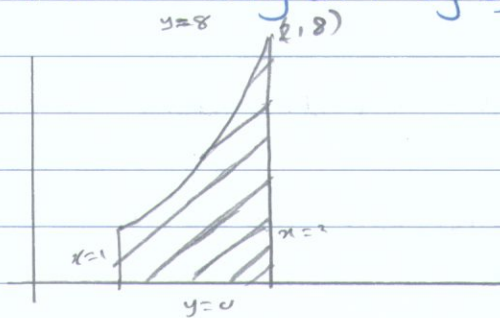
$$\int_{t_1}^{t_2} \vec{f}(t) dt = (\vec{F}(t) + c)_{t_1}^{t_2}$$

$$= \vec{F}(t_2) + c - \vec{F}(t_1) - c$$

$$\boxed{\int_{t_1}^{t_2} \vec{f}(t) dt = \vec{F}(t_2) - \vec{F}(t_1)}$$

prob (3)  $F = 5xy\mathbf{i} + 2y\mathbf{j}$ , evaluate  $\int_C F \cdot d\mathbf{r}$   $C$  is the part of the curve  $y = x^3$  between  $x=1$  to  $x=2$   
 $y=1$   $y=8$

Sol:  $F = 5xy\mathbf{i} + 2y\mathbf{j}$   
 $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$   
 $F \cdot d\mathbf{r} = 5xy\,dx + 2y\,dy$



$$y = x^3$$

$$dy = 3x^2 dx$$

$$\begin{aligned} \int_C F \cdot d\mathbf{r} &= \int_1^2 (5xy\,dx + 2y\,dy) \\ &= \int_1^2 (5x \cdot x^3\,dx + 2x^3 \cdot 3x^2\,dx) \\ &= \int_1^2 (5x^4 + 6x^5)\,dx \\ &= \int_1^2 5x^4\,dx + \int_1^2 6x^5\,dx \\ &= 94 \leftarrow \end{aligned}$$

prob (4) If  $F = (3x^2 + 6y)\mathbf{i} - 14yz\mathbf{j} + 20xz^2\mathbf{k}$ , evaluate  $\int_C F \cdot d\mathbf{r}$  from  $(0,0,0)$  to  $(1,1,1)$  along curve  $x=t$ ,  $y=t^2$ ,  $z=t^3$

Sol:  $F \cdot d\mathbf{r} = (3x^2 + 6y)dx - 14yz\,dy + 20xz^2\,dz$

$$x=t, \quad y=t^2, \quad z=t^3$$

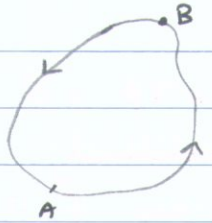
$$dx=dt, \quad dy=2t\,dt, \quad dz=3t^2\,dt$$

$$\begin{aligned} F \cdot d\mathbf{r} &= (3x^2 + 6y)dx - 14yz\,dy + 20xz^2\,dz \\ &= (3t^2 + 6t^2)dt - (14t^2 \cdot t^3 \cdot 3t^2)dt + 20 \cdot t \cdot t^6 \cdot 3t^2\,dt \\ &= (9t^2\,dt - 28t^7 + 60t^9)dt \\ \int_C F \cdot d\mathbf{r} &= \int_0^1 (9t^2\,dt - 28t^7 + 60t^9)dt \end{aligned}$$

$$= 5 \leftarrow$$

- Work done by force.

$$\vec{F}(x, y, z)$$



- Total work done by force is  $\int_A^B \vec{F} \cdot d\vec{r}$

Conservative vector field:

$\int_C \vec{F} \cdot d\vec{r}$  is independent of path iff  $\vec{F} = \nabla\phi$  → if and only if

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla\phi \cdot d\vec{r}$$

$$= \int_C \left( i \frac{\partial\phi}{\partial x} + j \frac{\partial\phi}{\partial y} + k \frac{\partial\phi}{\partial z} \right) \cdot (dx i + dy j + dz k)$$

$$= \int_C \left( \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \right)$$

$$\Rightarrow \int_C d\phi = [\phi]_A^B$$

$$= \phi(B) - \phi(A) \quad \leftarrow$$

Note: the total differential  $d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz$

$$\text{RHS: of } \textcircled{1} : \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dy dx = -6y - (-16y) dy dx$$

$$= \iint_R (-6y + 16y) dy dx$$

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=1-y} 10y dx dy$$

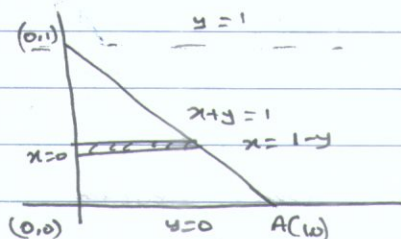
$$= \int_0^1 10y (x)_0^{1-y} dy$$

$$= \int_0^1 10y (1-y) dy$$

$$= \int_0^1 10(y - y^2) dy$$

$$= 10 \left( \frac{1}{2} - \frac{1}{3} \right)$$

$$= \frac{5}{3} \leftarrow$$

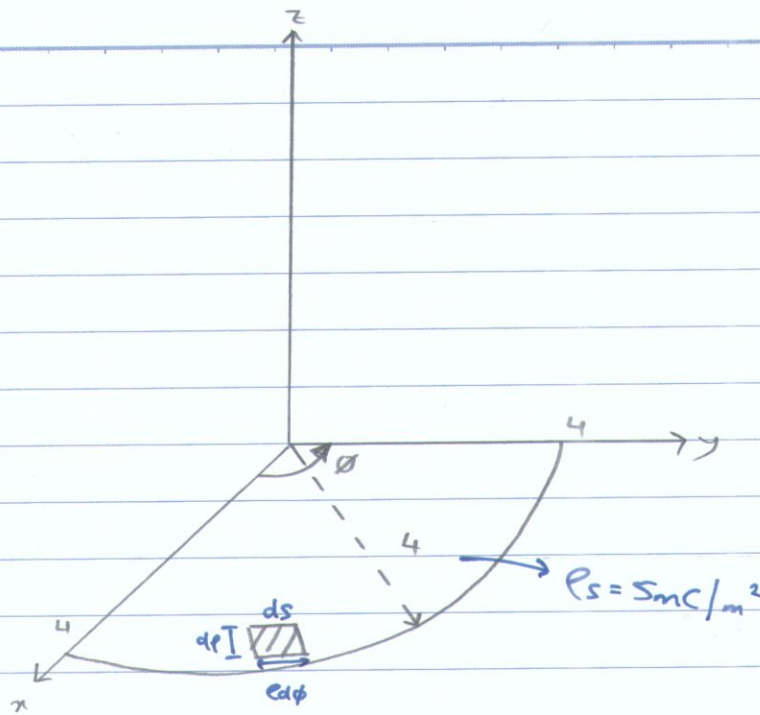


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Ex2) Verify that Green theorem for:  $\int_C (xy + y^2) dx + x^2 dy$  where  $C$  is area between  $y = x^2$  and  $y = x$ .

$$\int_C (P dx + Q dy) = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \rightarrow \textcircled{1}$$

$$Q_B =$$



$$Q_B = \iint_S \rho_s \, ds$$

$$ds = r \, d\rho \, d\phi$$

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$$Q_B = \int_{\rho=0}^4 \int_{\phi=0}^{\frac{\pi}{2}} \rho \, d\rho \, d\phi$$

$$= 5 \int_0^4 \rho \, d\rho \int_0^{\frac{\pi}{2}} d\phi$$

$$= 5 \cdot \left( \frac{\rho^2}{2} \right)_0^4 \cdot \left( \phi \right)_0^{\frac{\pi}{2}}$$

$$= 5 \cdot \left( \frac{16}{2} \right) \cdot \frac{\pi}{2}$$

$$= 62.83 \, \text{mC} \leftarrow$$

$$E_T = E_{Q_B} + E_{Q_C} + E_{Q_D}$$

$$= \frac{Q_A Q_B}{4\pi \epsilon_0 |R_1|^3} + \frac{Q_A Q_C}{4\pi \epsilon_0 |R_2|^3} + \frac{Q_A Q_D}{4\pi \epsilon_0 |R_3|^3}$$

$$R_1 = (2ax)$$

$$R_2 = (ax - ay)$$

$$R_3 = (ax + ay)$$

$$\therefore |R_1| = 2$$

$$|R_2| = \sqrt{2}$$

$$|R_3| = \sqrt{2}$$

$$E = \frac{50n \cdot 50n}{4\pi \epsilon_0 (2)^3} (2ax) + \frac{50n \cdot 50n}{4\pi \epsilon_0 (2)^{\frac{3}{2}}} (ax - ay) + \frac{50n \cdot 50n}{4\pi \epsilon_0 (2)^{\frac{3}{2}}} (ax + ay)$$

$$= (56.25 \cdot 2ax) + 159.1 ax - 159.1 ay + 159.1 ax + 159.1 ay$$

$$= 112.5 ax + 159.1 ax + 159.1 ax - 159.1 ay + 159.1 ay$$

$$= 430.7 ax + 0 ay$$

$$= \cancel{430.7 ax} \text{ mN} - (2.813 \times 10^{-6} ax) \text{ N}$$

$$= \frac{50n \cdot 50n (2ax)}{4\pi \epsilon_0 2^3} + \frac{50n \cdot 50n (ax)}{4\pi \epsilon_0 (2)^{\frac{3}{2}}} + \frac{50n \cdot 50n (ax)}{4\pi \epsilon_0 (2)^{\frac{3}{2}}}$$

$$= 5.625 \times 10^{-5} ax + 2(7.9545 \times 10^{-6}) \quad \therefore F =$$

$$= \cancel{13579 \times 10^{-5}}$$

$$= 2.15 \times 10^{-6}$$

Ans good!

$$\boxed{\text{Ans} = 21.5 \times 10^{-6} \text{ am.}} \quad \text{Re-do}$$

$$= 25 \int \frac{\sec^2 x}{(25 \sec^2 x)^{\frac{3}{2}}} dx$$

$$= 25 \int \frac{\sec^2 x}{5^3 \sec^2 x} dx$$

$$= \frac{1}{5} \int \frac{1}{\sec x} dx$$

$$= \frac{1}{5} \int_{x=?}^? \cos x dx$$

when  $y = -5$

$$x = \tan^{-1} \left( \frac{-5}{1} \right)$$

$$= -\frac{\pi}{4}$$

when  $y = 0$   $x = 0$

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$$\therefore \frac{1}{5} \int_{-\frac{\pi}{4}}^0 \cos x dx$$

$$= \frac{1}{5} \left( \sin x \right)_{-\frac{\pi}{4}}^0$$

$$= 0.141 \leftarrow$$

Second integration  $\Rightarrow -0.059$ .