$$\lim_{x \to -4} \sqrt{x} = \text{not real} = \text{DNE}$$

*be careful

$$\lim_{x \to 0} \sqrt{x} = \text{DNE} \text{ (no left side)}$$

$$\lim_{x \to 2} \frac{x^2+4}{x+1} = \frac{2^2+4}{2+1} = \frac{8}{3}$$

undefined = DNE

undetermined = don't know (0)

$$\lim_{x \to 3.} \frac{x^2-9}{x-3} = \frac{3^2-9}{3-3} = \frac{0}{0} = \text{undetermined}$$

indeterminate form

x	y
3.1	
3.001	
3.0001	
2.9	
2.999	
2.9999	

very close to 6

When you have indeterminate
- find rational expression

$$\frac{x^2-9}{x-3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}$$

$$= \lim_{x \to 3} x+3 = 6$$

Piecewise Functions

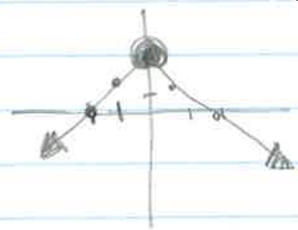
can be solved either graphically or algebraically.

$$f(x)=\begin{cases}2+x & x\le 0\\ 2-x & x>0\end{cases}$$

x	y=2+x
0	2
-1	1
-2	0

x	y=2-x
H 0	2
1	1
2	0

Dot win -> fill in hole



$\lim_{x\to 0^-} f(x)=2$

$\lim_{x\to 0^+} f(x)=2$

$\lim_{x\to 0} f(x)=2$

$f(0)=2+0=2$
cont. at $x=0$

Algebra:

1) $\lim_{x\to 0^-}(2+x)=2+0=2$

$\lim_{x\to 0^+}(2-x)=2-0=2$

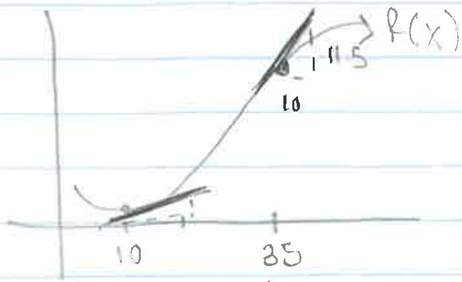
$\lim_{x\to 0} f(x)=2$

$f(0)=2+0=2$

equal = cont. at $x=0$

Last Day of Limits!

The rate of change of one quantity with respect to another is mathematically equivilant to the slope of the tangent line.



f(x) is # of social security beneficiaries in yr t
t = 0 → 1990 (in millions)

In yr 2000 → t = 10
$f(x) = {}^{10}/_{20} = .5$
= 500,000

In yr 2025 → t = 35
$f(x) = {}^{15}/_{10} = 1.5$
= 1,500,000

Average rate of change: of f(x) over the interval starting [x, x+h] is:

$$\frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

* difference quotient

Instantaneous rate of change: of f(x) w/ respect to x is: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Derivative of a function is defined: new function Domain is subset of domain f(x)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

f' → f prime of x
1st derivative

* or $\frac{dy}{dx}$ (integration) & $D_x(x, y)$

$f(x) = 3/2x$ x+h

$$f'(x) = \lim_{x\to 0} \frac{\left[\frac{3}{2(x+h)} - \frac{3}{2x}\right]}{h} \frac{(2x+2h)(2x)}{2x+2h(2x)}$$

$6x - 6x - 6h$

$$= \lim_{x\to 0} \frac{3(2x) - 3(2x+2h)}{h(2x+2h)(2x)}$$

$$= \lim_{x\to 0} \frac{-6\cancel{h}}{\cancel{h}(2x+2h)(2x)} \qquad \frac{-6}{(2x+2(0))(2x)}$$

$$= \frac{-6}{4x^2} \qquad f'(x) = -3/2x^2$$

Where is the tangent line horizontal?

top has no x

$-3/2x^2$ No where

Find equation of tangent line at $x=1$

y_1: $f(1) = 3/2(1) = 3/2$ $\qquad$ m: $f'(1) = -3/2(1)^2$

$(1, 3/2)$ $\qquad$ $m = -3/2$

$$y - 3/2 = -3/2(x-1)$$

What is the rate of change? (slope)

$-3/2$

Chain Rule for Power Functions

$y = [f(x)]^n$ $\qquad f(x) \neq$ plain x

$y' = n[f(x)]^{n-1} \cdot f'(x)$

ex. $f(x) = (x^4-5)^5$

$f'(x) = 5(x^4-5)^4 \cdot 4x^3$

$f'(x) = 20x^3(x^4-5)^4$

$f(x) = (5x+2)^3$

$f'(x) = 3(5x+2)^2 \cdot 5$

$f'(x) = 15(5x+2)^2$

$f(x) = \frac{1}{(x^2+4)^2}$

$f(x) = (x^2+4)^{-2}$

$f'(x) = -2(x^2+4)^{-3} \cdot 2x$

$f'(x) = -4x(x^2+4)^{-3}$

$f(x) = \sqrt{4-x}$

$f(x) = (4-x)^{1/2}$

$f'(x) = \frac{1}{2}(4-x)^{-1/2} \cdot (-1)$

$f'(x) = -\frac{1}{2}(4-x)^{-1/2}$

$f(x) = \overset{F}{(2x+1)^2}\overset{S}{(x^2-6)}$

$f'(x) = \overset{F}{(2x+1)^2} \cdot \overset{S'}{(2x)} + \overset{F'}{2(2x+1)(2)} \cdot \overset{S}{(x^2-6)}$

$= (2x+1)^2 \cdot (2x) + 4(2x+1)(x^2-6)$

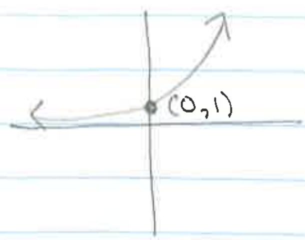
$= 2(2x+1)[(2x+1)(x) + 2(x^2-6)]$

$2x^2 + x + 2x^2 - 12$

doesn't factor

$f'(x) = 2(2x+1)(4x^2+x-12)$

$f(x) = b^x$ $\quad$ $f(x) = e^x$



D: $(-\infty, \infty)$

R: $(0, \infty)$

$f(x) = e^x$

$f'(x) = e^x$

$f(x) = 5e^x$

$f'(x) = 5e^x$

$f(x) = 4e^x + 8x^2 + 7x - 14$

$f'(x) = 4e^x + 16x + 7$

$f(x) = \overset{F}{x^2}\,\overset{S}{e^x}$

$f'(x) = x^2 \cdot e^x + 2x \cdot e^x$

$f'(x) = e^x(x^2 + 2x)$ or $e^x x(x+2)$

$f(x) = \dfrac{1-e^x}{1+e^x}$ H L

$f'(x) = \dfrac{(1+e^x)(-e^x) - (1-e^x)(e^x)}{(1+e^x)^2}$

$= \dfrac{-e^x - \cancel{e^{2x}} - e^x + \cancel{e^{2x}}}{(1+e^x)^2}$

$f'(x) = \dfrac{-2e^x}{(1+e^x)^2}$

Averages

Avg cost, rev, profit:

$\bar{C}(x) = \frac{C(x)}{x}$ $\quad \bar{R}(x) = \frac{R(x)}{x}$ $\quad \bar{P}(x) = \frac{P(x)}{x}$

marginals

$\bar{C}'(x)$ $\quad \bar{R}'(x)$ $\quad \bar{P}'(x)$

ex. $C(x) = 150{,}000 + 30x$

avg. cost & marginal cost function eval. $x = 1000$

$\bar{C}(x) = \frac{150{,}000 + 30x}{x}$

$\bar{C}(x) = 150{,}000x^{-1} + 30$

$\bar{C}(1000) = \$180$ on avg. the cost \$180

$\bar{C}'(x) = -150{,}000x^{-2}$

$\bar{C}'(1000) = -.15$

Average cost is decreasing by \$.15

When 1001 are produced the average cost is \$179.85

$R(x) = 300x - \frac{1}{30}x^2 \rightarrow$ avg. rev + marginal $x = 1000$

$\bar{R}(x) = \frac{300x - \frac{1}{30}x^2}{x}$

$\bar{R}(x) = 300 - \frac{1}{30}x$

$\bar{R}(1000) = \$266.67$

$\bar{R}'(x) = -\frac{1}{30}$

$\bar{R}'(1000) = -.03$

Average revenue is decreasing by \$.03 when 1001 are produced the average revenue is \$266.64

Cobb-Douglas Productivity Function

top # = 1 $f(x,y) = ax^b y^{1-b}$ a & b = #s

Economists use this to determine # of units produced from the utilization x units of labor & y units of capital

ex. The productivity of a steel manufacturing company is approximated by

$$f(x,y) = 10x^{.2}y^{.8}$$

a. If the company currently uses 1000 units of labor and 2000 units of capital find the # of units of steel produced

$$f(1000, 2000) = 10(1000)^{.2}(2000)^{.8}$$

$$= 17411.01 \approx 17{,}411 \text{ units of steel}$$

Partial Derivatives

When we take the derivative with respect to 1 of the variables and hold the other variable constant we obtain partial

$f_x(x,y) \rightarrow$ we take derivative of xs and treat ys as constants

$f_y(x,y) \rightarrow$ we take derivate of ys and treat xs as constants

ex. $f(x,y) = 3x^2y - 4y$

$f_x(x,y) = 3(2x)y - 0$

$= 6xy$

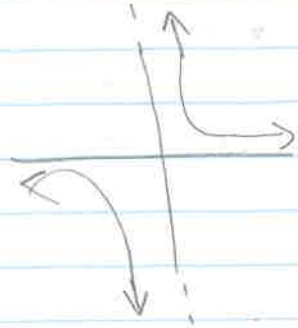
$f_y(x,y) = 3x^2(1) - 4(1)$

$= 3x^2 - 4$

Vertical Asymptotes: Another place where limit DNE is where a function has VA → really large ($+\infty$) or really small ($-\infty$) as $x \to a$.

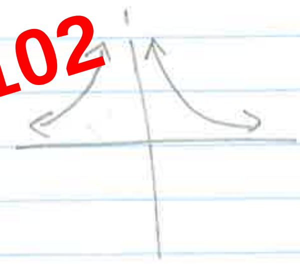
$f(x) = 1/x$, $x \neq 0$

x	-.1
f(x)	



$$\left.\begin{array}{l} \lim\limits_{x \to 0^-} 1/x = -\infty \\ \lim\limits_{x \to 0^+} 1/x = +\infty \end{array}\right\} VA = x = 0$$

$f(x) = 1/x^2$, $x \neq 0$



$$\left.\begin{array}{l} \lim\limits_{x \to 0^-} 1/x^2 = +\infty \\ \lim\limits_{x \to 0^+} 1/x^2 = +\infty \end{array}\right\} VA = x = 0$$

$f(x) = \dfrac{x-3}{x^2-4x+3} = \dfrac{\cancel{x-3}}{\cancel{(x-3)}(x-1)} = 1/(x-1)$, $x \neq 1$

$$\left.\begin{array}{l} .9 \quad \lim\limits_{x \to 1^-} 1/(x-1) = -\infty \\ 1.1 \quad \lim\limits_{x \to 1^+} 1/(x-1) = +\infty \end{array}\right\} VA: x = 1$$

Anti-Derivative

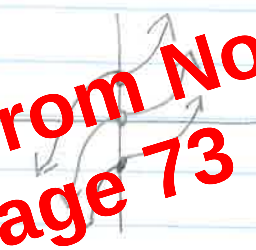
A function F is an antiderivative of f(x) if $F'(x) = f(x)$

$f'(x) = x^4$

$f(x) = \frac{1}{5}x^5$
$f(x) = \frac{1}{5}x^5 + 3$
$f(x) = \frac{1}{5}x^5 - \frac{1}{3}$

} family of antiderivative of $f'(x) = x^4$

If g(x) is the antiderivative of function f(x), every antiderivative, F(x) must be of the form $F(x) = g(x) + c$



The process of finding the antiderivative is called antidifferentiation or integration

$\int f(x)dx = F(x) + c$

Integrate w/ respect to x

$\rightarrow \int x^4 dx = \frac{1}{5}x^5 + c$

Rules:

Constant Rule: $\int k\,dx = kx + c$

$\int 1\,dx = 1x + c$

Power Rule : $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

$\int t^3\,dt = \frac{t^4}{4} + C = \frac{1}{4}t^4 + C$

$\int \frac{1}{x^2}\,dx = \frac{x^{-1}}{-1} + C = -x^{-1} + C$

Constant multiplier Rule : $\int k x^n\,dx = k\frac{x^{n+1}}{n+1} + C$

$\int 3x^4\,dx = 3\frac{x^5}{5} + C = \frac{3}{5}x^5 + C$

$\int 4\sqrt[5]{w^3}\,dx = \int 4w^{3/5}\,dw = 4\frac{w^{8/5}}{8/5} + C = 5/2\,w^{8/5} + C$

Sum / Diff Rule : $\int (f(x) \pm g(x) \pm \dots)\,dx$
$\int f(x)\,dx \pm \int g(x)\,dx \pm \dots$

$\int (2x^5 - 3x^2 + 1)\,dx = 2\frac{x^6}{6} - 3\frac{x^3}{3} + 1x + C$
$= 1/3\,x^6 - x^3 + x + C$

$\int (2\sqrt[3]{x^2} - \frac{3}{x^4})\,dx = \int (2x^{2/3} - 3x^{-4})\,dx =$
$= 2\frac{x^{5/3}}{5/3} - 3\frac{x^{-3}}{-3} + C = 6/5\,x^{5/3} + x^{-3} + C$

$\int \frac{x^4 - 8x^3}{x^2}\,dx = \int (x^2 - 8x)\,dx$
$= \frac{x^3}{3} - 8\frac{x^2}{2} + C = 1/3\,x^3 - 4x^2 + C$

$\frac{x^4}{x^2} - \frac{8x^3}{x^2}$
$x^2 - 8x$

$\int (8\sqrt[3]{x} - \frac{6}{\sqrt{x}})\,dx = \int (8x^{1/3} - 6x^{-1/2})\,dx$
$= \frac{3}{4}8\frac{x^{4/3}}{4/3} - 2\,6\frac{x^{1/2}}{1/2} + C = 6x^{4/3} - 12x^{1/2} + C$

U-Substitution

Reverse the Chain Rule

$$\int [\underbrace{f(x)}_{u}]^n \cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$\int e^{\overbrace{f(x)}^{u}} \cdot f'(x)\,dx = e^{f(x)} + c$$

$$\int [\underbrace{f(x)}_{u}]^{-1} \cdot f'(x)\,dx = \ln|f(x)| + c$$

ex.

1. $\int (\underbrace{2x^3-3}_{u})^{20} (\underbrace{6x^2\,dx}_{du}) = \int u^{20}\,du$

$u = 2x^3 - 3$

$dx \cdot \frac{du}{dx} = 6x^2 \cdot dx$

$du = 6x^2\,dx$

$= \frac{u^{21}}{21} + c$

$\frac{1}{21}(2x^3-3)^{21} + c$

2. $\int 5\boxed{e^{5x}}\,dx = \int e^{5x} \cdot \underbrace{5dx}_{du}$

$u = 5x$

$du = 5dx$

$= \int e^u\,du$

$= e^u + c = e^{5x} + c$

3. $\int \frac{2x}{4+x^2}\,dx = \int (\underbrace{4+x^2}_{u})^{-1}\,2x\,dx$

$u = 4+x^2$

$du = 2x\,dx$

$= \int u^{-1}\,du = \ln|u| + c$

$= \ln|4+x^2| + c$

4. $\int \underbrace{(x^2+2x+5)^5}_{u} \underbrace{(x+1)\,dx}_{1/2\,du}$

$u = x^2+2x+5$

$du = \frac{2x+2\,dx}{2}$

$\frac{1}{2}du = (x+1)\,dx$

$\int = u^5 \cdot \frac{1}{2}du$

$= \frac{1}{2}\frac{u^6}{6} + c$

$\frac{1}{12}(x^2+2x+5)^6 + c$

5. $\int (4x-6)(x^2-3x+7)^{-4}\,dx$

$\int \underbrace{(x^2-3x+7)^{-4}}_{u} \underbrace{(4x-6)\,dx}_{2du}$

$u = x^2-3x+7$

$2 \cdot du = 2x-3\,dx$

$2du = 4x-6\,dx$

$\int u^{-4} \cdot 2du$

$= 2\frac{u^{-3}}{-3} + c$

$-\frac{2}{3}(x^2-3x+7)^{-3} + c$

6. $\int e^{-3x}\,dx = \int e^u \cdot -\frac{1}{3}du$

$-\frac{1}{3}du$

$u = -3x$

$\frac{du}{-3} = \frac{-3\,dx}{-3}$

$= -\frac{1}{3}e^u + c$

$= -\frac{1}{3}e^{-3x} + c$

7. $\int \frac{x}{x^2-9}\,dx = \int \underbrace{(x^2-9)^{-1}}_{u} \underbrace{x\,dx}_{1/2\,du}$

$u = x^2-9$

$\frac{du}{2} = \frac{2x\,dx}{2}$

$\frac{1}{2}du = x\,dx$

$\int u^{-1} \cdot \frac{1}{2}du$

$= \frac{1}{2}\ln|u| + c$

$\frac{1}{2}\ln|x^2-9| + c$

1 cont. h. $\int x(x+4)^3\,dx = \int \underbrace{(x+4)^3}_{u} \overset{u-4}{(x)} \underbrace{dx}_{du}$

$u = x+4 \rightarrow u-4 = x$

$du = dx$

$\int u^3(u-4)\,du$

$\int u^4 - 4u^3\,du$

$\frac{1}{5}(x+4)^5 - (x+4)^4 + C = \frac{u^5}{5} - 4\frac{u^4}{4} + C$

#5

2. a. $f'(x) = \frac{-10}{x^2}$ $\quad f(1) = 20$

$f(x) = \int -10x^{-2}\,dx \rightarrow f(x) = 10x^{-1} + 10$

$f(x) = -10\frac{x^{-1}}{-1} + C$

$20 = 10(1) + C$

$10 = C$

b. $f'(x) = x^3 + x$ $\quad f(2) = 5$

$f(x) = \int x^3 + x\,dx \rightarrow f(x) = \frac{1}{4}x^4 + \frac{1}{2}x^2 - 1$

$f(x) = \frac{x^4}{4} + \frac{x^2}{2} + C$

$5 = \frac{2^4}{4} + \frac{2^2}{2} + C$

$4 + 2 + C$

$5 = 6 + C$

$-1 = C$

#6 & 7 (1u)

3. a. $\int_1^2 (5 - 16^{-3})\,dx$

$= 5x - 16\frac{x^{-2}}{-2}\Big|_1^2 = [5(2) + 8(2)^{-2}] - [5(1) + 8(1)^{-2}]$

$5x + 8x^{-2}$ $\quad 10 + 2 \quad - 13$

$= -1$

#8

4. $V'(t) = 500(t-12)$ $\quad 0 < t < 10$

a. Find total loss in value during the first 5 yr

$V(t) = \int_0^5 500(t-12)\,dt = \$-23{,}750$ Loss 5 yr

b. Find average loss over the first 5 yr

$\frac{1}{5-0} \cdot \frac{1}{5}\int_0^5 500(t-12)\,dt = \$-4750$

Average loss $4,750

a. Barrels of oil ... 10 years

$B(t) = \int_{10}^{20} 100t^{-1} + 5\,dt = 119{,}315$ barrels

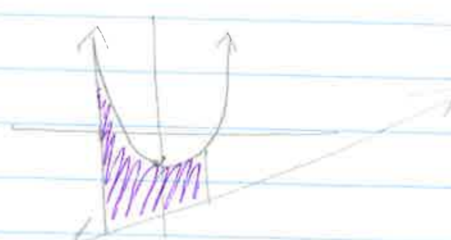
b. $\frac{1}{20-10} = \frac{1}{10}\int_{10}^{20} 100t^{-1} + 5\,dt = 11931.5$ barrels/yr

#9 & 10

5. a. $y = x^2 - 1$

$y = x - 2$

$-2 \leq x \leq 1$



$x^2 - 1 - x + 2$

$\int_{-2}^{1} [(x^2-1) - (x-2)]\,dx$

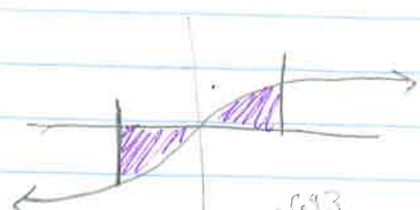
$x^2 - x + 1$

$= 7.5$

b. $y = \frac{2x}{x^2+4}$

$y = 0$

$-2 \leq x \leq 2$



$\int_{-2}^{0} \left[0 - \frac{2x}{x^2+4}\right] dx$ (.693) $+ \int_0^2 \left[\frac{2x}{x^2+4} - 0\right]$ (.693) $= 1.386$

You wish to have \$20,000 for a down payment on a home but you only have \$15,000. You invest the \$15,000 at 10% comp. quarterly. How long until you have \$20,000?

$A = P(1+c)^n \qquad c = R/m \qquad n = mt$

$20000 = 15{,}000\left(1+\frac{.10}{4}\right)^{(4t)}$

$\ln 1.3333 = \ln 1.025^{4t}$

$\frac{\ln(1.3333)}{4\ln(1.025)} = \frac{4t\ln(1.025)}{4\ln(1.025)}$

$\ln(1.333)/4 \rightarrow \cancel{/}\ln(1.025)$

2.91 yrs

Effective Rate (annual percentage yield) APY

Simple interest rate that produces same results as compounded rate in 1 yr

$R_{eff} = (1 + R/m)^m - 1 \qquad R_{eff} = e^R - 1$ cont.

Find the effective rate

1. 10% comp. monthly

$R_{eff} = (1 + .10/12)^{12} - 1$

$= .1047$ 10.47%

2. 9% comp. cont.

$R_{eff} = e^{.09} - 1$

$= 9.42\%$

Present value: How much should you deposit now in order to withdraw a set amt each period for a set # of yrs

$$PV = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

How much should you deposit now in order to withdraw $1000 each mo for 2 yrs At 12% comp. cont?

N = 24
I% = 12
☆ PV = 0 → $21,243.39
PMT = 1000
FV = 0
P/Y = 12
C/Y = 12
End

I = # of pay - PV
1000(24) - 21,243.39
= $2756.61

Amortization: when a loan is repaid in regular installments for a set # of yrs
use PV formula

ex. A family buys a $180,000 condo. They pay 10% and finance at 4.5% comp. monthly for 30 yrs What is the monthly payment

Down = $18,000   PV = $162,000

N = 12 · 30 = 360
I% = 4.5
PV = 162,000
☆ PMT = 0 → $820.83
FV = 0
P/Y = 12
C/Y = 12

I = 820.83(360) − 162,000 = $133,498.80