$$A = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

II y

$$M_{12} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{bmatrix}$$

E.g.

(ii) Let,

$$A = \begin{bmatrix} 2 & 5 & 8 \\ 1 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}, M_{12} = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}, M_{13} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}, M_{12} = \begin{bmatrix} 2 & 8 \\ 0 & 6 \end{bmatrix}, M_{23} = \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$$

(b) Cofactor of an element :

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is a square matrix of order n and a_{ij} denotes cofactor of the element a_{ij}.

$C_{ij} = (-1)^{i+j} . M_{ij}$ Where M_{ij} is minor of a_{ij}.

$$\text{If } A = \begin{bmatrix} a^1 & b^1 & c^1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{bmatrix}$$

A_1 = The cofactor of $A_1 = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

B_1 = The cofactor of $b_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

C_1 = The cofactor of $b_1 = (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

(iii) Adding non-zero scalar multitudes of all the elements of any row (or columns) into the corresponding elements of any another row (or column).

Definition:- Equivalent Matrix:

Two matrices A & B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order & the same rank. It can be denoted by

[it can be read as A equivalent to B]

Example 4: Determine the rank of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Solution:

$$\text{Given } A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - R_1 \;\; \& \;\; R_3 \Rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Here two column are Identical . hence 3rd order minor of A vanished

Hence 2nd order minor $\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1 \neq 0$

$\therefore e(A) = 2$

Hence the rank of the given matrix is 2.

1.5 CANONICAL FORM OR NORMAL FORM

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - C_1,\ C_3 - C_1,\ C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 6R_3,$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 56 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \times \frac{1}{28},\ R_3 \times (-1)$

$$\therefore\ \rho(AD) = 3$$

$$\rho(A) = 2$$

$$\therefore\ \rho(AD) \neq \rho(A)$$

$\therefore$ The system is inconsistent and it has no solution.

Example 5: Discuss the consistency of

$$3x + y + 2z = 3$$

$$2x - 3y - z = -3$$

$$x + 2y + z = 4$$

Solution: In the matrix form,

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$A \qquad X = D$$

Now we join matrices A and D

Consider

$$[A:D] = \begin{bmatrix} 3 & 1 & 2 & : & 3 \\ 2 & -3 & -1 & : & -3 \\ 1 & 2 & 1 & : & 4 \end{bmatrix}$$

We reduce to Echelon form

$$R_1 \to R_3$$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -3 & -1 & : & -3 \\ 3 & 1 & 2 & : & 3 \end{bmatrix}$$

$$R_2 \to R_2 - 2\,R_1$$

$$R_3 \to R_3 - 3\,R_1$$

$$[A:D] = \begin{bmatrix} 1 & 2 & 1 & : & 4 \\ 2 & -7 & -3 & : & -11 \\ 0 & -5 & -1 & : & -9 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{5}{7}\,R_2$$

$7x+2y+10z=5$

Solution:

Step (1) : In the matrix form

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$A \qquad X = D$$

Consider

$$[A:D] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$R_1 \to \frac{1}{5} R_1$$

$$[A:D] = \begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1$$

$$R_3 \to R_3 - 7R_1$$

$$[A:D] = \begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 0 & 121/5 & -11/5 & : & 33/5 \\ 0 & -11/5 & 1/5 & : & -3/5 \end{bmatrix}$$

$$R_3 \to R_3 + \frac{1}{11} R_2$$

$$[A:D] = \begin{bmatrix} 1 & 3/5 & 7/5 & : & 4/5 \\ 0 & 121/5 & -11/5 & : & 33/5 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$\therefore \quad \rho(AD) = 2$

$\rho(A) = 2$

$\therefore \quad \rho(AD) = \rho(A) = 2 < 3 = \textit{Number}$ of unknowns

The system is consistent and has infinitely many solutions.

Step (2) :- To find the solution we proceed as follows:

Let

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ *Ans : Rank = 2*

iv) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$ *Ans : Rank = 2*

2) Solve the following system of equations.

i) $x_1 + x_2 + x_3 = 3,\ x + 2x_2 + 3x_3 = 4,\ x_1 + 4x_2 + 9x_3 = 6$

Ans:- $x = 2, y = 1, z = 0.$

ii) $2x_1 - x_2 - x_3 = 0,\ x_1 - x_3 = 0,\ 2x_1 + x_2 - 3x_3 = 0$

Ans:- $x_1 = x_2 = x_3 = \lambda ..... \therefore \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

iii)
$$5x_1 - 3x_2 - 7x_3 + x_4 = 10$$
$$-x_1 + 2x_2 + 6x_3 - 3x_4 = -3$$
$$x_1 + x_2 + 4x_3 - 5x_4 = 0$$

iii)
$$2x_1 + 3x_2 - 2x_3 = 0$$
$$3x_1 - x_2 + 3x_3 = 0$$
$$7x_1 + 5x_2 - x_3 = 0.$$

iv)
$$x_1 - 4x_2 - x_3 = 3$$
$$3x_1 + x_2 - 2x_3 = 7$$
$$2x_1 - 3x_2 + x_3 = 10.$$

v)
$$x_1 - 4x_2 + 7x_3 = 8$$
$$3x_1 + 8x_2 - 2x_3 = 6$$
$$7x_1 - 8x_2 + 26x_3 = 31$$

*****

CHARACTERISTIC EQUATION

Let ‘A’ be a given matrix. Let λ be a scalar. The equation det $[A - \lambda I]$ or $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A.

1. Find the Characteristic equation of A = $\begin{pmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{3} \end{pmatrix}$

Solution: The Characteristic equation of A is $|A - \lambda I|$ =0 ie. $\lambda^2 - D_1\lambda + D_2 = 0$ Where D_1 = Trace of A & D_2 = $|A|$. Therefore D_1 = 4 & D_2 = -5 implies that $\lambda^2 - 4\lambda - 5 = 0$.

2. Find the Characteristic equation of A = $\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{1} \end{pmatrix}$

Solution: The Characteristic equation of A is $|A - \lambda I|$ =0 ie., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$ Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & D_3 =$|A|$ $\therefore$ D_1 = 3 & D_2 = -1 & $D_3 = -9$ implies that $\lambda^3 - 3\,\lambda^2 - \lambda + 9 = 0$.

EIGEN VALUE

The values of λ obtained from the characteristic equation $|A - \lambda I| = 0$ are called the Eigen values of A.

EIGEN VECTOR

Let A be a square matrix of order ‘n’ and λ be a scalar, X be a non-zero column vector such that AX = λX.

The non-zero column vector $X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$ which satisfies $[A - \lambda I]X = 0$ is called eigen vector or latent vector.

LINEARLY DEPENDENT AND INDEPENDENT EIGEN VECTOR

Let ‘A’ be the matrix whose columns are eigen vectors.

(i) If $|A| = 0$ then the eigen vectors are linearly dependent.

(ii) If $|A| \neq 0$ then the eigen vectors are linearly independent.

1. Find the eigen values and eigen vectors of $A = \begin{pmatrix} \mathbf{-2} & \mathbf{2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-6} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{0} \end{pmatrix}$

Solution: The Characteristic equation of A is $|A - \lambda I|$ =0 ie., $\lambda^3 - D_1\;\lambda^2 + D_2\lambda - D_3 = 0$ Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & D_3 =$|A|$

$$=> \quad A^{-1} = \left[\begin{pmatrix} 8 & 0 & -3 \\ -43 & 1 & 17 \\ 3 & 0 & -1 \end{pmatrix}\right]$$

6. Verify Cayley Hamilton theorem and also find A^5 interms of A^2 , A & I of A = $\begin{pmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{2} \end{pmatrix}$

SOLUTION : The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^3 - D_1 \lambda^2 + D_2\lambda - D_3 = 0$ Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$ ∴ $D_1 = 5$ & $D_2 = 7$ & $D_3 = 3$ implies that $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

(Every square matrix satisfies its own characteristic equation is the statement of Cayley Hamilton theorem.)

To verify C.H.T we have check : $A^3 - 5A^2 + 7A - 3I = 0 \; \ldots\ldots (i)$

Consider L.H.S of (I) : $A^3 - 5A^2 + 7A - 3I$

$$= \begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix} - \begin{pmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{pmatrix} + \begin{pmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \text{R.H.S of (i)}$$

Therefore C.H.T is verified.

By Cayley Hamilton theorem we have $A^3 - 5A^2 + 7A - 3I = 0$ .....(1) ,

$$=> \quad A^3 = [5A^2 - 7A + 3I]$$

$$= \left[\begin{pmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{pmatrix} - \begin{pmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix}$$

Premultiplying by A^1 on both sides of (1) we get

$$=> \quad A^4 = [5A^3 - 7A^2 + 3A^1] \quad \ldots\ldots\ldots(2)$$

$$= 5\begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix} - 7\begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + 3\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 41 & 40 & 40 \\ 0 & 1 & 0 \\ 40 & 40 & 41 \end{pmatrix}$$

Premultiplying by A^1 on both sides of (2) we get

$$=> \quad A^5 = [5A^4 - 7A^3 + 3A^2]$$

$$=> \quad A^5 = \left[\begin{pmatrix} 205 & 200 & 200 \\ 0 & 5 & 0 \\ 200 & 200 & 205 \end{pmatrix} - \begin{pmatrix} 98 & 91 & 91 \\ 0 & 7 & 0 \\ 91 & 91 & 98 \end{pmatrix} + \begin{pmatrix} 15 & 12 & 12 \\ 0 & 3 & 0 \\ 12 & 12 & 15 \end{pmatrix}\right]$$

$$=> \quad A^5 = \left[\begin{pmatrix} 122 & 121 & 121 \\ 0 & 1 & 0 \\ 121 & 121 & 122 \end{pmatrix}\right]$$

7. Verify Cayley Hamilton theorem and also find A^4 interms of A^2, A & I of A = $\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$

SOLUTION : The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$ Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$ $\therefore D_1 = 6$ & $D_2 = 8$ & $D_3 = 3$ implies that $\lambda^3 - 6\,\lambda^2 + 8\lambda - 3 = 0$

(Every square matrix satisfies its own characteristic equation is the statement of Cayley Hamilton theorem.)

To verify C.H.T we have check : $\quad A^3 - 6\,A^2 + 8A - 3I = 0 \;\ldots\ldots(i)$

Consider L.H.S of (I) : $\quad A^3 - 6\,A^2 + 8A - 3I$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - \begin{pmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{pmatrix} + \begin{pmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = \text{R.H.S of (i)}$$

Therefore C.H.T is verified.

By Cayley Hamilton theorem we have $A^3 - 6\,A^2 + 8A - 3I = 0 \quad \ldots..(1)$,

$A^3 = 6\,A^2 - 8A + 3I$.

$$=> \quad A^4 = [6A^3 - 8A^2 + 3A]$$

$$= 6\,(6A^2 - 8A + 3I) + [-8A^2 + 3A]$$

$$= 28A^2 - 45A + 18I$$

$$=> A^4 = \left[28\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - \begin{pmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{pmatrix} + \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}\right]$$

$$\therefore \quad A^4 = \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$$

8. **Verify the Cayley Hamilton Theorem and hence find A^{-1} *for* A =** $\begin{pmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}$

Ans: : The Characteristic equation of A is $|A - \lambda I| = 0$ ie., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$

1. A Q.F is positive definite if $D_1, D_2, D_3 \dots\dots\dots\dots D_n$ are all positive i.e., $D_n > 0$ for all n.
2. A Q.F is negative definite if $D_1, D_3, D_5 \dots\dots$ are all negative and $D_2, D_4, D_6 \dots\dots$ are all positive i.e.,$(-1)^n\, D_n > 0$ for all n.
3. A Q.F is positive semi- definite if $D_n \geq 0$ and atleast one $D_i = 0$.
4. A Q.F is negative semi- definite if $(-1)^n D_n \geq 0$ and atleast one $D_i = 0$.
5. A Q.F is indefinite in all other cases.

1. Without reducing to canonical form find the nature of the Quadratic form $x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

Solution: Matrix of the Quadratic form is A = $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

$D_1 = 1 > 0$, $D_2 = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 0$ & $D_3 = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$ = 0 -2-2 =-4

Since $D_1 > 0$, $D_2 = 0$ & $D_3 < 0$ $\therefore$ Nature of the Quadratic form is indefinite.

2. Reduce the quadratic form $x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ to canonical form using orthogonal transformation also find its nature, rank , index & signature.

Solution: Quadratic form = $x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

Matrix form of Quadratic form = $X^T AX \rightarrow (1)$ where X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & A = $\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - D_1\, \lambda^2 + D_2 \lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & D_3 =$|A|$

$\therefore D_1 = 3$ & D_2 =0 & $D_3 = -4$ implies that $\lambda^3 - 3\,\lambda^2 + 4 = 0$.

$\therefore$ *The* eigen values of the matrix A are -1, 2 & 2.

To find Eigen vector : By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=-1 , Substituting in (2) we get

$2x_1 - x_2 - x_3 = 0$

$-x_1 + 2x_2 - x_3 = 0$

$-x_1 - x_2 + 2x_3 = 0$

Signature of the Q.F (s) =2p-r = 2.

4. **Reduce the quadratic form $6\,x^2+3y^2+3z^2-4xy-2yz+4xz$ to canonical form using orthogonal transformation also find its nature, rank , index & signature.**

Solution: Quadratic form = $6\,x^2+3y^2+3z^2-4xy-2yz+4xz$

Matrix form of Quadratic form = $X^TAX \to (1)$ where X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & A = $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

The Characteristic equation of A is $|A-\lambda I|=0$

i.e., $\lambda^3 - D_1\,\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

$\therefore D_1 = 12$ & $D_2 = 36$ & $D_3 = 32$ implies that $\lambda^3 - 12\,\lambda^2 + 36\lambda - 32 = 0$.

$\therefore$ *The e*igen values of the matrix A are 8 , 2 & 2.

To find eigen vector : By the definition we have $AX=\lambda X$ ie., $(A-\lambda I)X=0$

$$=> \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \to (2)$$

CASE (I) : When λ=8 , Substituting in (2) we get

$-2x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 - 5x_2 - x_3 = 0$

$2x_1 - x_2 - 5x_3 = 0$

Solving using cross multiplication rule $\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6} = k$ => If $\lambda_1 = 8$ then $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

CASE(ii) : When λ=2 , Substituting in (2) we get

$4x_1 - 2x_2 + 2x_3 = 0$

$-2x_1 + x_2 - x_3 = 0$

$2x_1 - x_2 + x_3 = 0$

We have only one equation $2x_1 - x_2 + x_3 = 0$ with three unknowns, let $x_2 = 2x_1 + x_3$

if $x_1 = 0, x_3 = 1$ *then* $x_2 = 1$ => If $\lambda_2 = 2$ then $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

CASE(iii) : When λ=2 , Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ From orthogonal transformation we know that X_1, X_2 & X_3 must be mutually perpendicular to each other. => $X_1.{X_2}^T = 0$, $X_2.{X_3}^T = 0$ & $X_3.{X_1}^T = 0$

$2a - b + c = 0$

$$N^T A N = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{-\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} = D$$

Let X = NY be an orthogonal transformation which changes the quadratic form to canonical form.

where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ Substituting X = NY in (1) we get

Q.F $= X^T A X$

$=[(NY)^T A(NY)]$

$=Y^T[N^T A N]Y$

$= Y^T D Y$

$=(y_1 \quad y_2 \quad y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Q.F $= 0{y_1}^2 + {y_2}^2 + 4{y_3}^2$ which is the canonical form of the quadratic form

Nature of the Q.F = Positive semi definite

Rank of the Q.F (r) = 2

Index of the Q.F (p) = 2

Signature of the Q.F (s) =2p-r = 2.

6. Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2xz$ to canonical form using orthogonal transformation also find its nature, rank, index & signature.

Solution: Quadratic form = $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2xz$

Matrix form of Quadratic form = $X^T A X$ → (1) where X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ & A = $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

The Characteristic equation of A is $|A - \lambda I|$ =0

i.e., $\lambda^3 - D_1\lambda^2 + D_2\lambda - D_3 = 0$

Where D_1 = Trace of A , D_2=Sum of the minors of the major diagonal elements & $D_3 = |A|$

∴ $D_1 = 11$ & $D_2 = 36$ & $D_3 = 36$ implies that $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$.

∴ *The* eigen values of the matrix A are 2, 3 & 6.

To find eigen vector : By the definition we have $AX = \lambda X$ ie., $(A - \lambda I)X = 0$

$$=> \begin{pmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow (2)$$

CASE (I) : When λ=2 , Substituting in (2) we get

Z =x+iy is called a complex number, where $i=\sqrt{-1}$ and $x, y \in R$ and $\overline{Z}=x-iy$ is called a conjugate of the complex number Z

Let A be a mxn matrix having complex numbers as its elements, then the matrix is called a complex matrix.

Conjugate of a Matrix:

The matrix of order mxn is obtained by replacing the elements by their corresponding conjugate elements, is called conjugate of a matrix. It is denoted by $\overline{A}$

For e.g. $A=\begin{vmatrix} 2-3i & 1-i & 3 \\ 2i+1 & 2 & 2i-3 \end{vmatrix}$

$\overline{A}=\begin{vmatrix} 2+3i & 1+i & 3 \\ -2i+1 & 2 & -2i-3 \end{vmatrix}$

Properties of conjugate of matrix:

(1) $\overline{(\overline{A})}=A$

(2) $\overline{A+B}=\overline{A}+\overline{B}$

(3) $\overline{(AB)}=\overline{A}.\overline{B}$

Conjugate Transpose:

Transpose of the conjugate matrix A is called conjugate transpose. It is denoted by A^{θ}.

For e.g. $A=\begin{vmatrix} 1+i & -i & 1 \\ 3 & i+2 & 3i-2 \end{vmatrix}$

$\overline{A}=\begin{vmatrix} 1-i & i & 1 \\ 3 & -i+2 & -3i-2 \end{vmatrix}$ then $A^{\theta}=\begin{bmatrix} 1-i & 3 \\ i & -i+2 \\ 1 & -3i-2 \end{bmatrix}$

Properties of Transpose of Conjugate of a matrix:

(1) $\left(A^{\theta}\right)^{\theta}=A$

(2) $(A+B)^{\theta}=A^{\theta}+B^{\theta}$

(3) $(AB)^{\theta}=B^{\theta}.A^{\theta}$

Hermitian matrix:

$$= \begin{vmatrix} 2i & 1+i & 2-2i \\ i-1 & -2 & 5+i \\ -2-2i & -5-i & 0 \end{vmatrix}$$

$$\frac{1}{2}\left(A - A^{\theta}\right) = \frac{1}{2}\begin{vmatrix} 2i & 1+i & 2-2i \\ i-1 & -2 & 5+i \\ -2-2i & -5-i & 0 \end{vmatrix} ......(IV)$$

Now, A=B+iC

$$A = \frac{1}{2}\begin{vmatrix} 4 & 1-i & 4-4i \\ i+1 & 2 & i-1 \\ 4+4i & -i-1 & 10 \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 2i & 1+i & 2-2i \\ i-1 & -2 & 5+i \\ -2-2i & -5-i & 0 \end{vmatrix}$$

Example 14:

Prove that the matrix, $A = \frac{1}{\sqrt{2}}\begin{vmatrix} 1 & i \\ -i & -1 \end{vmatrix}$

Solution:

Let $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$

$$A^{\theta} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

$$A^{\theta}A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1-i^2 & i-i \\ -i+i & -i^2+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore AA^{\theta} = I$

Hence A is Unitary.

Check Your Progress:

(1) Show that the following matrices are Skew –Hermitian.

(i) $A = \begin{bmatrix} 2i & 2 & -3 \\ -2 & 4i & -6 \\ 3 & 6 & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 4i & 1+i & 2+2i \\ i-1 & i & 5i \\ 2-2i & -5i & 3i \end{bmatrix}$

(2) Show that the following matrices are Unitary matrices.

(i) $A=\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ i-1 & -1 \end{bmatrix}$ (ii) $A=\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ i+1 & 1-i \end{bmatrix}$

(3) If A is Hermitian matrix, then show that iA is Skew- Hermitian matrix.

4.8 LET US SUM UP

In this chapter we have learn

- ❖ Cayley Hamilton theorem & it application like Higher power of matrix & Inverse of matrix.
- ❖ Minimal .polynomial & derogatory & non-derogatory matrix.
- ❖ Complex matrix.
- ❖ Hermitian matrix. i.e $A = A^{\theta}$
- ❖ Skew Hermitian matrix. i.e $A^{\theta} = -A$
- ❖ Unitary matrix= $AA^{\theta} = I$.

4.9 UNIT END EXERCISE

1. Show that the given matrix A satisfies its characteristics equation.

i) $A=\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

ii) $A=\begin{bmatrix} 2 & 4 & 3 \\ 0 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$

iii) $A=\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$

2. Using Cayley Hermitian theorem find inverse of the matrix A.

i) $A=\begin{bmatrix} -2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

ii) $A=\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$= cos\left(\frac{x}{y}\right).\frac{1}{y}e^t + cos\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right)2t \quad = \cos\left(\frac{e^t}{t^2}\right)\left(\frac{e^t}{t^2} - \frac{2e^t}{t^3}\right)$

5. If z be a function of x and y and u and v are other two variables, such that

$u = lx + my, v = ly - mx$ show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$

Solution: z may be represented as the function of u,v

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}l + \frac{\partial z}{\partial v}(-m)$$

$$\frac{\partial}{\partial x} = l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}.\frac{\partial z}{\partial x} = \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right)\left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right) = l^2\frac{\partial^2 z}{\partial u^2} - 2lm\frac{\partial^2 z}{\partial u \partial v} + m^2\frac{\partial^2 z}{\partial v^2} \qquad (1)$$

Similarly

$$\frac{\partial^2 z}{\partial x^2} = m^2\frac{\partial^2 z}{\partial u^2} + 2lm\frac{\partial^2 z}{\partial u \partial v} + l^2\frac{\partial^2 z}{\partial v^2} \qquad (2)$$

(1)+(2) => $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2 + m^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$

6. ***If $z = f(x,y)$, Where $x = u^2 - v^2,\ y = 2uv,$ PT $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2 + v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$***

Ans: Here Z is a composite function of u and v

$Hence\ \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$……………………………………(1)

$and\ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$……………………………………….(2)

Now $\frac{\partial x}{\partial u} = 2u,\ \ \frac{\partial x}{\partial v} = -2v,\ \frac{\partial y}{\partial u} = 2v,\ \ \frac{\partial y}{\partial v} = 2u$

Sub these values in (1) & (2) We get

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}2u + \frac{\partial z}{\partial y}2v$………………………………..(3)

Now $\frac{\partial}{\partial u}(z) = \left(2u\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial y}\right)(z)$

Which implies $\frac{\partial}{\partial u} = \left(2u\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial y}\right)$……………..(4)

(3)x(4) We get

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(2u\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial y}\right)\left(\frac{\partial z}{\partial x}2u + \frac{\partial z}{\partial y}2v\right)$$

$\frac{\partial^2 z}{\partial u^2} = 4u^2\frac{\partial^2 z}{\partial x^2} + 4v^2\frac{\partial^2 z}{\partial y^2} + 8uv\frac{\partial^2 z}{\partial x \partial y}$……………………………….(A)

Similarly we get $\frac{\partial^2 z}{\partial v^2} = 4u^2\frac{\partial^2 z}{\partial y^2} + 4v^2\frac{\partial^2 z}{\partial x^2} - 8uv\frac{\partial^2 z}{\partial x \partial y}$……………………….(B)

(A)+(B) Gives $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4u^2\frac{\partial^2 z}{\partial x^2} + 4v^2\frac{\partial^2 z}{\partial y^2} + 4u^2\frac{\partial^2 z}{\partial y^2} + 4v^2\frac{\partial^2 z}{\partial x^2}$

$\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0;\;\; 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0$ ---------------- (3)

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0;\;\; 8xz + \lambda\left(\frac{2y}{b^2}\right) = 0$ ---------------- (4)

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0;\;\; 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0$ ---------------- (5)

Solve the equation

(3)x+ (4)y+(5)z $\Rightarrow$

$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0 \;\; => \;\; \lambda = -12xyz$

Put in (3) => $8yz - 12xyz\left(\frac{2x}{a^2}\right) = 0 => x = \frac{a}{\sqrt{3}}$

Similarly, $y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$

Put in (2) ∴ Max volume = $\frac{8abc}{3\sqrt{3}}$.

4. Find the dimensions of the rectangular box without a top of maximum capacity, whose surface is 108 sq. cm

Solution: Given Surface area

$\varphi(x, y, z) = xy + 2xz + 2yz = 108$ ---------------- (1)

The volume is $f(x, y, z) = xyz$ ---------------- (2)

At the max point or min point

$\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0;\;\; yz + \lambda(y + 2z) = 0$ ---------------- (3)

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0;\;\; xz + \lambda(x + 2z) = 0$ ---------------- (4)

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0;\;\; 8xy + \lambda(2x + 2y) = 0$ ---------------- (5)

Solve the equation

(3)x - (4)y =>

$\lambda(2zx - 2zy) = 0 \;\; => \;\; x = y$

(3)x - (5)z =>

$y(x - 2z) = 0 \;\; => \;\; z = \frac{x}{2}$

Put in (1) $xy + 2xz + 2yz = 108 => x = 6$

∴ y =6, z =3

∴ The dimension of the box, having max capacity is Length=6cm, Breadth = 6cm, Height = 3cm.

5. The temperature T at any point (x, y, z) in space is T = 400xyz^2. Find the highest temperature on the surface of the unit sphere $x^2 + y^2 + z^2 = 1$

Solution: $\varphi(x, y, z) = x^2 + y^2 + z^2 - 1$ ---------------- (1)

$f(x, y, z) = 400\mathrm{xyz}^2$ ---------------- (2)

At the max point or min point

$$\frac{\partial^2 f}{\partial \lambda^2}\cdot\frac{\partial^2 f}{\partial \mu^2}-\left(\frac{\partial^2 f}{\partial \lambda \partial \mu}\right)^2>0$$

$\therefore$ *At* $(1,-1)$ *the function* $f(\lambda,\mu)$ *has minimum.*

i.e., At $(1,-1)$, PQ^2 *has minimum which gives the shortest length.*

$$At(1,-1),\qquad PQ^2 = 17+41+32-66-114+99$$
$$=9$$

$$\therefore Shortest\ length = PQ = \sqrt{9} = 3$$

10. *Find the minimum value of* $x^2+y^2+z^2$ *subject to the condition* $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Solution

$$Let\ f = x^2+y^2+z^2$$
$$g=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1$$

Let the auxillary function 'F' be

$$F(x,y,z)=(x^2+y^2+z^2)+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right) \qquad ...(1)$$

By Lagranges method the values of x,y,z *for which 'f' is minimum are obtained by the following eqations*

$$\frac{\partial F}{\partial x}=0 \Rightarrow 2x-\frac{\lambda}{x^2}=0 \Rightarrow \frac{\lambda}{2}=x^3 \qquad ...(2)$$
$$\frac{\partial F}{\partial y}=0 \Rightarrow 2y-\frac{\lambda}{y^2}=0 \Rightarrow \frac{\lambda}{2}=y^3 \qquad ...(3)$$
$$\frac{\partial F}{\partial z}=0 \Rightarrow 2z-\frac{\lambda}{z^2}=0 \Rightarrow \frac{\lambda}{2}=z^3 \qquad ...(4)$$
$$\frac{\partial F}{\partial \lambda}=0 \Rightarrow \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1=0 \qquad ...(5)$$

From (2), (3) *and* (4) *we get*

$$x^3=y^3=z^3=\frac{\lambda}{2}$$

i.e.,
$$x=y=z=\left(\frac{\lambda}{2}\right)^{\frac{1}{3}} \qquad ...(6)$$

Substituting (6) *in* (5) *we get*

$$\frac{3}{x} = 1 \ \ or\ \ x=3$$

$$= 2\left(\sqrt{\pi}\ / 2\right) = \sqrt{\pi}$$

$$\Rightarrow \sin(\theta+\frac{\pi}{4}) = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \qquad \Rightarrow \theta+\frac{\pi}{4} = \frac{\pi}{4}(or)\pi-\frac{\pi}{4}$$

$$\Rightarrow \theta = 0(or)\theta+\frac{\pi}{4} = \pi-\frac{\pi}{4} \qquad \Rightarrow \theta = \pi-\frac{2\pi}{4} = \pi-\frac{\pi}{2} = \frac{\pi}{2}$$

$$\Rightarrow \theta = 0(or)\theta = \frac{\pi}{2}$$

$$\therefore \text{The required area} = \int_0^{\pi/4}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta-(1+\cos^2\theta-2\cos\theta)\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta-\cos^2\theta-1+2\cos\theta\right)d\theta = \frac{a^2}{2}\left[\int_0^{\pi/2}\left(1-\cos^2\theta-\cos^2\theta-1+2\cos\theta\right)d\theta\right]$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(2\cos\theta-2\cos^2\theta\right)d\theta \qquad = \frac{a^2}{2}.2\int_0^{\pi/2}\left(\cos\theta-\cos^2\theta\right)d\theta$$

$$= a^2\left[(\sin\theta)_0^{\pi/2}-\int_0^{\pi/2}\cos^2\theta\,d\theta\right] \qquad = a^2\left[1-\int_0^{\pi/2}\left(\frac{1+\cos2\theta}{2}\right)d\theta\right]$$

$$= a^2\left[1-\frac{1}{2}\left[\theta+\frac{\sin2\theta}{2}\right]_0^{\pi/2}\right] \qquad = a^2\left[1-\frac{1}{2}\left(\frac{\pi}{2}+0\right)-0\right]$$

$$= a^2\left[1-\frac{\pi}{4}\right] = \frac{a^2(4-\pi)}{4}$$

6.Find by double integration the area enclosed by the curves $y^2 = 4ax$ and $x^2 = 4ay$

Ans: $y^2 = 4ax$ ……………(1) $x^2 = 4ay$ ……………(2)

Sub (1) in (2) we get $\left[\frac{y^2}{4a}\right]^2 = 4ay$

$$\frac{y^4}{16a^2} = 4ay \Rightarrow y^4 = 64a^3y$$

$$y^4-64a^3y = 0 \Longrightarrow y(y^3-64a^3) = 0$$

$$y = 0 \ or \ y^3-64a^3 = 0 \Longrightarrow y^3 = 64a^3 \Longrightarrow y = \sqrt[3]{64a^3} \Longrightarrow y = 4a$$

$$If \ y = 0 \Longrightarrow x = 0; \quad y = 4a \Longrightarrow x = 4a$$

Therefore the point of intersection of (1)&(2) is (0,0) and (4a,4a)

x Varies from 0 to 4a and **y** varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$

The required Area $=\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dydx = \int_0^{4a}[y]_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx = \int_0^{4a}\left[2\sqrt{ax}-\frac{x^2}{4a}\right] dx$

$dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$

$dxdydz = |J| d\rho d\varphi d\theta$

$$J = \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} sin\varphi cos\theta & \rho cos\varphi cos\theta & -\rho sin\varphi sin\theta \\ sin\varphi sin\theta & \rho cos\varphi sin\theta & \rho sin\varphi cos\theta \\ cos\varphi & -\rho sin\varphi & 0 \end{vmatrix}$$

$=\rho^2 sin\varphi$

Hence $dV = dxdydz = \rho^2 sin\varphi d\rho d\varphi d\theta$

Therefore the integral will become,

$$\iiint_R f(x,y,z)\, dV = \int_\delta^\gamma \int_\alpha^\beta \int_a^b \rho^2 \sin\varphi \; f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, d\rho\, d\theta\, d\varphi$$

PROBLEMS BASED ON TRIPLE INTEGRATION

PART – A

1. **Evaluate: $\int_0^a \int_0^b \int_0^c (x+y+z)\, dzdydx$**

Solution: Let I = $\int_0^a \int_0^b \left(xz + yz + \frac{z^2}{2}\right)_0^c dydx$

$$= \int_0^a \int_0^b \left(cz + cy + \frac{c^2}{2}\right) \; dydx$$

$$= \int_0^a \left(cxy + \frac{cy^2}{2} + \frac{c^2 y}{2}\right)_0^b dx$$

$$= \int_0^a \left(bcx + \frac{cb^2}{2} + \frac{c^2 b}{2}\right) \; dx$$

$$= \left(\frac{bcx^2}{2} + \frac{cb^2 x}{2} + \frac{c^2 bx}{2}\right)_0^a$$

$$= \frac{bca^2}{2} + \frac{cb^2 a}{2} + \frac{c^2 ba}{2}$$

$$= \frac{abc}{2}(a+b+c)$$

X varies from 0 to a

Y varies from 0 to b$\left(1-\frac{x}{a}\right)$

Z varies from o to c$\left(1-\frac{x}{a}-\frac{y}{b}\right)$

Required Volume = $\iiint dzdydx$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx$$

$$= c \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \left[\left(1-\frac{x}{a}\right)-\frac{y}{b}\right] dydx$$

$$= c \int_0^a \left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c \int_0^a \left[b\left(1-\frac{x}{a}\right)^2-\frac{b}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx = \frac{abc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 \left(\frac{dx}{a}\right)$$

$$= \frac{abc}{2}\left[\frac{\left(1-\frac{x}{a}\right)}{-3}\right]_0^a = \frac{abc}{6}$$

4. Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integral.

Solution: Required Volume = 8 x volume in the positive octant = $8\iiint dxdydz$

Limits of integration are:

Z varies from 0 to $\sqrt{a^2-x^2-y^2}$

Y varies from 0 to $\sqrt{a^2-x^2}$

X varies from 0 to a

$$\text{Volume} = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx \quad = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} (z)_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dydx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2}+\frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left(\frac{a^2-x^2}{2}\right)\sin^{-1}(1)dx \;= 2\pi\int_0^a a^2-x^2\, dx = 2\pi\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= 2\pi\left[\frac{2a^3}{3}\right] \quad = \frac{4\pi a^3}{3}\text{cu.units}$$

5. Find the Volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c}1$

Solution: Required Volume = 8 x Volume in the first octant

Limits of Integration are:

DIFFERENTIAL EQUATIONS

UNIT STRUCTURE

6.1 OBJECTIVE

After going through this chapter you will able to

i. Define differential equation
ii. Order & degree of differential equation
iii. Formulate the differential equation

6.2 INTRODUCTION

We have already learned differential equation in XIIth . Hence we are going to discuss differential equation in brief. In this chapter we discuss only formulation of differential equation.

6.3 DIFFERENTIAL EQUATION

Definition:-

An equation involving independent and dependent variables and the differential coefficients or differentials is called a differential equation.

e.g. 1 $\frac{dy}{dx}=9$

x=independent variable

y= depedent variable

2 $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

3 $\frac{d^ny}{dx^n}+y=0$

These are all examples of differential equations.

The differential equation is said to be ordinary if it contains only one independent variable. All the examples of above are of ordinary differential equations.

Order and Degree of a Differential Equations:-

(i) Order:-

The order of the differential equations is the order of the highest orderal derivatives present in the function or equation.

If y = f (x) is a function, then

$\frac{dy}{dx}$ is the first order derivative,

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ is the second order derivative.

e.g 1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Order = 2

2) E = Ri + L $\frac{di}{dt}$

Order = 1

Degree:-

The degree of differential equation is the degree of the highest ordered derivative in the equation when it is made free from radicals and fractions.

e.g.

1 $\frac{d^2y}{dx^2} + k^2 y = 0$

order = 2, degree = 1

2 $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + Y = 0$

Order =2, degree =1

3 $y = \left(\frac{dy}{dx}\right)x + \frac{1}{\frac{dy}{dx}}$

Order=1, degree=2

4 $\sqrt[3]{\frac{dy^2}{dx^2}} = \sqrt{\frac{dy}{dx}}$

$\therefore \left(\frac{d^2y}{dx^2}\right)^{1/3} = \left(\frac{dy}{dx}\right)^{1/2}$

Cubing both sides

$$\therefore\ x=\int dt+\int\frac{a^2}{t^2-a^2}\cdot dt+c$$

$$\therefore\ x=t+\cancel{a}^2\cdot\frac{1}{2\cancel{a}}\cdot\log\left(\frac{t-a}{t+a}\right)+c$$

$$\therefore\ x=t+\frac{a}{2}\cdot\log\left(\frac{t-a}{t+a}\right)+c$$

t=x-y

$$\therefore\ x=x-y+\frac{a}{2}\cdot\log\left(\frac{x-y-a}{x-y+a}\right)+c$$

$$y=\frac{a}{2}\cdot\log\left(\frac{x-y-a}{x-y+a}\right)+c$$

This is the required general solution

Example 7: Solve $\frac{dy}{dx}=\cos(x+y)$

Solution: We have $\frac{dy}{dx}=\cos(x+y)$ ---------- (1)

Put $x+y=t$

Differentiating above with respect to x, we get

$$\therefore\ 1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\therefore\ \frac{dy}{dx}=\frac{dt}{dx}-1$$

Using equation (1)

$$\therefore\ \frac{dt}{dx}-1=\cos t$$

$$\therefore\ \frac{dt}{dx}=1+\cos t$$

$$\therefore\ \frac{1}{1+\cos t}\cdot dt=dx$$

$$\therefore\ \frac{1}{2\cos^2 t/2}dt=dx$$

This is invariable separable form,

Integrating we get

$$\therefore\ \int\frac{1}{2\cos^2 t/2}\cdot dt=\int dx+cons\tan t$$

$$\therefore\ \frac{1}{2}\cdot\int\sec^2 t/2\cdot dt=x+c$$

Substitute $\frac{y}{x} = v$

$\therefore$ y=vx

Differentiating above with respect to x, we get

$$\therefore \frac{\mathrm{dy}}{\mathrm{dx}} = v + x \cdot \frac{dv}{dx}$$

But the above equation can be written as

$$\therefore \frac{\mathrm{y}}{\mathrm{x}} \cdot \cos\frac{y}{x} - \left(\frac{x}{y} \cdot \sin\frac{y}{x} + \cos\frac{y}{x}\right) \cdot \frac{dy}{dx} = 0$$

$$\therefore \mathrm{v\ cosv} - \left(\frac{1}{\mathrm{v}} \cdot \sin v + \cos v\right) \cdot \left(v + x \cdot \frac{dy}{dx}\right) = 0$$

By rearranging the terms, we have

$$\therefore \frac{1}{\mathrm{x}} \cdot dx = -\frac{\sin v + v\cos v}{v\sin v} dv$$

$$\therefore \frac{1}{\mathrm{x}} \cdot dx + \frac{\sin v + v\cos v}{v\sin v} dv = 0$$

This is in variable separable form

Integrating we get,

$$\therefore \int \frac{1}{\mathrm{x}} \cdot dx + \int \frac{\sin v + v\cos v}{v\sin v} \cdot dv = constant$$

$$\therefore \mathrm{logx + log(v\,sinv) = c}$$

$$\mathrm{log(x\ vsinv) = log\,c}$$

$$\mathrm{xv \cdot sinv = c}$$

$$\mathrm{v} = \frac{\mathrm{y}}{\mathrm{x}}$$

$$\therefore \mathrm{x} \cdot \frac{\mathrm{y}}{\mathrm{x}} \sin\frac{\mathrm{y}}{\mathrm{x}} = c$$

$$\therefore \mathrm{ysin}\frac{\mathrm{y}}{\mathrm{x}} = c$$

This is the required general solution

Check Your Progress:

Solve the following

1) $\frac{dy}{dx} + e^{y/x} = \frac{y}{x}$ Ans : log cx=$e^{-y/x}$

2) $\left(1 + e^{x/y}\right) + e^{x/y}\left(1 - \frac{x}{y}\right) \cdot \frac{dy}{dx} = 0$ Ans : x+y$\cdot e^{x/y} = c$

$$\therefore\ \int\left(5x^4+6x^2y^2\text{-}8xy^3\right)dx+\int\left(-5y^4\right)\cdot dy=c$$

$$\therefore\ \cancel{5}\cdot\frac{x^5}{\cancel{5}}+\cancel{6}^2 y\cdot\frac{x^3}{\cancel{3}}-\cancel{8}^4 y^3\cdot\frac{x^2}{\cancel{2}}-\cancel{5}\cdot\frac{y^5}{\cancel{5}}=c$$

$$x^5+2x^3y^2-4x^2y^3-y^5=c$$

This is the required general solution

Example 16: Solve $\dfrac{dy}{dx}=-\dfrac{4x^3y^2+y\cos xy}{2x^4y+x\cos xy}$

Solution**:**

The given equation is

$$\frac{dy}{dx}=-\frac{4x^3y^3+y\cos xy}{2x^4y+x\cos xy}$$

$$\therefore\left(4x^3y^2+y\cos xy\right)\,dx+\left(2x^4+y\cos xy\right)\,dy=0\text{...........(1)}$$

Comparing with Mdx+Ndy=0; we have

$$M=4x^3y^2+y\cos xy$$

$$N=2x^4y+x\cos xy$$

$$\therefore\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}\left(4x^3y^2+y\cos xy\right)$$

$$\frac{\partial M}{\partial y}=8x^3y^2+\cos xy-xy\sin xy$$

$$\therefore\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}\left(2x^4y+y\cos xy\right)$$

$$\frac{\partial N}{\partial x}=8x^3y\ +\cos xy-xy\sin xy$$

$$\therefore\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Hence differential equation (1) is exact

Its solution is given by

$$\int Min\left(treat\ \text{y constant}\right)dx+\int N\left(terms\ \text{free from x}\right)\cdot dy=c$$

$$\left(4x^3y^2+y\cos xy\right)\ \text{dx}+\int\text{ody}=\text{c}$$

$$4y^2\int x^3dx+y\int\cos\ \text{xy}=\text{c}$$

$$4\text{y}^2\cdot\frac{x^4}{4}+y\frac{\sin xy}{y}=c$$

$$\therefore\quad x^4y^2+\sin xy=c$$

This is the required general solution

Example 17: Solve $\left(x-2e^y\right)dy+\left(y+x\sin x\right)dx=0$

7.5 LET US SUM UP

In this chapter we have learn

- ❖ solution of D.E:- general solution, particular solution
- ❖ variable separable form:- dx

$\varsigma f(x)dx = \varsigma f(y)dy + c$

- ❖ Equations reducible to variable separable form.
- ❖ Homogeneous differential equation i.e $\frac{dy}{dx} = \frac{f(xy)}{g(xy)}$

With substituting Y=Yx.

7.6 UNIT END EXERCISE

Solve the following differential equation.

i. $\frac{dy}{dx} = \frac{sinx + xcosx}{Y(1+2logu)}$

ii. $\frac{dy}{dx} + x^2 = x^2 e^3 y$

iii. $2x\,Cosy\,dx - (1+x^2)siny\,dy = 0$

iv. $(x+1)\frac{dy}{dx} + 1 = e^{x-y}$

v. $\frac{dy}{dx} = ax + by + c$

vi. $\frac{dy}{dx} = sin(x+y) + cos(x+y)$

vii. $\frac{dy}{dx} = e^{y/x} + y/x$

viii. $\frac{dy}{dx} = (4x + y + 1)^2$

ix. $\frac{dy}{dx} = y/x + sin\left(y/x\right)$

x. $\frac{dy}{dx} = (x + y + 1)^2$

xi. $\frac{dy}{dx} = 1 + y/x - Cos\,y/x$

xii. $(x^3 + y^3)\frac{dy}{dx} = x^2 y$

xiii. $\left(4 - \frac{y^2}{x^2}\right)dx + \frac{2y}{x}dy = 0$

The first order and first degree linear -
Differential equation is of the type

$$\frac{dy}{dx} + py = Q$$

Where y is dependent variable and x is independent variable. and p& Q are functions of x only. (may be constant)

The above differential equation is known as Leibnitz's linear differential equation.

Working Rule:

1) Consider linear differential equation.

$$\frac{dy}{dx} + py = Q$$

Where P and Q are function of x or constants only

Its integrating factor is given by

$$I.F. = e^{\int p dx}$$

Its solution is given by

$$y \cdot (\text{I.F.}) = \int Q \cdot (I.F.) dx + c$$

Where c is arbitrary constant.

2) For linear differential equation

$$\frac{dx}{dy} + p_1 x = Q_1$$

Where p_1 and Q_1 are functions of y or constants only

Its integrating factor is given by

$$\therefore \text{I.F.} = e^{\int p_1 dy}$$

Its solution is given by

$$x \cdot (IF) = \int Q \ (IF) \ dy + c$$

Where c is arbitrary constant.

Solved Examples:-

Example 4: Solve $(x+1) \dfrac{dy}{dx} - y = e^x (x+1)^2$

Solution: The given equation is

$$(x+1) \frac{dy}{dx} - y = e^x (x+1)^2$$

Dividing throughout by $(x+1)$ we have

$$\therefore \frac{dy}{dx} - \frac{1}{(x+1)} \cdot y = e^x \ (x+1) \text{.........(1)}$$

This is of the type

$$\therefore \frac{dy}{dx} + py = Q$$

Hence equation (1) is linear differential equation.

ii) $\dfrac{dy}{dx} - x^3 \cos^2 y = -x \sin 2y$

$Hint$ $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

$\div$ through out by $\cos^2 y$

$IF = e^{x^2}$

$2\tan y = x^2 - 1 + c_1 \cdot e^{-x^2}$

8.6 LET US SUM UP

In this chapter we have learned

- ❖ Integrating factor for non-exact equation.
- ❖ Using integrating factor find the solution of non-exact equation.
- ❖ Using integrating factor find the solution of linear differential equation.
- ❖ Bernoulli's equation.

8.7 UNIT END EXERCISE

Solve the following D.E:

i. $\dfrac{dy}{dx} + \dfrac{4x}{(x^2+1)} y = \dfrac{1}{(x^2+1)^3}$

ii. $\dfrac{dy}{dx} + x^2 y = x^5$

iii. $\dfrac{dy}{dx} + \dfrac{(1-2x)}{x^2} y = 1$

iv. $(1+y^2)dx = (\tan^{-1} y - x)dy$

v. $(x^2 + y^2 + 1)dx - 2xy\, dy = 0$

vi. $(4xy + 3y^2 - x)dx + x(x+2y)dy = 0$

vii. $(x^2 + y^2)dx - (x^2 + xy)dy = 0$

viii. $y(1+xy)dx + (1-xy) \times dy = 0$

ix. $(2y^2 + 4x^2 y)dx + (4xy + 3x^3)dy = 0$

x. $\dfrac{dy}{dx} + (\cot x)y = \cos x$

xi. $\dfrac{dy}{dx} + y \sec x = \tan x$

xii. $(1+x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$

xiii. $(1+x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1} x}$

xiv. $\dfrac{dy}{dx} + \dfrac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

xv. $\sec x\, dy = (y + \sin x)dx$

xvi. $(y \log x - 1)y\, dx = x\, dy$

xvii. $\frac{dy}{dx} + xy = x^3y^3$

xviii. $\frac{dy}{dx} + \frac{xy}{1-x^2} = xy\ {}^{1}\!/_{2}$

xix. $y - Cosx\frac{dy}{dx} = y^2(1 - Sinx)Cosx$

xx. $y\,dx + x(1 - 3x^2y^2)dy = 0$

*****

APPLICATIONS OF DIFFERENTIAL EQUATIONS

UNIT STRUCTURE

9.1 Objective
9.2 Introduction
9.3 Geometrical
9.4 Physical Application
9.5 Simple Electric Circuits
9.6 Newton's Law of Cooling
9.7 Let Us Sum Up
9.8 Unit End Exercise

9.1 OBJECTIVE

After going through this chapter you will able to

❖ Use differential equation to find the equation of any curve.

❖ Use differential equation physics like projectile motion, S.H.M's, Rectilinear motion, Newton's law of cooling.

❖ Use differential equation in electric circuits.

9.2 INTRODUCTION

In previous chapter we have learn to solve differential equations. We differ type. Now here we are going use differential equation in different field its useful to geometrical, physical, and electronic circuits, civil engineering and so on we are going to discuss few application of differential equation.

$$E = L \cdot \frac{di}{dt} + Ri$$

$$\therefore \; L\frac{di}{dt} + Ri = E$$

$$\therefore \; \frac{di}{dt} + \frac{R}{L} \cdot i = \frac{E}{L} \longrightarrow (1)$$

Which is a linear equation in i .

$\therefore$ P = $\frac{R}{L}$ · $Q = \frac{E}{L}$

$\therefore$ I.F. = $e^{\int pdt}$

= $e^{\int \frac{R}{L} \cdot dt}$

I.F. = $e^{\frac{R}{L} \cdot t}$

$\therefore$ The general solution is given by

$$i \, .(\, IF \,) = \int Q \, .(\, IF \,) \cdot dt + \text{constant}$$

$$\text{i. } e^{\frac{R}{L} \cdot t} = \int \frac{E}{L} \cdot e^{\frac{R}{L} \cdot t} \cdot dt + c$$

$$= \frac{E}{\cancel{L}} \cdot \frac{\cancel{L}}{R} \cdot e^{\frac{R}{L} \cdot t} + c$$

$$i. \, e^{\frac{R}{L} \cdot t} = \frac{E}{R} \cdot e^{\frac{R}{L} \cdot t} + c$$

$$\therefore \; i = \frac{E}{R} + c \cdot e^{\frac{-R}{L} \cdot t} \longrightarrow (2)$$

To find c we impose initial conditions

i.e. at t = o , i = o

$\therefore$ O = $\frac{E}{R} + C$

$\therefore$ C = $-\frac{E}{R}$

$\therefore$ *Equation* (2) becomes

$$i = \frac{E}{R} - \frac{E}{R} \cdot e^{-\frac{R}{L} \cdot t}$$

$$\therefore \; i = \frac{E}{R}\left(1 - e^{-\frac{R}{L} \cdot t}\right) \longrightarrow (3)$$

This is the expression for i at any time t.

Now as t increases decreases $e^{-\frac{R}{L} \cdot t}$ increases and its maximum value is $\frac{E}{R}$

Step (2)

Let the current in the circuit be half its theoretical maximum after a time T seconds then.

ORDINARY DIFFERENTIAL EQUATIONS

Higher order differential equations with constant coefficients – Method of variation of parameters – Cauchy's and Legendre's linear equations – Simultaneous first order linear equations with constant coefficients.

The study of a differential equation in applied mathematics consists of three phases.

(i) Formation of differential equation from the given physical situation, called modeling.
(ii) Solutions of this differential equation, evaluating the arbitrary constants from the given conditions, and
(iii) Physical interpretation of the solution.

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

General form of a linear differential equation of the nth order with constant coefficients is

$$\frac{d^n y}{dx^n} + K_1\frac{d^{n-1}y}{dx^{n-1}} + K_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots\ldots + K_n y = X \quad \ldots\ldots\ldots\ldots (1)$$

Where $K_1, K_2 \ldots\ldots\ldots K_n$ are constants.

The symbol D stands for the operation of differential

(i.e.,) Dy = $\frac{dy}{dx}$, similarly $D^2 y = \frac{d^{2y}}{dx^2}, D^3 y = \frac{d^3 y}{dx^3}, etc...$

The equation (1) above can be written in the symbolic form

$(D^n + K_1 D^{n-1} + \ldots\ldots\ldots + K_n)y = X$ i.e., f(D)y = X

Where f (D) = $D^n + K_1 D^{n-1} + \ldots\ldots\ldots + K_n$

Note

1. $\frac{1}{D}X = \int X dx$

2. $\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx$

3. $\frac{1}{D+a}X = e^{-ax}\int Xe^{ax}dx$

(i) The general form of the differential equation of second order is

9. Find the particular integral of $(D^2+1)y = \sin 2x \sin x$

Solution: Given $(D^2+1)y = \sin 2x \sin x$

$$= -\frac{1}{2}(\cos 3x - \cos x)$$

$$= -\frac{1}{2}\cos 3x + \frac{1}{2}\cos x$$

$$P.I_1 = \frac{1}{D^2+1}\left[-\frac{1}{2}\cos 3x\right]$$

$$= -\frac{1}{2}\frac{1}{-9+1}\cos 3x$$

$$= \frac{1}{16}\cos 3x$$

$$P.I_2 = \frac{1}{D^2+1}\left[\frac{1}{2}\cos x\right]$$

$$= \frac{1}{2}\frac{1}{-1+1}\cos x$$

$$= \frac{1}{2}x\frac{1}{2D}\cos x$$

$$= \frac{x}{4}\int \cos x dx$$

$$= \frac{x}{4}\sin x$$

$$\therefore P.I = \frac{1}{16}\cos 3x + \frac{x}{4}\sin x$$

Problems based on P.I. R.H.S = $e^{ax} + \cos ax (or) e^{ax} + \cos ax$

10. Solve $(D^2-4D+4)y = e^{2x} + \cos 2x$

Solution: Given $(D^2-4D+4)y = e^{2x} + \cos 2x$

The Auxiliary equation is $m^2-4m+4=0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$C.F = (Ax+B)e^{2x}$

$$P.I_1 = \frac{1}{D^2-4D+4}e^{2x}$$

$$= \frac{1}{4-8+4}e^{2x}$$

$$= \frac{1}{0}e^{2x}$$

$$= x\frac{1}{2D-4}e^{2x}$$

$$= x\frac{1}{0}e^{2x}$$

$$= \left[x - \frac{2}{(D+1)}\right]\frac{1}{(D^2+2D+1)}(\cos x)$$

$$= \left[x - \frac{2}{D+1}\right]\frac{1}{(-1+2D+1)}(\cos x)$$

$$= \left[x - \frac{2}{D+1}\right]\frac{\sin x}{2}$$

$$= \frac{x\sin x}{2}$$

$$= \frac{x\sin x}{2} - \frac{\sin x}{D+1}$$

$$= \frac{x\sin x}{2} - \frac{(D-1)\sin x}{D^2-1}$$

$$= \frac{x\sin x}{2} + \frac{\cos x - \sin x}{2}$$

The solution is y = $(A+Bx)e^{-x} + \frac{x\sin x}{2} + \frac{\cos x - \sin x}{2}$

17. Solve $(D^2+1)y = \sin^2 x$

Solution: A.E : $\mathbf{m}^2 + 1 = 0$

$\mathbf{m} = \pm i$

C.F =A cosx +B sinx

P.I = $\frac{1}{D^2+1}\sin^2 x$

$$= \frac{1}{D^2+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left\{\frac{1}{D^2+1}e^{0x} - \frac{1}{D^2+1}\cos 2x\right\}$$

$$= \frac{1}{2}\left\{1 + \frac{1}{3}\cos 2x\right\}$$

$$= \frac{1}{2} + \frac{1}{6}\cos 2x$$

$\therefore y =$ A cosx +B sinx $+ \frac{1}{2} + \frac{1}{6}\cos 2x$

18. Solve $\frac{d^2y}{dx^2} - y = x\sin x + (1+x)e^x$

Solution: A.E : $m^2 - 1 = 0$

$m = \pm 1$

C.F = $Ae^{-x} + Be^{x}$

$$\text{P.I}_1 = \frac{1}{f(D)}(xV) = \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}(V)$$

$$= -\int \frac{\sin 2x \sec 2x}{2}\,dx$$

$$= -\frac{1}{2}\int \sin 2x \frac{1}{\cos 2x}\,dx$$

$$= \frac{1}{4}\int \frac{-2\sin 2x}{\cos 2x}\,dx$$

$$= \frac{1}{4}\log(\cos 2x)$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2}\,dx$$

$$= \int \frac{\cos 2x \sec 2x}{2}\,dx$$

$$= \frac{1}{2}\int \cos 2x \frac{1}{\cos 2x}\,dx$$

$$= \frac{1}{2}\int dx$$

$$= \frac{1}{2}x$$

$$\text{P.I} = \text{P}f_1 + Qf_2$$

$$= \frac{1}{4}\log(\cos 2x)\,(\cos 2x) + \frac{1}{2}x\sin 2x$$

2. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = x\sin x$

Solution: The A.E is $m^2+1=0$

$$m = \pm i$$

$$\text{C.F} = C_1\cos x + C_2\sin x$$

Here $f_1 = \cos x \qquad f_2 = \sin x$

$$f_1' = -\sin x \qquad f_2' = \cos x$$

$$f_1 f_2' - f_1' f_2 = \cos^2 x + \sin^2 x = 1$$

$$P = -\int \frac{f_2 X}{f_1 f_2^1 - f_1' f_2}\,dx$$

$$= -\int \frac{\sin x(x\sin x)}{1}\,dx$$

$$= -\int x\sin^2 x\,dx$$

$$= -\int x\frac{(1-\cos 2x)}{2}\,dx$$

$$= -\frac{1}{2}\int (x - x\cos 2x)\,dx$$

$$= -\frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx$$

6. **Solve** $(D^2+a^2)y=\tan ax$ **by the method of variation of parameters**.

Solution: Given $(D^2+a^2)y=\tan ax$

The A.E is $m^2+a^2=0$

$$m=\pm ai$$

C.F = $c_1\cos ax+c_2\sin ax$

$f_1=\cos ax,\ f_2=\sin ax$

$f_1'=-a\sin x,\ f_2'=a\cos ax$

$$f_1f_2'-f_2f_1'=a\cos ax\cos ax-\sin ax(-a\sin ax)$$
$$=a\cos^2 ax+a\sin^2 ax$$
$$=a(\cos^2 ax+\sin^2 ax)$$
$$=\text{a}$$

P.I = Pf_1+Qf_2

$$P=-\int\frac{f_2X}{f_1f_2^1-f_1'f_2}dx$$
$$=-\int\frac{\sin ax\tan ax}{a}dx$$
$$=-\frac{1}{a}\int\frac{\sin^2 ax}{\cos ax}dx$$
$$=\frac{-1}{a}\int\frac{1-\cos^2 ax}{\cos ax}dx$$
$$=\frac{-1}{a}\int(\sec ax-\cos ax)dx$$
$$=\frac{-1}{a}\left[\frac{1}{a}\log(\sec ax+\tan ax)-\frac{\sin ax}{a}\right]$$
$$=\frac{-1}{a^2}[\log(\sec ax+\tan ax)-\sin ax]$$
$$=\frac{1}{a^2}[\sin ax-\log(\sec ax+\tan ax)]$$

$$Q=\int\frac{f_1X}{f_1f_2'-f_1'f_2}dx$$
$$=\int\frac{\cos ax\tan ax}{a}dx$$
$$=\frac{1}{a}\int\sin ax dx$$
$$=-\frac{1}{a^2}\cos ax$$

$$\therefore P.I=Pf_1+Qf_2$$
$$=\frac{1}{a^2}\cos ax[\sin ax-\log(\sec ax+\tan ax)]-\frac{1}{a^2}\sin ax[\cos ax]$$

$$\left(D^2-1\right)y=-2\sin t\text{..............(3)}$$

$$m^2-1=0, m=\pm 1$$

C.F = $Ae^{t}+Be^{-t}$

P.I $=-2\dfrac{\sin t}{D^2-1}=(-2)\dfrac{\sin t}{-1-1}=\sin t$

y = $Ae^{t}+Be^{-t}+\sin t$

(2) : x = cost –D(y)

x = cost - $\dfrac{d}{dt}\left(Ae^{t}+Be^{-t}+\sin t\right)$

x = $\cos t-Ae^{t}+Be^{-t}-\cos t$

x = $-Ae^{t}+Be^{-t}$

Now using the conditions given, we can find A and B

$$t=0, x=1\Rightarrow 1=-A+B$$

$$t=0, y=0\Rightarrow 0=A+B$$

B = $\dfrac{1}{2}$**, A =** $-\dfrac{1}{2}$

Solution is

x = $\dfrac{1}{2}e^{t}+\dfrac{1}{2}e^{-t}=\cosh t$

y = $-\dfrac{1}{2}e^{t}+\dfrac{1}{2}e^{-t}+\sin t=\sin t-\sinh t$

3. Solve $\dfrac{dx}{dt}+2y=-\sin t,\ \dfrac{dy}{dt}-2x=\cos t$

Solution: Dx + 2y = -sin t ………………(1)

- 2x +Dy = cos t ………………(2)

(1) $\times 2+(2)\times D\Rightarrow 4y+D^2y=-2\sin t+D(\cos t)$

$$\Rightarrow\left(D^2+4\right)=-3\sin t$$

$$m^2+4=0, m=\pm i2$$

C.F = $A\cos 2t+B\sin 2t$

P.I = $-\dfrac{3\sin t}{D^2+4}=\dfrac{-3\sin t}{-1+4}=-\sin t$

y = $A\cos 2t+B\sin 2t$ **- sin t**

(2) : x = $\dfrac{1}{2}\left[Dy-\cos t\right]$

x = $\dfrac{1}{2}\left[\dfrac{d}{dt}\left(A\cos 2t+B\sin 2t-\sin t\right)-\cos t\right]$

x = A cos2t +Bsin2t – cost

Solution is :

x = A cos2t +Bsin2t – cost

y = $A\cos 2t+B\sin 2t$ **- sin t**

Define function of class A.

Solution : A function which is sectionally continuous over any finite interval and is of exponential order is known as a function of class A.

♦ Important Result

(1) $L[1] = \dfrac{1}{s}$ where $s > 0$

(2) $L[t^n] = \dfrac{n!}{s^{n+1}}$ where $n = 0, 1, 2, \ldots$

(3) $L[t^n] = \dfrac{\Gamma n+1}{s^{n+1}}$ where n is not a integer.

(4) $L[e^{at}] = \dfrac{1}{s-a}$ where $s > a$ or $s - a > 0$

(5) $L[e^{-at}] = \dfrac{1}{s+a}$ where $s + a > 0$

(6) $L[\sin at] = \dfrac{a}{s^2+a^2}$ where $s > 0$

(7) $L[\cos at] = \dfrac{s}{s^2+a^2}$ where $s > 0$

(8) $L[\sinh at] = \dfrac{a}{s^2-a^2}$ where $s > |a|$ or $s^2 > a^2$

(9) $L[\cosh at] = \dfrac{s}{s^2-a^2}$ where $s^2 > a^2$

(10) $L[af(t) \pm bg(t)] = a\, L[f(t)] \pm b\, L[g(t)]$ [Linearity property]

Note : (1) $e^x = 1 + \dfrac{x}{\lfloor 1} + \dfrac{x^2}{\lfloor 2} + \ldots$

$e^{\infty} = 1 + \dfrac{\infty}{\lfloor 1} + \dfrac{\infty^2}{\lfloor 2} + \ldots$

(2) $e^{-\infty} = \dfrac{1}{e^{\infty}} = \dfrac{1}{\infty} = 0$

(3) $\Gamma_{n+1} = n!$

Result (2) : Prove that $L[t^n] = \frac{n!}{s^{n+1}}$ $[n = 0, 1, 2, \ldots]$

Proof : We know that

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\,dt = \int_0^\infty t^n\, d\left[\frac{e^{-st}}{-s}\right]$$

$$= t^n \left(\frac{e^{-st}}{-s}\right)\Bigg]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, n\, t^{n-1}\,dt$$

$$= (0 - 0) + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\,dt$$

i.e., $$L[t^n] = \frac{n}{s} L\left[t^{n-1}\right]$$

Similarly $$L[t^{n-1}] = \frac{n-1}{s} L\left[t^{n-2}\right]$$

$$L[t^{n-2}] = \frac{n-2}{s} L\left[t^{n-3}\right]$$

....

....

$$L[t^{n-(n-1)}] = \frac{n-(n-1)}{s} L\left[t^{\{n-(n-1)\}-1}\right]$$

$$= \frac{1}{s} L\left[t^0\right] = \frac{1}{s} L[1] = \frac{1}{s}\frac{1}{s}$$

$$\therefore L[t^n] = \frac{n}{s}\,\frac{n-1}{s} \ldots \frac{2}{s}\,\frac{1}{s}\,\frac{1}{s} = \frac{n!}{s^n}\,\frac{1}{s}$$

$$= \frac{n!}{s^{n+1}} \text{ where } [n = 0, 1, 2, \ldots]$$

Result (3) Prove that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$ **where** n **is not a integer.**

Proof : We know that $$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[t^n] = \int_0^\infty e^{-st} t^n\,dt$$

Put $st = x$ as $t \to 0 \Rightarrow x \to 0$

$s\,dt = dx$ as $t \to \infty \Rightarrow x \to \infty$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{s^{n+1}}\, dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x}\, dx$$

i.e., $L[t^n] = \dfrac{\Gamma_{n+1}}{s^{n+1}}$ $\left[\because \int_0^\infty x^n e^{-x}\, dx = \Gamma_{n+1}\right]$

when n is a positive integer.

we get $\Gamma_{n+1} = n!$

$$L\left[t^n\right] = \frac{n!}{s^{n+1}}$$

II. PROBLEMS BASED ON TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Example 1 **Find L[t]**

Solution: $L[t^n] = \dfrac{n!}{s^{n+1}}$ [we know that]

$$L[t] = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$$

Example 2 **Find L [t³]**

Solution : We know that $L\,[t^n] = \dfrac{n!}{s^{n+1}}$

$$L\,[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

Example 3 **Find L[√t]**

Solution : We know that $L[t^n] = \dfrac{\Gamma_{n+1}}{s^{n+1}}$

$$L[\sqrt{t}] = L[t^{1/2}] = \frac{\Gamma_{1/2+1}}{s^{1/2+1}}$$

$$L\left[e^{at}\right] = \int_0^{\infty} e^{-st}\ e^{at}\,dt = \int_0^{\infty} e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = -\frac{1}{s-a}\left[e^{-(s-a)t}\right]_0^{\infty}$$

$$= \frac{-1}{s-a}[0 - 1] = \frac{1}{s-a} \text{ where } s - a > 0$$

Example 6. Find the value $L\left[e^{3t}\right]$

Solution : We know that

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[e^{3t}] = \frac{1}{s-3}$$

Example 7 Find $L[e^{3t+5}]$

Solution :

W.K.T $\quad L[e^{at}] = \frac{1}{s-a}$

$$L[e^{3t+5}] = L[e^{3t}\ e^{5}]$$

$$= e^{5}\ L\ [e^{3t}] = e^{5}\left[\frac{1}{s-3}\right] = \frac{e^{5}}{s-3}$$

Example 8 Find $L\left[\frac{e^{at}}{a}\right]$

Solution : W.K.T $\quad L[e^{at}] = \frac{1}{s-a}$

$$L\left[\frac{e^{at}}{a}\right] = \frac{1}{a}L\,[e^{at}] = \frac{1}{a}\left[\frac{1}{s-a}\right]$$

Example 9 Find $L\ [2^{t}]$

Solution: W.K.T. $\quad L[e^{at}] = \frac{1}{s-a}$

$$L[2^{t}] = L\left[e^{\log 2^{t}}\right]$$

$$= L\left[e^{t \log 2}\right]$$

$$= L\left[e^{(\log 2)\,t}\right]$$

$$= \frac{1}{s - \log 2}$$

Example 15 **Prove that L [cos at] $= \dfrac{s}{s^2 + a^2}$ and L [sin at] $= \dfrac{a}{s^2 + a^2}$**

Solution : By Euler's theorem

$$e^{ix} = \cos x + i \sin x$$

$$e^{iat} = \cos at + i \sin at$$

$$L[e^{iat}] = L[\cos at + i \sin at]$$

$$= L[\cos at] + i\, L[\sin at]$$

$$L[\cos at] + i\, L[\sin at] = L[e^{iat}]$$

$$= \frac{1}{s - ia}$$

$$= \left[\frac{1}{s - ia}\right] \left[\frac{s + ia}{s + ia}\right]$$

$$= \frac{s + ia}{s^2 + a^2}$$

Equating real & Imaginary parts we get

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

$$L[\sin at] = \frac{a}{s^2 + a^2}$$

Example 16 **Find L [cos (at + b)]**

Solution : $L[\cos (at + b)]$

$$= L[\cos at \cos b - \sin at \sin b]$$

$$= \cos b\, L[\cos at] - \sin b\, L[\sin at]$$

$$= \cos b \left[\frac{s}{s^2 + a^2}\right] - \sin b \left[\frac{a}{s^2 + a^2}\right]$$

$$= \frac{s \cos b - a \sin b}{s^2 + a^2}$$

Example 17 **Find L [$\sin^2 2t$]**

Solution : $L[\sin^2 2t] = L\left[\dfrac{1 - \cos 4t}{2}\right] = \dfrac{1}{2} L[1 - \cos 4t]$

$$= \frac{1}{2}\Big[L[1] - L[\cos 4t]\Big]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2 + 16}\right]$$

Result 11. Prove that $L[f'(t)] = s\,L[f(t)] - f(0)$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty e^{-st}\,d[f(t)]$$

$$= e^{-st} f(t)\Big]_0^\infty - \int_0^\infty f(t)(-s)\,e^{-st}\,dt$$

$$= [0 - f(0)] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,L[f(t)]$$

$$= s\,L[f(t)] - f(0)$$

Result 12. Prove that $L[f''(t)] = s^2 L[f(t)] - s\,f(0) - f'(0)$

Proof : W.K.T. $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L[f''(t)] = \int_0^\infty e^{-st} f''(t)\,dt$$

$$= \int_0^\infty e^{-st}\,d[f'(t)]$$

$$= e^{-st} f'(t)\Big]_0^\infty - \int_0^\infty f'(t)(-s)\,e^{-st}\,dt$$

Example 6 show that $\int_0^\infty e^{-t}\, t\cos t\, dt = 0$

Solution : $\int_0^\infty e^{-t}\, t\, dt = \left[L\,[t\cos t]\right]_{s=1}$

$$= \left[-\frac{d}{ds}L(\cos t)\right]_{s=1} = \left[-\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s=1} = \left[-\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]\right]_{s=1}$$

$$= \left[-\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]\right]_{s=1} = \left[-\left[\frac{1-s^2}{(s^2+1)^2}\right]\right]_{s=1} = [-(0)] = 0$$

Example 7 Find $L\,[t e^{-t}\cosh t]$

Solution : $L\,[t e^{-t}\cosh t] = -\frac{d}{ds}L\,[e^{-t}\cosh t]$

$$= -\frac{d}{ds}\left[\frac{s+1}{(s+1)^2-1}\right] = -\left[\frac{[(s+1)^2-1]-(s+1)\,2\,(s+1)]}{[(s+1)^2-1]^2}\right]$$

$$= -\left[\frac{(s+1)^2-1-2\,(s+1)^2}{[(s+1)^2-1]^2}\right] = \frac{(s+1)^2+1}{(s^2+2s)^2} = \frac{s^2+2s+2}{s^4+4s^2+4s^3}$$

Result 18. Integrals of transform

If $L\,[f(t)] = \varphi(s)$ and $\frac{1}{t}f(t)$ has a limit as $t \to 0$ then

$$L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \varphi(s)\, ds$$

Proof : $\varphi(s) = L\,[f(t)]$

$$\int_s^\infty \varphi(s)\, ds = \int_s^\infty L\,[f(t)]\, ds$$

$$= \int_s^\infty \int_0^\infty e^{-st} f(t)\, dt\, ds = \int_0^\infty \int_s^\infty e^{-st} f(t)\, ds\, dt$$

[since s and t are independent variables and hence the order of integration in the double integral can be interchanged]

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\, ds\right] dt = \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty f(t)\left[0+\frac{e^{-st}}{t}\right] dt = \int_0^\infty f(t)\frac{e^{-st}}{t}\, dt$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}\, dt = L\left[\frac{1}{t}f(t)\right]$$

Example 1 Find the Laplace transform of the Half wave rectifier function

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \dfrac{\pi}{\omega} \\ 0, & \dfrac{\pi}{\omega} < t < \dfrac{2\pi}{\omega} \end{cases}$$

Solution : This function is a periodic function with period $\dfrac{2\pi}{\omega}$ in the interval $\left(0, \dfrac{2\pi}{\omega}\right)$

$$L[f(t)] = \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\, dt$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} \sin \omega t \, dt + 0 \right]$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2 + \omega^2} \left[-s \sin \omega t - \omega \cos \omega t\right] \right]_0^{\pi/\omega}$$

$$= \frac{1}{1 - e^{\frac{-2\pi s}{\omega}}} \left[\frac{e^{-s\pi/\omega}\,\omega + \omega}{s^2 + \omega^2} \right]$$

$$= \frac{\omega \left[1 + e^{\frac{-s\pi}{\omega}}\right]}{\left[1 - e^{-s\pi/\omega}\right] \left[1 + e^{-s\pi/\omega}\right] (s^2 + \omega^2)}$$

$$= \frac{\omega}{(s^2 + \omega^2)\left(1 - e^{-s\pi/\omega}\right)}$$