results we get

$$f\Big(f\Big(x,\frac{1}{2}\Big),y\Big)=f\Big(x\Big(\frac{1}{2}\Big)^{k-1},y\Big)=f\Big(x,f\Big(\frac{1}{2}\Big)\Big)=f\Big(x,\frac{1}{2}y^{k-1}\Big).$$

Let us now consider $x\le 2^{k-1}y$ in order to simplify the expression to the form $f\Big(x,\frac{1}{2}y^{k-1}\Big)=x\Big(\frac{y}{2}\Big)^{k-1}$, and if we take $x$ for which $2x\le y^{k-1}$ we get $k-1=(k-1)^2$, i.e. $k=1$ or $k=2$. For $k=1$ the solution is $f(x,y)=\min(x,y)$, and for $k=2$ the solution is $f(x,y)=xy$. It is easy to verify that both solutions satisfy the given conditions. $\triangle$

**Problem 23.** *(APMO 1989) Find all strictly increasing functions $f:\mathbb{R}\to\mathbb{R}$ such that*

$$f(x)+g(x)=2x,$$

*where $g$ is the inverse of $f$.*

**Solution.** Clearly every function of the form $x+d$ is the solution of the given equation. Another useful idea appears in this problem. Namely denote by $S_d$ the the set of all numbers $x$ for which $f(x)=x+d$. Our goal is to prove that $S_d=\mathbb{R}$. Assume that $S_d$ is non-empty. Let us prove that for $x\in S_d$ we have $x+d\in S_d$ as well. Since $f(x)=x+d$, according to the definition of the inverse function we have $g(x+d)=x$, and the given equation implies $f(x+d)=x+2d$, i.e. $x+d\in S_d$. Let us prove that the sets $S_{d'}$ are empty, where $d'<d$. From the above we have that each of those sets is infinite, i.e. if $x$ belongs to some of them, then each $x+kd$ belongs to it as well. Let us use this to get the contradiction. More precisely we want to prove that if $x\in S_d$ and $x\le y\le x+(d-d')$, then $y\notin S_{d'}$. Assume the contrary. From the monotonicity we have $y+d'=f(y)\ge f(x)=x+d$, which is a contradiction to our assumption. By further induction we prove that every $y$ satisfying

$$x+k(d-d')\le y<x+(k+1)(d-d')$$

can't be a member of $S_{d'}$. However this is a contradiction with the previously established properties of the sets $S_d$ and $S_{d'}$. Similarly if $d'>d$ switching the roles of $d$ and $d'$ gives a contradiction.

Simple verification shows that each $f(x)=x+d$ satisfies the given functional equation. $\triangle$

**Problem 24.** *Find all functions $h:\mathbb{N}\to\mathbb{N}$ that satisfy*

$$h(h(n))+h(n+1)=n+2.$$

**Solution.** Notice that we have both $h(h(n))$ and $h(n+1)$, hence it is not possible to form a recurrent equation. We have to use another approach to this problem. Let us first calculate $h(1)$ and $h(2)$. Setting $n=1$ gives $h(h(1))+h(2)=3$, therefore $h(h(1))\le 2$ and $h(2)\le 2$. Let us consider the two cases:

1° $h(2)=1$. Then $h(h(1))=2$. Plugging $n=2$ in the given equality gives $4=h(h(2))+h(3)=h(1)+h(3)$. Let $h(1)=k$. It is clear that $k\ne 1$ and $k\ne 2$, and that $k\le 3$. This means that $k=3$, hence $h(3)=1$. However from $2=h(h(1))=h(3)=1$ we get a contradiction. This means that there are no solutions in this case.

2° $h(2)=2$. Then $h(h(1))=1$. From the equation for $n=2$ we get $h(3)=2$. Setting $n=3,4,5$ we get $h(4)=3, h(5)=4, h(6)=4$, and by induction we easily prove that $h(n)\ge 2$, for $n\ge 2$. This means that $h(1)=1$. Clearly there is at most one function satisfying the given equality. Hence it is enough to guess some function and prove that it indeed solves the equation (induction or something similar sounds fine). The solution is

$$h(n)=\lfloor n\alpha\rfloor+1,$$

where $\alpha=\dfrac{-1+\sqrt{5}}{2}$ (this constant can be easily found $\alpha^2+\alpha=1$). Proof that this is a solution uses some properties of the integer part (although it is not completely trivial). $\triangle$

15. (IMO 2002, shortlist) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$f(f(x)+y)=2x+f(f(y)-x).$$

16. (Iran 1997) Let $f:\mathbb{R}\to\mathbb{R}$ be an increasing function such that for all positive real numbers $x$ and $y$:
$$f(x+y)+f(f(x)+f(y))=f(f(x+f(y))+f(y+f(x))).$$
Prove that $f(f(x))=x$.

17. (IMO 1992, problem 2) Find all functions $f:\mathbb{R}\to\mathbb{R}$, such that $f(x^2+f(y))=y+f(x)^2$ for all $x,y\in\mathbb{R}$.

18. (IMO 1994, problem 5) Let $S$ be the set of all real numbers strictly greater than -1. Find all functions $f:S\to S$ that satisfy the following two conditions:

    (i) $f(x+f(y)+xf(y))=y+f(x)+yf(x)$ for all $x,y\in S$;

    (ii) $\dfrac{f(x)}{x}$ is strictly increasing on each of the intervals $-1<x<0$ and $0<x$.

19. (IMO 1994, shortlist) Find all functions $f:\mathbb{R}^+\to\mathbb{R}$ such that $f(x)f(y)=y^\alpha f(x/2)+x^\beta f(y/2)$, za sve $x,y\in\mathbb{R}^+$.

20. (IMO 2002, problem 5) Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$(f(x)+f(z))(f(y)+f(t))=f(xy-zt)+f(xt+yz).$$

21. (Vietnam 2005) Find all values for a real parameter $\alpha$ for which there exists exactly one function $f:\mathbb{R}\to\mathbb{R}$ satisfying
$$f(x^2+y+f(y))=f(x)^2+\alpha\cdot y.$$

22. (IMO 1998, problem 3) Find the least possible value for $f(1998)$ where $f:\mathbb{N}\to\mathbb{N}$ is a function that satisfies
$$f(n^2f(m))=mf(n)^2.$$

23. Does there exist a function $f:\mathbb{N}\to\mathbb{N}$ such that
$$f(f(n-1))=f(n+1)-f(n)$$
for each natural number $n$?

24. (IMO 1987, problem 4) Does there exist a function $f:\mathbb{N}_0\to\mathbb{N}_0$ such that $f(f(n))=n+1987$?

25. Assume that the function $f:\mathbb{N}\to\mathbb{N}$ satisfies $f(n+1)>f(f(n))$, for every $n\in\mathbb{N}$. Prove that $f(n)=n$ for every $n$.

26. Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$, that satisfy:

    (i) $2f(m^2+n^2)=f(m)^2+f(n)^2$, for every two natural numbers $m$ and $n$;

    (ii) If $m\geq n$ then $f(m^2)\geq f(n^2)$.

27. Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ that satisfy:

    (i) $f(2)=2$;

    (ii) $f(mn)=f(m)f(n)$ for every two relatively prime natural numbers $m$ and $n$;

49. Let $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}^+$ be a function satisfying

$$f(xy, z) = f(x, z)f(y, z), \quad f(z, xy) = f(z, x)f(z, y), \quad f(x, 1 - x) = 1,$$

for all rational numbers $x, y, z$. Prove that $f(x, x) = 1$, $f(x, -x) = 1$, and $f(x, y)f(y, x) = 1$.

50. Find all functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ that satisfy

$$f(x, x) = x, \quad f(x, y) = f(y, x), \quad (x + y)f(x, y) = yf(x, x + y).$$